Linear Algebra MAT313 Fall 2021
Professor Sormani
Lesson 3

Please be sure to mark down the date and time that you start this lesson. Carefully take notes on pencil and paper while watching the lesson videos. Pause the lesson to try classwork before watching the video going over that classwork. If you work with any classmates, be sure to write their names on the problems you completed together. Please wear masks when meeting with classmates even if you meet off campus.

You will cut and paste the photos of your notes and completed classwork and a selfie taken holding up the first page of your work in a googledoc entitled:

MAT313F21-lesson3-lastname-firstname

and share editing of that document with me sormanic@gmail.com and with our graders. If you have a question, type QUESTION in your googledoc next to the point in your notes that has a question and email me with the subject MAT313 QUESTION. I will answer your question by inserting a photo into your googledoc or making an extra video.

Watch Playlist 313F21-3-1to9. The homework is at the end.

Linear Algebra Lesson 3

- Augmented Matrices
- Reduced Echelon Form

Classwork:

(1)
$$\boxed{\begin{aligned} 4x + 8y + 4z &= 16 \\ x + 2y + z &= 4 \\ x + y + z &= 3 \end{aligned}}$$

(2)
$$\boxed{\begin{aligned} 2x + 2y &= 4 \\ x + 3y &= 1 \\ 3x + 5y &= 6 \end{aligned}}$$

(3)
$$\boxed{\begin{aligned} x + y + 4z &= 12 \\ x + 2y + 4z &= 12 \end{aligned}}$$

The solution of one of these is a line. Find position and direction for it.

① Linear System

$$\boxed{\begin{aligned} 4x + 8y + 4z &= 16 \\ 1x + 2y + 1z &= 4 \\ 1x + 1y + 1z &= 3 \end{aligned}}$$

Notice we put 1 in front of variables with no coefficient:

$x = 1x$ and so on

Augmented Matrix Form

$$\left[\begin{array}{ccc|c} 4 & 8 & 4 & 16 \\ 1 & 2 & 1 & 4 \\ 1 & 1 & 1 & 3 \end{array}\right]$$

open bracket for a matrix

close bracket

represents the equals signs

② Linear System

$$\boxed{\begin{aligned} 2x + 2y &= 4 \\ 1x + 3y &= 1 \\ 3x + 5y &= 6 \end{aligned}}$$

Augmented Matrix Form

$$\left[\begin{array}{cc|c} 2 & 2 & 4 \\ 1 & 3 & 1 \\ 3 & 5 & 6 \end{array}\right]$$

(3) Linear System

$$\boxed{\begin{aligned} 1x + 1y + 4z &= 12 \\ 1x + 2y + 4z &= 12 \end{aligned}}$$

Augmented Matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 4 & 12 \\ 1 & 2 & 4 & 12 \end{array}\right]$$

don't forget $x = 1x$

(4)

$$\boxed{\begin{aligned} x + z &= 10 \\ y + z &= 8 \end{aligned}}$$

fill in missing variables with coefficient zero

$$\boxed{\begin{aligned} 1x + 0y + 1z &= 10 \\ 0x + 1y + 1z &= 8 \end{aligned}}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 10 \\ 0 & 1 & 1 & 8 \end{array}\right]$$

$$\boxed{
\begin{aligned}
\boxed{1x} + 2y + 1z &= 4\\
0x + \boxed{1y} + 0z &= 1\\
0x + 0y + 0z &= 0
\end{aligned}}
\qquad
\left[\begin{array}{ccc|c}
\boxed{1} & 2 & 1 & 4\\
0 & \boxed{1} & 0 & 1\\
0 & 0 & 0 & 0
\end{array}\right]$$

In Echelon Form!

Subup from Echelon Form

Last line $0x + 0y + 0z = 0$ is just $0 = 0$

leaders are: $\boxed{x}$, $\boxed{y}$ free variables: $\boxed{z = z}$

solve for the leaders from bottom row up

$0x + \boxed{1y} + 0z = 0 \Rightarrow \boxed{y = 0}$

$\boxed{1x} + 2y + 1z = 4 \Rightarrow x = 4 - 2y - z$

sub in the previous leader $x = 4 - 2\cdot 0 - z = 4 - z$

$$\left\{ \begin{pmatrix} x\\ y\\ z \end{pmatrix} = \begin{pmatrix} 4-z\\ 0\\ z \end{pmatrix} : z \in \mathbb{R} \right\}$$

It can be tricky to sub in previous leaders if they have long formulas

It would be nice to avoid the subup. of the previous leaders

We would prefer not to have any leaders on our formulas for other leaders. So we use:

Reduced Echelon Form

* Reduced Echelon Form *

$$\begin{aligned} \boxed{1x} + 2y + 1z &= 4 \\ 0x + \boxed{1y} + 0z &= 1 \\ 0x + 0y + 0z &= 0 \end{aligned}$$

Our Echelon Form

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 4 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

avoid previous leaders in the other leader's row

continue row reduction so that we have zeroes above each leader starting with the bottom leader

use skew actions to get zeroes above it

to remove 2y from row 1

$\rho_1 \to \rho_1 - 2\rho_2$

$\rho_1 \to \rho_1 - 2\rho_2$

$$\begin{aligned} \boxed{1x} + 0y + 1z &= 2 \\ 0x + \boxed{1y} + 0z &= 1 \\ 0x + 0y + 0z &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 2 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

zeroes above all leaders so this is reduced echelon form!

Now we solve for leaders & we are done

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2-z \\ 1 \\ z \end{pmatrix} : z \in \mathbb{R} \right\}$$

because solve each leader

$x = 2 - z$ (no y to sub)

$y = 1$

$0 = 0$

free: $z = z$

② Linear System — Solve it using Augmented Matrices and Reduced Echelon Form

$$\boxed{\begin{aligned} 2x + 2y &= 4 \\ 1x + 3y &= 1 \\ 3x + 5y &= 6 \end{aligned}}$$

convert to an Augmented Matrix

$$\left[\begin{array}{cc|c} \boxed{2} & 2 & 4 \\ 1 & 3 & 1 \\ 3 & 5 & 6 \end{array}\right]$$

next do row reduction to Echelon Form using only this

change 1st leader into a 1 using scale (or switch if zero)

$$\xrightarrow{\rho_1 \to \frac{1}{2}\rho_1} \left[\begin{array}{cc|c} \boxed{1} & 1 & 2 \\ 1 & 3 & 1 \\ 3 & 5 & 6 \end{array}\right]$$

← all entries in row 1 are divided by 2

← rest are copies

box the leader

make all entries below the leader into zeroes using skew $\rho_i \to \rho_i - k\rho_1$ because leader in row 1

$$\xrightarrow[\rho_3 \to \rho_3 - 3\rho_1]{\rho_2 \to \rho_2 - \rho_1} \left[\begin{array}{cc|c} \boxed{1} & 1 & 2 \\ 0 & \boxed{2} & -1 \\ 0 & 2 & 0 \end{array}\right]$$

copy row 1

$3-3(1)$ $5-3(1)$ $6-3(2)$

box the next leader

Change the second leader into a 1 by scaling (or switch if needed)

$$\xrightarrow{\rho_2 \to \frac{1}{2}\rho_2} \left[\begin{array}{cc|c} \boxed{1} & 1 & 2 \\ 0 & \boxed{1} & -1/2 \\ 0 & 2 & 0 \end{array}\right]$$

Make all entries below the leader into zeroes using skew $\rho_i \to \rho_i - k\rho_2$ because leader is in row 2.

$$\xrightarrow{\rho_3 \to \rho_3 - 2\rho_2} \left[\begin{array}{cc|c} \boxed{1} & 1 & 2 \\ 0 & \boxed{1} & -1/2 \\ 0 & 0 & 1 \end{array}\right]$$

$0-2(\frac{-1}{2}) = 1$

Echelon Form

Continue to Reduced Echelon Form

Make all entries above the last leader into zeroes using skew $\rho_i \to \rho_i - k\rho_2$ because last leader is in row 2

$$\xrightarrow{\rho_1 \to \rho_1 - \rho_2} \left[\begin{array}{cc|c} \boxed{1} & 0 & 5/2 \\ 0 & \boxed{1} & -1/2 \\ 0 & 0 & 1 \end{array}\right]$$

$2-(\frac{-1}{2}) = \frac{4}{2}+\frac{1}{2} = \frac{5}{2}$

Reduced Echelon Form

Rewrite as a linear system

$$1x + 0y = 5/2$$
$$0x + 1y = -1/2$$
$$0x + 0y = 1$$

Solve for leaders $x = \frac{5}{2}$

$y = -1/2$

that's all our variables (no free variables)

But wait! Final Line is $\boxed{0=1}$

No solution

$\emptyset$

can notice this earlier and so no solution sooner if you wish.

See $\left[\begin{array}{cc|c} \sim & & - \\ \sim & & - \\ 0 & 0 & 1 \end{array}\right]$

at any time during row reduction and there is no solution!

