Vector Calculus MAT226 Fall 2021
Professor Sormani
Lesson 24: More Integration 14.2

Please be sure to mark down the date and time that you start this lesson. Carefully take notes on pencil and paper while watching the lesson videos. Pause the lesson to try classwork before watching the video going over that classwork. If you work with any classmates, be sure to write their names on the problems you completed together. Please wear masks when meeting with classmates even if you meet off campus.

You will cut and paste the photos of your notes and completed classwork and a selfie taken holding up the first page of your work in a googledoc entitled:

MAT226F21-lesson24-lastname-firstname

and share editing of that document with me sormanic@gmail.com and with our graders. If you have a question, type QUESTION in your googledoc next to the point in your notes that has a question and email me with the subject MAT226 QUESTION. I will answer your question by inserting a photo into your googledoc or making an extra video.

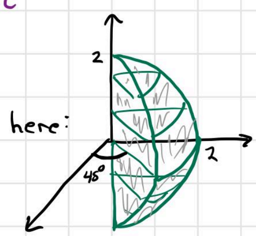
Watch the Playlist 226F21-24-0to4 which includes three classwork problems.

Double Integration to Find Volumes and Averages

Classwork ① Find the volume of the region bounded by the following graphs

$y = 16 - x^2 \quad z = 16 - x^2$ first octant

Classwork ② Find the volume of the shaded region depicted here: which is a sector of a ball



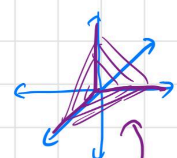
Classwork ③ Find the average of the function $f(x,y) = e^{5x+2y}$ over the triangle with vertices:

$(2,-1) \quad (1,4) \quad (0,3)$

Classwork ① Find the volume of the region bounded by the following graphs

$y = 16 - x^2 \quad z = 16 - x^2$ first octant

$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \begin{array}{c} y = 16 - x^2 \\ z \in \mathbb{R} \end{array} \right\} \quad \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \begin{array}{c} z = 16 - x^2 \\ y \in \mathbb{R} \end{array} \right\}$



The first octant is the region where $x \geq 0$ $y \geq 0$ and $z \geq 0$

$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \begin{array}{c} y = 16 - x^2 \\ z \in \mathbb{R} \end{array} \right\}$ is a parabolic cylinder

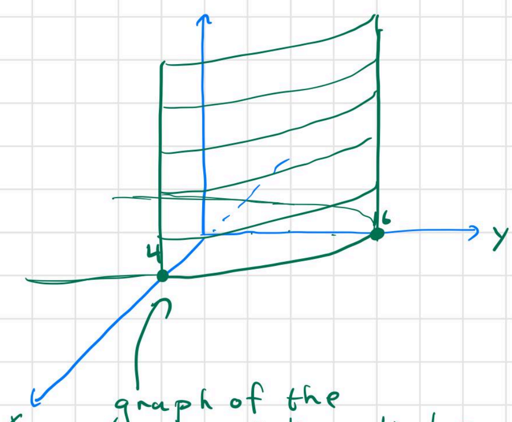
$y = 16 - x^2$ a parabola in xy plane

when $x = 0$, $y = 16 - 0^2 = 16$

when $y = 0$, $0 = 16 - x^2 \Rightarrow 16 = x^2 \Rightarrow x = \pm 4$

Since z is free we have vertical lines up from the parabola.

pause + try to graph $z = 16 - x^2$

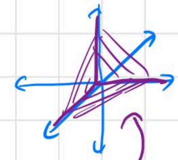


graph of the parabolic cylinder restricted to the first octant

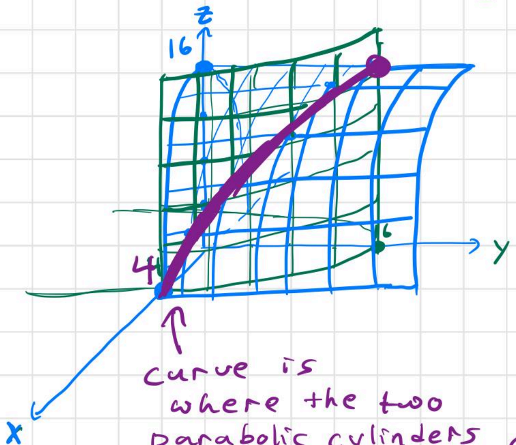
Classwork ① Find the volume of the region bounded by the following graphs

