Gabriel's Horn Paradox (Trombeta de Gabriel) and a New Cosmology

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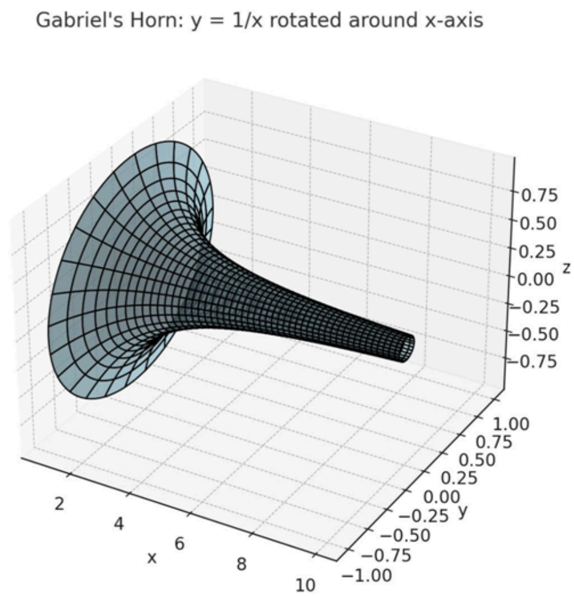
We believe that the TIF (Fibraged Informational Web / *Teia Informacional Fibrada*) model can offer an interesting conceptual interpretation for the so-called Gabriel's Horn Paradox, but it does not "solve" it in a mathematical sense, because the paradox is not a mathematical inconsistency, but a perfectly consistent consequence of classical mathematical analysis.



The so-called Gabriel's Horn paradox is obtained by rotating a curve around the x-axis where the values of the x and y coordinates obey two requirements:

$y = x/1$, $x \geq 1$

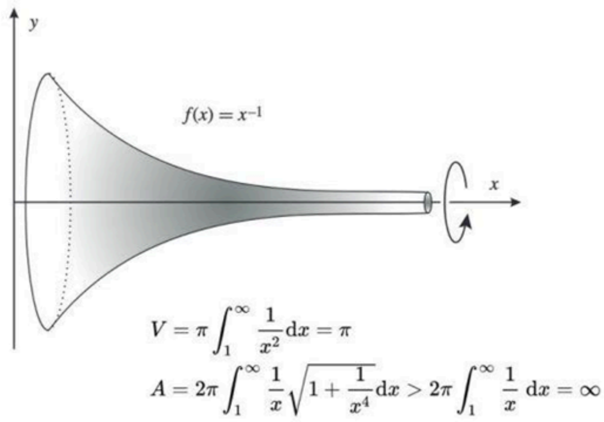
A solid is thus obtained with two seemingly contradictory properties, where the volume is finite but the surface area is infinite. In mathematical terms, we can formalize the values of the volume (*V*) and area (*A*) as follows:

$$V = \pi \int_1^{\infty} \frac{1}{x^2}\, dx = \pi,$$

and

$$A = 2\pi \int_1^{\infty} \frac{1}{x} \sqrt{1 + \frac{1}{x^4}}\, dx = \infty.$$

Thus, in principle, it would be possible to "fill" the horn with a finite amount of information, but an infinite amount of information would be required to cover its entire surface.



This happens because mathematically, as we have seen, Gabriel's Horn develops around two different geometric properties so that there is no contradiction whatsoever, because the volume depends on an integral of $\frac{1}{x^2}$, which converges, while the area depends essentially on an integral of 1/*x*, which diverges. The "paradox" arises only when we try to literally interpret these results and convert them into a physical object.

Often, this idea of finiteness versus infiniteness also generates confusion regarding the value of *π*, characterized by its infinite decimal expansion in the value of *V*. The fact that *π* is equal to 3.14159265358979323846… does not mean that its value is infinite. It merely means that it possesses an infinite number of decimal places and that this expansion never terminates nor becomes periodic. The value of *π* is perfectly finite, situated between $3<\pi<4$.

