A Literature Review of the Product and Process of Abstraction

Dan Meyer
Stanford University
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Abstraction is a skill that means different things to different fields — whether cognitive psychology, computer science, or mathematics — but each agrees on its importance. This paper will review the literature on abstraction with the goal of first defining abstraction broadly across several disciplines and then narrowly in mathematics alone. It will then narrow its scope even further to examine abstraction as it exists in school mathematics. It will close by suggesting several open avenues for further study.

Abstraction is a worthy investment for our scholarship. As Hershkowitz quotes Plato and Russell:

> Not only did Plato and his followers see in abstraction a way to reach "eternal truths," but modern philosophers such as Russell (1926) characterized abstraction as one of the highest human achievements.

It is a skill that also extends from childhood to adulthood. Piaget noted (1975) it's an early-stage skill that's essential for coordinating sensori-motor structures but it also forms the foundation on which advanced mathematics is built. "[Abstraction] alone supports and animates the immense edifice of logico-mathematical construction" (1974, p. 92). Abstraction is a skill exercised from early childhood through adulthood and also across different vocational tracks. As Devlin writes, "an important prerequisite for writing (good) computer programs is the ability to handle abstractions in a precise manner" (2003, p. 38). We find that abstraction is important for both intellectual and vocational development.

While these researchers agree on the importance of abstraction, it is difficult to arrive at consensus around any one definition of the term. This review initially searched Google Scholar for "abstraction" and "mathematical abstraction," in particular, and then made a secondary search for references that recurred throughout that initial search. Cognitive psychologists such as Rosch and Mervis (1975) have defined abstraction as the process of extracting the commonalities from a set of objects and then naming those defining features. In their formulation, if we consider a chair and a table as two concrete objects, their commonality is that they are both members of a superordinate set called "furniture." Furniture is the "superordinate" abstraction of those two "subordinate" objects and may then include other objects not previously listed, like a lamp or a desk. Of course, we can choose a different abstraction for the same subordinate set. If the superordinate set is "words with five letters" then that abstraction would include "chair" and "table" but wouldn't include "lamp" or "desk" after all.

Webster's (1966) defines abstraction as

- "The act of withdrawing or removing something," and;
- "The act or process of leaving out of consideration one or more properties of a complex

object so as to attend to others."

In our first consideration of the chair and table, we "left out of consideration" all the non-furniture properties of those objects (eg. their color, their weight, their texture, etc.) so we could attend to their categorization as furniture. This example trivializes somewhat the power of abstraction (it isn't obvious why we'd care about this particular abstraction) but as we move from cognitive psychology into the fields of mathematics and computer science, the power of abstraction becomes more evident.

In her 2006 paper on "computational thinking," Jeannette Wing wrote that, "Thinking like a computer scientist means more than being able to program a computer. It requires thinking at multiple levels of abstraction" (p. 35). In other words, successful computer scientists can regard the chair as both "a chair" and as "furniture" and as "an object you find in a house" and onwards to other levels of abstraction." She further defines abstraction. "It is choosing an appropriate representation for a problem or modeling the relevant aspects of a problem to make it tractable" (p. 33). This highlights the importance of a problem or, more generally, a *purpose* for an abstraction. Abstraction doesn't occur in a purposeless vacuum. Two different purposes for the same context might suggest two different contexts. In Kramer's 2007 paper, "Is Abstraction the Key to Computing?" he makes the point that a railway map is an abstraction of London's geography that's useful for rail passengers but useless to drivers on the surface streets who require a different abstraction of the same context, one that includes precise geography. A computer scientist, both Kramer and Wing suggest, must be sensitive to the different purposes of an abstraction.

In mathematics, consensus around the definition of "abstraction" has proven even more elusive. Hazzan and Zazkis write:

> There is no consensus with respect to a unique meaning for abstraction; however, there is an agreement that the notion of abstraction can be examined from different perspectives, that certain types of concepts are more abstract than others, and that the ability to abstract is an important skill for a meaningful engagement with mathematics (2005, p.102).

Janvier (1981) writes that mathematical abstraction, "does not depend on the use of an abstract vocabulary but on a slow and gradual 'drawing-out' (*abs-trahere*) of the main features guided by some precise intentions," a definition that bears some resemblance to those we saw in cognitive psychology (the drawing-out of main features) and computer science (the precise intentions).

Dienes (1963), another mathematics researcher, put similarly broad boundaries around "abstraction," defining it as, "the extraction of what is common to a number of different situations … the formation of a class, the end-point being the realisation of the attribute or attributes which make elements eligible or not for membership of the class" (p. 57).