$y = 16 - x^2$ $\quad z = 16 - x^2$ $\quad$ first octant

$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \begin{matrix} y = 16 - x^2 \\ z \in \mathbb{R} \end{matrix} \right\}$ $\quad \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \begin{matrix} z = 16 - x^2 \\ y \in \mathbb{R} \end{matrix} \right\}$



The first octant is the region where $x \geq 0$ $y \geq 0$ and $z \geq 0$



curve is where the two parabolic cylinders meet

$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \begin{matrix} y = 16 - x^2 \\ z \in \mathbb{R} \end{matrix} \right\}$ is a parabolic cylinder

$y = 16 - x^2$ a parabola in xy plane

when $x = 0$, $y = 16 - 0^2 = 16$

when $y = 0$, $0 = 16 - x^2 \Rightarrow 16 = x^2 \Rightarrow x = \pm 4$

since z is free we have lines $\updownarrow$

$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \begin{matrix} z = 16 - x^2 \\ y \in \mathbb{R} \end{matrix} \right\}$ is a parabolic cylinder

$z = 16 - x^2$ is a parabola in the xz plane

when $x = 0$, $z = 16 - 0^2 = 16$

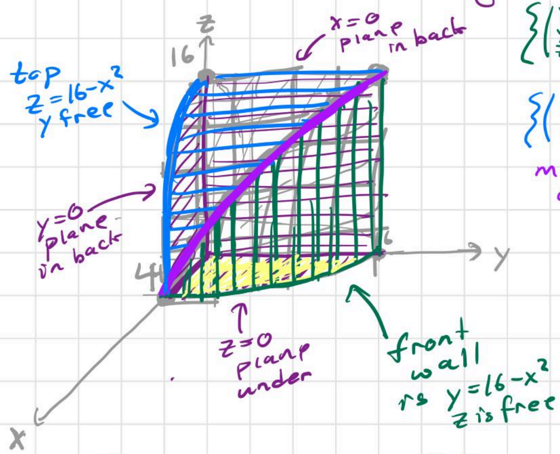
when $z = 0$, $0 = 16 - x^2 \Rightarrow x = \pm 4$

since y is free we have $\longleftrightarrow$ lines

$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \begin{matrix} y = 16 - x^2 & x \geq 0 \\ z = 16 - x^2 & y \geq 0 \\ & z \geq 0 \end{matrix} \right\}$ Notice $y = z$ on this curve

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 16 - x^2 \\ 16 - x^2 \end{pmatrix} \middle| 0 \leq x \leq 4 \right\}$$

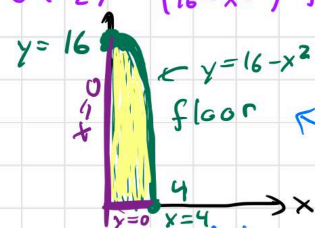
The first octant is the region where $x \geq 0$ $y \geq 0$ and $z \geq 0$



$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \begin{array}{l} y = 16 - x^2 \\ z \in \mathbb{R} \end{array} \right\}$ is a parabolic cylinder

$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \begin{array}{l} z = 16 - x^2 \\ y \in \mathbb{R} \end{array} \right\}$ is a parabolic cylinder

meet along: $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 16 - x^2 \\ 16 - x^2 \end{pmatrix} \middle| 0 \leq x \leq 4 \right\}$



We can consider this as an integral of the function $z = f(x,y) = 16 - x^2$ because $z = 16 - x^2$ is the top and the bottom is just $z = 0$

$$\iint_R 16 - x^2 \, dA$$

Our region is the floor

min $x = ?$ max $x = ?$

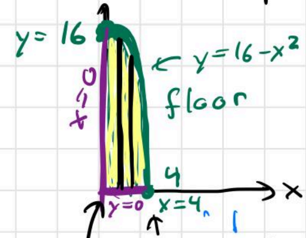
what about stripes of y

$$\iint_R 16 - x^2 \, dA$$

Our region is the floor

min x = ? max x = ?