Finally, the crucial question involves the distinction between two mathematical concepts: on one hand, the infinite number ∞ is not a real number, and on the other hand, a number with an infinite decimal expansion, such as *π*, 2, or *e*, are perfectly finite real numbers. For example, 1/3=0.333… has infinite decimal places, but is worth exactly 1/3; or the square root of 2=1.4142135… also has infinite decimal places, but is smaller than 2. The same happens with *π*.

As many might be led to think, the apparent strangeness does not lie in the fact that *π* has an infinite decimal expansion, but rather in the different behavior of the two integrals. In other words, the volume integral involves

$$\frac{1}{x^2},$$

which decreases fast enough for the accumulated area under the curve to remain bounded, but the surface area essentially involves 1/*x*, whose integral is:

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \ln b = \infty.$$

The difference between 1/*x* and $1/x^2$ is precisely the threshold between convergence and divergence of the so-called p-series:

- $\int_1^\infty \frac{1}{x^p}dx$ converge se $p > 1$;
- diverge se $p \leq 1$.

This is why the volume is finite while the area is infinite.

However, this observation suggests an interesting philosophical question: can a number whose representation requires an infinite amount of information be considered "physically finite"?

Here, the Fibrated Informational Web (TIF) model could introduce a distinction between what is considered to be the definition of mathematical value, which is finite (π), and informational complexity, which can be infinite if arbitrary precision is required to represent it.

In that theoretical model where information is the fundamental element, it could be argued that no physical quantity contains an infinite amount of information. Thus, the physical value of π would always be a finite approximation determined by the resolution of the fundamental network of qutrits. This idea does not alter classical mathematics, but it offers an interesting perspective on the difference between ideal mathematical objects and physically realizable quantities. This line of thought can be developed in a rigorous way within TIF, explicitly distinguishing the finiteness of the value from the complexity of the description—two notions that, as we have seen, are frequently confused, but belong to different conceptual levels.

TIF departs from the idea that the spatio-temporal domain is not continuous at arbitrarily small sub-Planckian scales, because it emerges from a discrete network of fundamental units: triplets of qutrit spins. Consequently, in this context, an infinite horn would never be a physically realizable object in the Universe, because always being endowed with a minimum resolution with a minimum length, a minimum unit of area, and a minimum unit of volume, it would mean that the continuous description ceases to be valid below a certain scale. Therefore, the Horn could only represent the phenomenology of a spatio-temporal event according to the mathematical principles that define it, if we consider the outer surface area to function as an interface with the pre-spatiotemporal domain.

In the language of TIF, both volume and surface represent different informational resources, but allocated to an interface that lives between two domains: the spatio-temporal and the pre-spatiotemporal. Thus, the volume would measure the storage capacity of states, while the surface would measure the complexity of the encoding interface of those states. In this way, as the radius decreases, the additional storage capacity would rapidly tend to zero, but the complexity of the boundary would continue to grow. Here, the mathematics of Gabriel's Horn is fulfilled. In an ideal continuous, this complexity could diverge; in a discrete network, it would eventually saturate when the fundamental scale is reached.

This is close to an idea present in holographic approaches to gravity where the physical information associated with a region is related to its boundary, but that boundary cannot carry an infinite density of information because there is a physical limit associated with the discretization scale (for example, the Planck scale in many theories). In a qutrit-based TIF, each boundary element, being the interface of two domains, would be capable of carrying an infinite number of states, as if the "infinite area" of the continuous description were then understood as an artifact of mathematical extrapolation to arbitrarily small scales.

Based on this formulation, a new principle could be enunciated: **The Principle of Informational Saturation**: *any emerging surface possesses a maximum capacity of information per fundamental unit*. When a continuous description predicts infinite complexity, it indicates only that the model has surpassed the domain of validity of the emerging geometry and now extends to the pre-spatiotemporal domain. This framework means that the finite volume remains a robust property, and that the infinite area ceases to represent a physically realizable quantity, becoming the ideal limit of the Newtonian continuous model and the interface to the pre-spatiotemporal domain.