③ Linear System

$$\boxed{\begin{aligned} x + y + 4z &= 12 \\ x + 2y + 4z &= 12 \end{aligned}}$$

Solve using Augmented Matrices and Reduced Echelon Form

Convert to Augmented Matrix:

$$\boxed{\begin{aligned} 1x + 1y + 4z &= 12 \\ 1x + 2y + 4z &= 12 \end{aligned}}$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 4 & 12 \\ 1 & 2 & 4 & 12 \end{array}\right]$$

Row reduction to Echelon Form

1^{st} leader to a 1 ✓

zeroes under first leader using skew $\rho_2 \to \rho_2 + k\rho_1$

$$\xrightarrow{\rho_2 \to \rho_2 - \rho_1} \left[\begin{array}{ccc|c} \boxed{1} & 1 & 4 & 12 \\ 0 & \boxed{1} & 0 & 0 \end{array}\right]$$

2^{nd} leader to a 1 ✓

nothing under 2^{nd} leader

already Echelon Form

Row Reduction to Reduced Echelon Form

zeroes above last leader $\boxed{1}$ using skew $\rho_1 \to \rho_1 + k\rho_2$

$$\xrightarrow{\rho_1 \to \rho_1 - \rho_2} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 4 & 12 \\ 0 & \boxed{1} & 0 & 0 \end{array}\right]$$

This is reduced Echelon Form

Change back into a linear system

$$\boxed{\begin{aligned} \boxed{1x} + 0y + 4z &= 12 \\ 0x + \boxed{1y} + 0z &= 0 \end{aligned}}$$

leaders: x, y

free $z = z$

Solve for leaders

$x = 12 - 4z$

$y = 0$

$z = z$ (free)

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12-4z \\ 0 \\ z \end{pmatrix} : z \in \mathbb{R} \right\}$$

Check:

original system

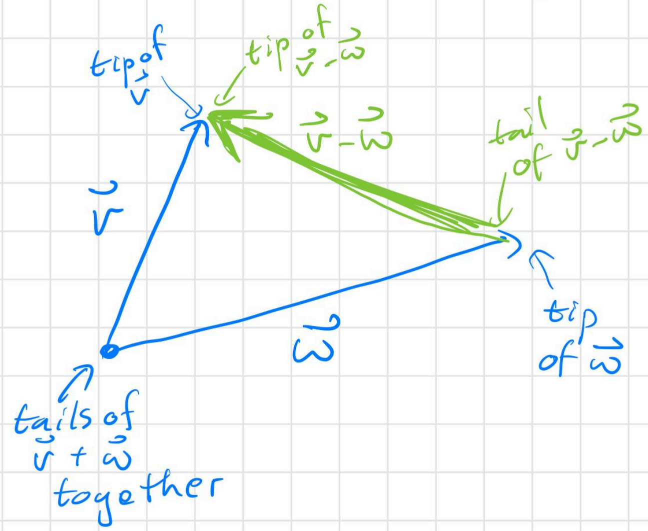
$x + y + 4z = 12$

$(12-4z) + 0 + 4z = 12$ ✓

$x + 2y + 4z = 12$

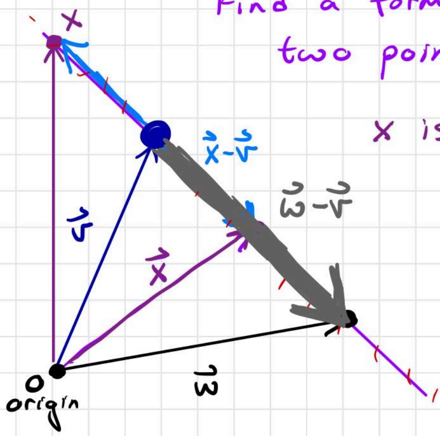
$(12-4z) + 2 \cdot 0 + 4z = 12$ ✓

So next we want to write this in line form.



Lines written in Vector Notation

Find a formula for a line through two points $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix}$



x is a typical point on the line

$$\vec{x}-\vec{v} = t\,(\vec{w}-\vec{v})$$

$$\Big\{\, \vec{x} = \underset{\substack{\text{position vector } \vec{v} \\ \text{on the line}}}{\vec{v}} + t\,\underbrace{(\vec{w}-\vec{v})}_{\substack{\text{direction vector } (\vec{w}-\vec{v}) \\ \text{where } \vec{w} \text{ is also on the line}}} \;\Big|\; t \in \mathbb{R} \,\Big\}$$

as we change the value of t we get different points $\vec{x}$ lying on the line

$$\left\{ \begin{pmatrix} x\\ y\\ z\end{pmatrix} = \begin{pmatrix} 12-4z\\ 0\\ z\end{pmatrix} \;\middle|\; z\in\mathbb{R} \right\}$$