what about stripes of y



min $x = 0$ max $x = 4$

bottom of each stripe is $y=0$

top of each stripe is $y = 16 - x^2$

$$= \int_0^4 \int_{y=0}^{y=16-x^2} \underbrace{16 - x^2}_{\text{const in } y} \, dy \, dx$$

$$= \int_0^4 (16 - x^2)\, y \Big|_0^{16-x^2} dx$$

$$= \int_0^4 (16-x^2)(16-x^2) - \cancel{(16-x^2)\cdot 0}_{\,0} \, dx$$

$$= \int_0^4 256 - 32x^2 + x^4 \, dx = 256x - 32\frac{x^3}{3} + \frac{x^5}{5} \Big|_0^4$$

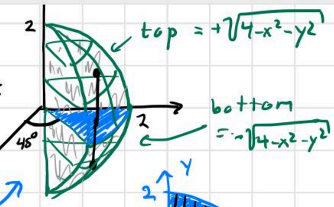
$$= 256 \cdot 4 - 32 \cdot \frac{4^3}{3} + \frac{4^5}{5} = 4^5 - \frac{2 \cdot 4^2 4^3}{3} + \frac{4^5}{5} = 4^5\left(1 - \frac{2}{3} + \frac{1}{5}\right)$$

$$= 4^5\left(\frac{8}{15}\right)$$

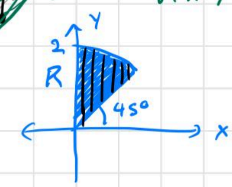
= use a calulator!

Also compute with dx inside and dy outside.

Classwork ② Find the volume of the shaded region depicted here: which is a sector of a ball



Let R = xy-region in the xy plane



pause + try!

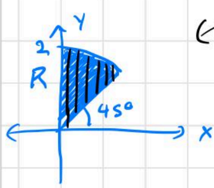
$$\iint_R \text{top} - \text{bottom } dA$$

The top and bottom lie on the sphere $x^2+y^2+z^2=2^2$

Solve for z because we are interested in height

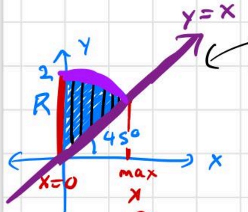
$$z^2 = 2^2 - x^2 - y^2 \qquad z = \pm\sqrt{4-x^2-y^2}$$

$$= \iint_R \sqrt{4-x^2-y^2} - \left(-\sqrt{4-x^2-y^2}\right) dA = \iint_R 2\sqrt{4-x^2-y^2}$$



Find the bounds of integration using vertical stripes.

pause + try!



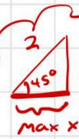
Find the bounds of integration using vertical stripes.

$\boxed{\text{Inside integral is with respect to } y}$

$\boxed{\text{Outside integral is with respect to } x}$

min $x = 0$

max $x = \sqrt{2}$



$\cos 45° = \frac{\max x}{2}$

$\frac{\sqrt{2}}{2} = \frac{\max x}{2}$

$\sqrt{2} = \max x$

for each fixed x (each vertical stripe)

min y is from a line of slope 1 because of 45° through the origin $y = x$

So min $y = x$

max y is from the curve of circle of radius 2

$x^2 + y^2 = 2^2$

$y^2 = 2^2 - x^2$

$y = \pm\sqrt{4 - x^2}$

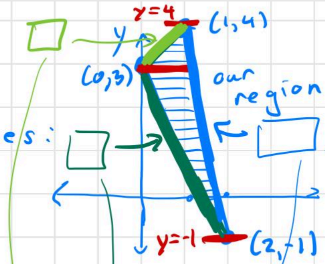
$y = \sqrt{4 - x^2}$ because $y > 0$ on the curve

So max $y = \sqrt{4 - x^2}$

$$= \int_0^{\sqrt{2}} \int_x^{\sqrt{4-x^2}} 2\sqrt{4 - x^2 - y^2}\, dy\, dx$$

Incredibly difficult to Integrate so skip!