But Hershkowitz, et al, (2001) localize abstraction to mathematics as "an activity of vertically reorganizing previously constructed mathematics into a new mathematical structure" (p. 202). This borrows from Piaget's concept of "reflective abstraction" (1974, further explicated by Dubinsky in 1991) where ideas themselves become objects for abstraction. When we add two apples to four apples, for example, we abstract away the “appleness” of the apples, leaving behind their numerical abstraction, “2 + 4.” But addition, itself, according to these scholars, is an object we can abstract for other purposes.

This is by no means an exhaustive sample of the different ways different mathematics researchers have defined abstraction. They may have more similarities than differences, but it would be worth our while to explore the source of this definitional conflict, which is largely syntactical: “abstract” can be used as a verb or an adjective or we can refer to the noun “abstraction,” which is the product of the process of abstraction. All three parts of speech are value-laden in this discussion.

Frorer, et al, (1997) relate a list of words students associated with the term “abstract,” a term which those students instinctively parsed as an adjective and universally pejorative: “hidden, complex, requiring deep thought, not concrete, apart from actual substance or experience, not easily understood, a mental construction, a theoretical consideration” (p. 218). Ferguson (1986) sought to assess a student’s anxiety around abstraction and added ten questions to the Mathematics Anxiety Rating Scale (Rounds and Hendel, 1980), a selection of which follows:

- Trying to read a sentence full of symbols such as $A = \{x\colon |x - 2| = 3,\ x \epsilon I\}$.
- Being asked to solve the equation $x^2 - 5x + 6 = 0$.
- Being told that *everyone* is familiar with the Pythagorean Theorem.
- Having to work a math problem that has x’s and y’s instead of 2’s and 3’s.

A factor analysis revealed that “abstraction anxiety” is an important component of mathematical anxiety, in addition to those already identified by Rounds and Hendel.

Sfard (1991) acknowledges this anxiety but finds it too prosaic to be of much use. “Saying what people usually say, namely that mathematics is the most *abstract* of sciences, does not help very much. Being almost a cliché, this claim has little explanatory power.”

We will bracket the question “what should educators do about all this anxiety about abstraction?” for a later section of this review. But let’s note that this anxiety confounds the definitions already reviewed in this paper, definitions which have treated abstract as a process – a verb – not an adjective.

Every researcher reviewed so far would agree that abstraction can be defined procedurally. It’s something you *do*. These researchers also identify abstraction as a noun, an object that can be operated upon and abstracted even further. These researchers often disagree, however, on the relationship between the verb and the noun.

Mitchelmore (1995, p. 56) describes those who believe that an abstraction can be considered apart from the process of abstraction that produced it:

> Another manifestation of the belief in abstraction as a product without a process is the constant search among curriculum developers for perfect concrete models to illustrate abstract structures. [..] Meanwhile, textbooks rely on the single best models the authors can devise. Using one model, however good, is completely contrary to the notion of

abstraction as the recognition of similarities between different situations (Truran, 1992).

Hazzan (2003) describes some of those "best" abstractions in the computer sciences. "For example, a queue is associated with a line of waiting people; a stack is associated with a stack of cafeteria trays, and so on" (p. 99). A decade after Hazzan, I was personally introduced to queues and stacks with these same abstractions in my own computer science coursework. They have power, but how much power is in dispute.

Wilensky (1991) advances a definition of "concreteness" (the inverse of "abstractness") that includes an individual's particular relationship to the context being abstracted. "Concreteness, then, is that property which measures the degree of our relatedness to the object, (the richness of our representations, interactions, connections with the object), how close we are to it, or, if you will, *the quality of our relationship with the object*" (p. 198). Sfard and Thompson (1994) affirm that definition. "[Concreteness] is not a property of an object but rather a property of a person's relationship with the object" (p. 23). In other words, there is nothing inherently concrete or abstract about cafeteria trays or a line of waiting people. Wilensky illustrates this vividly by pointing to the fact that for most Americans, "snow" is a very concrete thing. You can abstract it higher, into "weather" and then "atmospheric phenomena," but it's hard to make snow more concrete. Meanwhile, "snow, for an Eskimo, is a vast generalization, combining together twenty-two different substances, some of which may be as different to an Eskimo as "pens" and "language" are to us" (p. 197). Or, as one of the discussants in Frorer, et al, notes: "Thus 'number' is concrete for me, but not for a 3-year old child in a Piagetian experiment" (p. 223).

Hayakawa (1949) proposed the ladder as a metaphor for this relationship between the process of abstraction, the people who perform that process, and the product of that process. For North Americans, "snow" would occupy one of the lowest rungs of that ladder, with "weather" on a rung above and "atmospheric phenomena" somewhere above that. The Inuit, however, might ascend and descend an entirely different ladder with respect to snow, one that put snow on a higher rung with room for subordinate definitions beneath it.

In mathematics that ladder presents itself when a student represents a context symbolically, abstracting two runners into a system of linear equations, for instance. Those linear equations exist on a higher rung than the runners. Those equations have nothing inherently to do with running. They are just symbols, which could be used to describe any number of other contexts beneath them on the ladder aside from running, in the same way that "weather" could be used to describe more than just "snow."