For this proposal to have scientific value, it would be necessary to develop a quantitative formalism showing how TIF discretization modifies the integrals of volume and area. To do so, it would be necessary to define a fundamental scale (sub-Planckian) corresponding to the smallest geometric unit of the network, replacing at the same time the continuous integrals with sums over discrete elements, demonstrating that the sum associated with the area no longer diverges and converges to a value dependent on the sub-Planckian fundamental scale. It would show that, in the limit, the exact results of classical geometry are recovered. TIF would not "correct" the mathematics of Gabriel's Horn—which is already consistent in itself—but would propose a physical criterion for the applicability of the continuous, explaining why the divergence of the area would not manifest in any physical object emerging from a discrete substrate. It would be a physical reinterpretation, and not a refutation of the mathematical result.

We now launch a new hypothesis: does Gabriel's Horn have the same topology as a Black Hole (or a White Hole, thinking of Rovelli)?

This hypothesis seems to bring together some ideas that are normally treated separately by physics, namely the mathematical analysis of Gabriel's Horn with the geometry of Black Holes and their respective General Relativity, as well as the possible direct consequences for Roger Penrose's new cosmology (CCC – Conformal Cyclic Cosmology).

Gabriel's Horn is a Euclidean solid, it does not represent a spacetime; and a Black Hole is a solution to Einstein's equations whose geometry is a four-dimensional Lorentzian manifold, meaning the topology in both frameworks is also not equivalent. Therefore, we cannot directly identify both objects. However, there is a profound analogy. What makes the Horn fascinating is that it separates two quantities: on one side, the internal volume, and on the other, the surface. In Black Holes, something similar happens, since all information of the interior seems to be encoded on the horizon area, given by the equality:

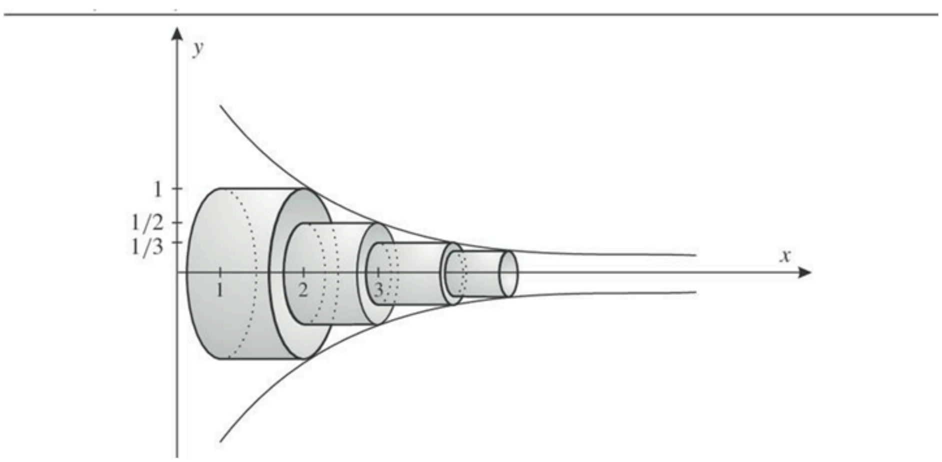
$$S = \frac{kA}{4l_P^2}.$$

This is precisely the basis of the holographic principle, with the Horn also presenting an extreme dissociation between area and volume. A conceptual analogy is reinforced.

In Roger Penrose's Conformal Cyclic Cosmology, the Universe is extremely diluted at the end of an aeon, being identified, through a conformal transformation, with the Big Bang of the next aeon. There is a rescaling of the geometry in which we can imagine an interesting analogy with the Horn, whose mouthpiece corresponds to the observable Universe, the tube narrows indefinitely where the infinite distance corresponds to the conformal limit of the aeon, and the asymptotic end represents the point where the identification with the new Big Bang occurs. It is certainly not Penrose's CCC, but it constitutes a very suggestive mathematical image.

In our perspective, which informs TIF, space is not fundamental. What is fundamental is the informational network, where "volume" ceases to represent physical space and starts representing informational capacity.