Classwork 3 Find the $\boxed{\text{average}}$ of the function $f(x,y) = e^{5x+2y}$ over the triangle with vertices: $(2,-1)$ $(1,4)$ $(0,3)$.



average of f over R = $\dfrac{1}{\text{Area}(R)} \iint_R f(x,y)\, dx\, dy$

pause + try to find bounds of integration with horizontal stripes

min $y = -1$ and max $y = 4$

warning the left end of the stripes has two segments

switch at $y = 3$

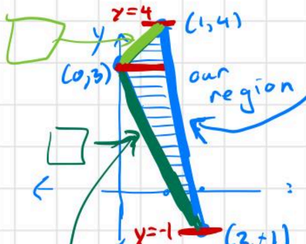
$$\iint_R f\, dx\, dy = \int_{y=-1}^{3} \int_{\square}^{\square} f\, dx\, dy + \int_{y=3}^{4} \int_{\square}^{\square} f\, dx\, dy$$

two parts

Fill in between videos!

$$\iint_R f\,dx\,dy = \int_{y=-1}^{3}\int_{\square}^{\square} f\,dx\,dy + \int_{y=3}^{4}\int_{\square}^{\square} f\,dx\,dy$$

Fill in between videos!



line through (1,4) and (2,-1)

so slope is $\frac{\Delta y}{\Delta x} = \frac{-1-4}{2-1} = \frac{-5}{1} = -5$

$y - y_0 = m(x - x_0)$

$y - 4 = -5(x-1)$

solve for x $\quad \frac{y-4}{-5} = x - 1$

$x = \boxed{1 + \left(\frac{-1}{5}\right)(y-4)}$

pause + try

line through (0,3) and (2,-1) so slope is $\frac{\Delta y}{\Delta x} = \frac{3-(-1)}{0-2} = \frac{4}{-2} = -2$

$y - 3 = -2(x-0)$

solve for x $\quad y-3 = -2x \qquad x = \boxed{\frac{y-3}{-2}}$

pause + try last box between videos.

line through (0,3) and (1,4) slope is $\frac{\Delta y}{\Delta x} = \frac{4-3}{1-0} = 1 \quad y-3 = 1(x-0)$

$x = \boxed{y-3}$

$$\iint_R f\,dx\,dy = \int_{-1}^{3}\int_{\boxed{\frac{y-3}{-2}}}^{\boxed{1+\left(\frac{-1}{5}\right)(y-4)}} f\,dx\,dy + \int_{3}^{4}\int_{\boxed{y-3}}^{\boxed{1+\left(\frac{-1}{5}\right)(y-4)}} f\,dx\,dy$$

For the area of Region

$$\text{Area}(R) = \iint_R 1\, dA = \int_{-1}^{3} \int_{\left(\frac{-1}{2}\right)(y-3)}^{1+\left(\frac{-1}{5}\right)(y-4)} 1 \, dx\, dy + \int_{3}^{4} \int_{y-3}^{1+\left(\frac{-1}{5}\right)(y-4)} 1 \, dx\, dy$$

$$= \text{pause + try}$$

$$\text{Average of } f = \frac{1}{\text{Area}(R)} \left(\int \int e^{5x+2y}\, dA + \int \int e^{5x+2y}\, dA \right)$$

$$= \text{pause + try.}$$

For the area of Region

$$\text{Area}(R) = \iint_R 1\,dA = \int_{-1}^{3} \int_{\left(\frac{-1}{2}\right)(y-3)}^{1+\left(\frac{-1}{5}\right)(y-4)} 1 \, dx\,dy + \int_{3}^{4} \int_{y-3}^{1+\left(\frac{-1}{5}\right)(y-4)} 1 \, dx\,dy$$

$$= \int_{-1}^{3} x \Big|_{\left(\frac{-1}{2}\right)(y-3)}^{1+\left(\frac{-1}{5}\right)(y-4)} dy + \int_{3}^{4} x \Big|_{(y-3)}^{1+\left(\frac{-1}{5}\right)(y-4)} dy$$

don't forget parenthesis!

$$= \int_{-1}^{3} 1+\left(\tfrac{-1}{5}\right)(y-4) - \left(\tfrac{-1}{2}\right)(y-3)\,dy + \int_{3}^{4} 1+\left(\tfrac{-1}{5}\right)(y-4) - (y-3)\,dy$$

$$= y + \left(\tfrac{-1}{5}\right)\left(\tfrac{y^2}{2} - 4y\right) + \tfrac{1}{2}\left(\tfrac{y^2}{2} - 3y\right)\Big|_{-1}^{3} + y + \left(\tfrac{-1}{5}\right)\left(\tfrac{y^2}{4} - 4y\right) - \left(\tfrac{y^2}{2} - 3y\right)\Big|_{3}^{4}$$

$$= 3 + \left(\tfrac{-1}{5}\right)\left(\tfrac{9}{2} - 12\right) + \tfrac{1}{2}\left(\tfrac{9}{2} - 9\right) + 4 + \left(\tfrac{-1}{5}\right)\left(\tfrac{16}{4} - 16\right) - \left(\tfrac{16}{2} - 12\right)$$

$$- (-1) - \left(\tfrac{-1}{5}\right)\left(\tfrac{1}{2} + 4\right) - \tfrac{1}{2}\left(\tfrac{1}{2} + 3\right) \qquad - 3 - \left(\tfrac{-1}{5}\right)\left(\tfrac{9}{4} - 12\right) + \left(\tfrac{9}{2} - 9\right)$$