Given this complex interaction between the person, process, and product of abstraction, let's turn our attention to how abstraction is treated in school mathematics. The existence of Ferguson's abstraction anxiety instrument (1986) suggests that abstraction is something students are occasionally anxious about. What accounts for that anxiety?

Ferrari (2003) notes that modern algebraic notation "is generally recognized as a powerful tool to support generalization processes in algebra" (p. 1227). Algebraic notation is an abstraction that lets us answer questions about a context — the runners, let's say — without having to bother ourselves with every aspect of that context itself — the color of the runners' shoes, for instance. Harel and Tall (1991) use the same language as Ferrari, describing abstraction as “a powerful tool” (p. 5). Richard Courant (1981) describes how David Hilbert wielded that power to solve a problem. “If you want to solve a problem, first strip the problem of everything that is not essential. Simplify it, specialize it as much as you can without sacrificing its core. Thus it becomes simple, as simple as it can be made, without losing any of its punch, and then you solve it” (p. 161). We’ve seen that “stripping away of the inessential” recur through different definitions of “abstraction.”

We should wonder, though: do students of school mathematics see abstraction as “a powerful tool” or as something else? The literature reveals very little on the question of “how do students define abstraction?” Ferguson’s anxiety instrument (1986) is able to tell us whether or not abstraction makes students anxious but not how they define it when it doesn’t. It’s easier to find researchers speculating on the origin of that anxiety than it is to find other student conceptions of abstraction.

Mason (1989) writes:

> It is easy to sympathise with the student's sense of abstract as *removed from* or *divorced from* reality (or perhaps, more accurately, from meaning, since our reality consists in that which we find meaningful). But perhaps this sense of being out of contact arises because there has been little or no participation in the process of abstraction, in the movement of drawing away. Perhaps all the students are aware of is the *having been drawn away* rather than the *drawing* itself (p. 2).

This offers a direction for teachers who would like their students to become unintimidated by and proficient with abstraction: let them participate in that process, rather than asking them to stand by as it happens or simply observe its result *ex post facto* . Mason describes abstraction as “a delicate shift.” He writes that “those who begin to make the move to abstraction spontaneously go on to study mathematics further, while the others remain mystified, though sometimes intrigued” (p. 7). His prescription is for teachers to be explicit about abstraction. “By being explicit, by focussing attention on the brief but important moments of abstraction movement, more students could be helped to appreciate the power and pleasure of mathematics, and to get more from their investigating” (p. 7).

Hazzan (2008) instantiates Mason’s recommendation by describing a computer science teacher’s explicit narration of her process of abstraction:

> In practice, for example, with respect to abstraction applied in front of the class, instructors can specifically make statements such as: "I am ignoring this aspect here because... ", "Now, let's move one level of abstraction down and elaborate on... ", "Similarly to what we did last week, abstraction is expressed here because... ", "If we hadn't used abstraction, the solution would have been... ", and so on. This can be done, for example, while developing a solution in stages in front of the class, specifying how abstraction is expressed and how it guides the solution process, instead of presenting the students with complete solutions (pp. 41-42).

Both of these researchers make the point that, once that shift in attention has been made, it becomes what Hazzan calls “a soft idea,” something that is difficult to define but easy to do unconsciously. The discussants in Frorer, et al, (1997) note the depth to which abstraction has sunk in as a soft idea when they recognize, “we rarely find [abstraction] explicitly discussed let alone defined. You can pick up a book entitled *Abstract Algebra* and not find a real discussion of abstraction as a process, or of abstractions as objects” (p. 222). Without being conscious of and explicit about her own proficiency with abstraction, a teacher may undermine her students’ nascent powers of abstraction.

Hazan (2003, 2005) also observed several different ways that students attempt to reduce abstraction in their mathematical tasks. In his 2005 paper, he watched students attempt to solve this problem:

> A length of 3 cm on a scale model corresponds to a length of 10 m in a park. A lake in the park has an area of 3600 m^2. What is the area of the lake in the model? (p. 115).

Rather than create an abstraction like “scale factor” to describe the relationship of 10 meters and 3 centimeters, students created a hypothetical park with an area of 3600 m^2, attempting to maintain their rung on the ladder of abstraction. In light of that habit, Hazan recommends “educators continually seek that thin line between abstract (for the students) and concrete (for the students) in order to avoid working too abstractly or too concretely, respectively” (2003, p. 118). Hayakawa (1949) expands on this sensitivity to the appropriate level of abstraction for a given learner:

> The interesting writer, the informative speaker, the accurate thinker, and the sane individual operate on all levels of the abstraction ladder, moving quickly and gracefully and in orderly fashion from higher to lower, from lower to higher, with minds as lithe and deft and beautiful as monkeys in a tree (p. 95).