So next we want to write this in line form $\{\vec{x} = \vec{v} + t(\vec{w}-\vec{v}) \mid t\in\mathbb{R}\}$

t is the free variable $t\in\mathbb{R}$

position vector $\vec{v}$ on the line

direction vector $(\vec{w}-\vec{v})$ where $\vec{w}$ is also on the line

Our solution set $z\in\mathbb{R}$ so $z=t\in\mathbb{R}$

position is the constant terms without the free variable $t=z$

$$\left\{ \begin{pmatrix} x\\ y\\ z\end{pmatrix} = \begin{pmatrix} 12-4z\\ 0+0z\\ 0+1z\end{pmatrix} \;\middle|\; z\in\mathbb{R} \right\}$$

$\vec{x}$ $\qquad \vec{v} = \begin{pmatrix} 12\\ 0\\ 0\end{pmatrix}$ $\qquad$ direction $(\vec{w}-\vec{v}) = \begin{pmatrix} -4\\ 0\\ 1\end{pmatrix}$

The solution in line form:

$$\left\{ \vec{x} = \begin{pmatrix} 12\\ 0\\ 0\end{pmatrix} + z\begin{pmatrix} -4\\ 0\\ 1\end{pmatrix} \;\middle|\; z\in\mathbb{R} \right\}$$

Try classwork ① solution in lineform.

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2-z \\ 1 \\ z \end{pmatrix} : z \in \mathbb{R} \right\}$$

Classwork ①

Convert to line form

$$\left\{ \vec{x} = \vec{v} + t(\vec{w} - \vec{v}) : t \in \mathbb{R} \right\}$$

What is our free variable?

In this case it is $z \in \mathbb{R}$

(we have only one free variable) ✓

(Only a line if we have a single free variable)

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2-z \\ 1 \\ z \end{pmatrix} = \begin{pmatrix} 2-1z \\ 1+0z \\ 0+1z \end{pmatrix} : z \in \mathbb{R} \right\}$$

position $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ direction $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} : z \in \mathbb{R} \right\}$$ ← line form

student who took vector calc can graph this.

④ $\boxed{\begin{array}{l} x + z = 10 \\ y + z = 8 \end{array}}$ → to convert to an augmented matrix add in missing variables with zero coefficients

$$\boxed{\begin{array}{l} 1x + 0y + z = 10 \\ 0x + 1y + 1z = 8 \end{array}}$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 10 \\ 0 & \boxed{1} & 1 & 8 \end{array}\right]$$ Already in Reduced Echelon Form

two leaders with coeff = 1 already and zeroes below and above.

Convert back to a system:

$$\boxed{\begin{array}{l} \boxed{1x} + 0y + z = 10 \\ 0x + \boxed{1y} + 1z = 8 \end{array}}$$ leaders: x and y

free: z

Solve for leaders
$x = 10 - z$
$y = 8 - z$
$z = z$ (free)

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 - z \\ 8 - z \\ z \end{pmatrix} = \begin{pmatrix} 10 - 1z \\ 8 - 1z \\ 0 + 1z \end{pmatrix} \middle| \ z \in \mathbb{R} \right\}$$ only one free so we can write it as a line

position $\begin{pmatrix} 10 \\ 8 \\ 0 \end{pmatrix}$ direction $\begin{pmatrix} -1 \\ -1 \\ +1 \end{pmatrix}$

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 8 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ -1 \\ +1 \end{pmatrix} \middle| \ z \in \mathbb{R} \right\}$$

Homework: Convert to an Augmented Matrix
Do Row Reduction to Reduced Echelon Form
Solve for leader and write Solution Set
If there is one free variable write the
solution set as a line with position and direction.

HW1

$$\begin{aligned} 2x + 2y + 4z &= 12 \\ x + y + z &= 5 \\ x - y + z &= 1 \end{aligned}$$

HW2

$$\begin{aligned} 2x + 2y + 4z &= 12 \\ x + y + z &= 5 \\ 2x + 2y + 2z &= 8 \end{aligned}$$

HW3

$$\begin{aligned} x + y + z + w &= 0 \\ x + y - z + w &= 0 \\ 2x + 3y + 2z + w &= 0 \end{aligned}$$

HW4

$$\begin{aligned} 1x_1 + 2x_2 + 3x_3 + 4x_4 &= 0 \\ 1x_1 + 2x_2 + 1x_3 + 1x_4 &= 0 \\ 2x_1 + 4x_2 + 6x_3 + 8x_4 &= 0 \\ 2x_1 + 4x_2 + 4x_3 + 5x_4 &= 0 \\ 3x_1 + 6x_2 + 3x_3 + 3x_4 &= 0 \end{aligned}$$