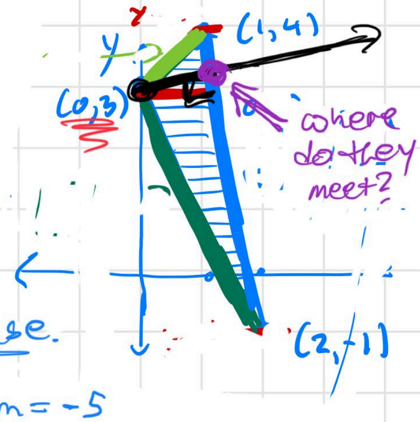
= use a calculator to get a final answer.

Notice this is the area of a triangle so we could also compute it using $\frac{1}{2}$(base)(height) but be careful determining what the base and the height are!

$$\text{Area} = \frac{1}{2}(\text{base})(\text{height})$$

base from $(2,-1)$ to $(1,4)$

$$\text{base} = \sqrt{(2-1)^2 + (-1-4)^2} = \sqrt{26}$$



What is the height?

distance from $(0,3)$ to the base.

The base is a line segment $m = -5$

$$\boxed{y - 4 = -5(x-1)}$$

$\perp$ line has slope $\frac{-1}{m} = \frac{1}{5}$ $\quad \boxed{y - 3 = \frac{1}{5}(x - 0)}$

solve together to find where they meet (,)

height = distance from $(0,3)$ to (,)

Finish on your own!

$$\text{Average of } f = \frac{1}{\text{Area}(R)}\left(\underbrace{\int_{-1}^{3}\int_{\left(\frac{-1}{2}\right)(y-3)}^{1+\left(\frac{-1}{5}\right)(y-4)} e^{5x+2y}\,dx\,dy + \int_{3}^{4}\int_{y-3}^{1+\left(\frac{-1}{5}\right)(y-4)} e^{5x+2y}\,dx\,dy}_{\text{Hint for these integrals}}\right)$$

fill in area

$$\int e^{5x+2y}\,dx = \int e^{u}\,\frac{1}{5}\,du$$

$u = 5x+2y$

$du = 5\,dx$

$\frac{1}{5}du = dx$

$$= e^{u}\,\frac{1}{5} + C$$

$$= \frac{1}{5}e^{5x+2y} + C$$

$$= \frac{1}{\text{Area}(R)}\left(\int_{-1}^{3} \frac{1}{5}e^{5x+2y}\Big|_{\left(\frac{-1}{2}\right)(y-3)}^{1+\left(\frac{-1}{5}\right)(y-4)} dy + \int_{3}^{4} \frac{1}{5}e^{5x+2y}\Big|_{y-3}^{1+\left(\frac{-1}{5}\right)(y-4)} dy\right)$$

$$= \frac{1}{\text{Area}(R)}\left(\int_{-1}^{3} \frac{1}{5}e^{5\left(1+\left(\frac{-1}{5}\right)(y-4)\right)+2y} - \frac{1}{5}e^{5\left(\frac{-1}{2}\right)(y-3)+2y}\,dy\right.$$

$$\left.+ \int_{3}^{4} \frac{1}{5}e^{5\left(1+\left(\frac{-1}{5}\right)(y-4)\right)+2y} - \frac{1}{5}e^{5(y-3)}\,dy\right)$$

simplify the exponents before integrating.

Finish it yourselves!

Before doing the homework verify that you have watched the entire Playlist 226F21-24-0to4 which includes three classwork problems.

When doing homework, draw in the stripes as done in the classwork.

14.2/ do four volumes of sketched regions, do three set up and evaluate double integral, Putnam challenge,

Review HW: Polar Coordinates 10.4

Include a selfie holding a page of your work.