The skilled teacher of abstraction, in other words, knows whether or not “a line of waiting people” is the right concretization of a queue in computer science for the right student at the right time. If the student seems intellectually complacent at lower levels of abstraction, the teacher can challenge her with a higher one.

The literature has noted repeatedly that abstraction is a tool of some power. If we want students to share that perception, it will be important to assign tasks that *require* abstraction. In Dreyfus’ (2007) formulation:

> … task designs using conflicts, surprise and uncertainty (see, e.g. Hadas, Hershkowitz and Schwarz, 2000) are conducive to intra-mathematical motivation. If such a design creates the need for a new construct, it is likely to be conducive to a process of abstraction. The question whether or not the need for a new construct arises depends of the learners’ previous experience, and therefore, within a curriculum, on the sequencing of tasks.

Task designers and teachers should therefore ask themselves, “Is this abstraction needed?" (p. 7).

Let's drop down the ladder of abstraction and offer an example. The sequence, 5, 8, 11, 14, 17, … , can be rendered more abstractly by the notation, $A_n = 3n + 2$ for $n \geq 0$. If I asked you to tell me the number following 17 in that sequence, that symbolic abstraction might seem intimidating and unnecessary. You'd just add three again. But if I asked you for the 2012th number in that sequence, suddenly that abstraction becomes potent and perhaps enticing. It promises you a way to avoid adding three repeatedly for the next several hours.

We should be mindful then of assigning the following class of tasks to students, which asks for an abstraction without motivating its need:

> Find a general expression for the arithmetic sequence: 5, 8, 11, 14, 17, … ,

Hayakawa (1949) was a semiotician but recognized the use of mathematical abstraction "in predicting occurrences" (p. 87). It's one thing to hold your hand out a window and decide, concretely, if it's raining. It's another to use an abstract model to predict rainfall three days from now. When the meteorologist uses abstraction to predict a future day's rainfall, it's plausible that he's very interested in whether or not he was right or wrong about that prediction. He tests his abstraction, in other words, by comparing it to the concrete data it tried to predict.

Students are asked to make predictions with abstractions frequently in math class but rarely are they given the opportunity to test out those abstractions. They use a parabolic abstraction to calculate the time a ball is in the air, for instance, but the teacher proves the power of that abstraction by referring to an answer key in a textbook, which may not be convincing. To verify the power of abstraction, a student would need to step down a rung from that abstraction and watch the ball sail through the air, timing it to see how close the abstraction came to predicting the concrete. This occurrence may be rare in the math classroom.

Educators may wonder where to pose such concrete problems in their curricula — at the end of a unit, after having introduced abstractions, or at the beginning, in order to introduce the abstractions. Harel and Tall (1991) and Mitchelmore and White (1995) recommend starting with concrete contexts and ascending to abstraction. Mitchelmore and White write:

> Teaching mathematics through the process of abstraction would look very different from traditional mathematics teaching. It would start with applications instead of leaving them to the end. Students would build up their understanding of each context separately, gradually becoming aware of similarities, and symbols would be used to summarise and clarify the similarities. Finally, ways would be found to manipulate the symbols abstractly in order to deal more efficiently with the original applications and investigate new ones (p. 66).

Harel and Tall (1991) describe the possible development of misconceptions that arises when students are shown concretizations of an abstraction that they haven't yet developed for themselves:

> To the teacher these ideas represent instantiations of the abstract concept, but the student has not yet performed the abstraction, and so these prototypes may function in a seriously erroneous way in which the student abstracts the wrong properties. This seems to happen with the introduction of the function concept in mathematics. So difficult is this abstract concept that it seems not possible to present it in a sufficiently generic manner. Instead we see that pupils presented with an informal introduction to the function concept develop a menagerie of examples from which they abstract inappropriate properties (p. 6).

Mitchelmore and White (1995) admit what is implicit throughout these studies. “There is very little empirical evidence on which to base such an approach” (p. 66). It is much easier to measure the product of abstraction (either the abstraction itself or the anxiety it provokes) than it is the process. Kramer (2007) proposes an annual assessment of a student’s ability to abstract, but writes, “Unfortunately, we have been unable to find any existing appropriate tests” (p. 42).

This review of the literature has found many definitions of abstraction as well as a great deal of informed speculation about how it is taught well and poorly in school math classrooms. Very little of that speculation has been validated by empirical or experimental research. This leaves a wide opening for a future empirical study of either the methods reviewed in this paper or alternative curricula or experimental assessments or any other innovation that might declaw the process and product of abstraction for students. If I may abstract over all these papers about abstraction and extract out their essential feature, it is the opinion of these researchers that abstraction is a tool that’s too powerful and too useful to be disregarded or feared by our mathematics students.

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