In this case, Gabriel's Horn can be reinterpreted as a network whose total storage and development capacity is finite (the Universe) but with an unlimited hierarchical depth (aeons followed by aeons), resembling certain infinite trees used in computer science theory. The "depth" of the Horn would not represent spatial distance, but rather a causal or informational depth. It would imply that each Universe would undergo a progressive degradation over aeons, eventually being absorbed by the pre-spatiotemporal domain: the definitive return to origins, reminiscent of the Manvantaras/Pralayas cycles of Vedic philosophy.



Imagine replacing $1/x$ with an information density $I(x)$. Here, the narrowing ceases to be geometric because it becomes informational compression, implying that the horizon is only an encoding interface. That is, the entire Universe could be compressed into a structure whose external "width" is limited, and the interior would correspond not to more space, but to more levels of organization of information. This idea simultaneously approaches both holography and the system of quantum error-correcting codes that would work over aeons toward final perfection, as well as the functional concept of emergent geometry in TIF philosophy.

Instead of interpreting Gabriel's Horn as a geometric object, it could be interpreted as a cosmic informational compression operator.

The "infinite area" would cease to mean an infinite physical surface and would start representing a potentially unlimited complexity of encoding.

The "finite volume" would correspond to the effectively stored content, which is informationally infinitely complex in its description (just like π). Black Holes would then be seen as the approximate physical realization of the described mechanism: a finite boundary encoding content whose internal organization can be much richer than classical geometry suggests.

In a simpler way with TIF, we would give another formulation to Gabriel's Horn. Instead of $r(x)=1/x$, we would define $r(I)=1/I$, where I is the informational depth of the qutrit network. Thus, the "narrowing" would represent an increase in information compression, while the "informationally infinite length" would represent unlimited hierarchical refinement, and the finite volume, the total capacity of information. The Black Hole would then be an informational compressor, and the White Hole, proposed by Carlo Rovelli [1], the decompressor.

In Penrose's CCC, the Universe ends in an extremely diluted state. We could imagine that this state would correspond to a Horn completely "traversed" and that the asymptotic end would not be a singularity but a region where all information became compressed and geometry disappears, leaving only the informational network. Consequently, Rovelli's White Hole would then initiate the expansion of the next aeon. Although different from Penrose's proposal, this is conceptually compatible with the idea of a conformal transition.

The reinterpretation of the Horn emerges as the state space of a process of information compression/decompression. In this framework, gravitational collapse would correspond to a path along the Horn toward increasingly compressed states, where Rovelli's "bounce" would reverse this path and observable geometry would emerge only as a projection of this informational flow.

This conceptual structure naturally brings TIF close to three contemporary ideas, namely: information conservation in Black Holes, emergent geometry from quantum entanglement, and the elimination of the singularity by quantum effects. The big difference is that, in TIF, the fundamental element ceases to be geometry and becomes the organization of information, of which geometry would be an emergent manifestation.

This interpretation, compatible with the spirit of TIF, would require substantial mathematical development. It would be necessary to define an informational metric on the network of qutrits and demonstrate that a geometry of the "informational Gabriel's Horn" type emerges naturally from this metric. If this were achieved, the Horn would cease to be merely a curiosity of mathematical analysis and would play the role of an abstract model for the hierarchical compression of fundamental information, offering a conceptual link between TIF, holography, and cosmological scenarios like CCC. It would be an original proposal needing mathematical and physical validation. This mathematics can be internally consistent by developing a set of rigorous definitions, theorems, and proofs describing an "Informational Horn". However, to validate that this mathematics could describe the real Universe, testable predictions and experimental confirmation would be required.

So, how to build a mathematical model of the "Informational Horn"? Is it possible to replace geometry with information?

In the classical Horn, $r(x)=1/x$, but in TIF we could define $I(x)$ as informational density. Thus, instead of:

$$dV = \pi r^2 dx$$

we would have:

$$d\mathcal{I} = \rho_I(x)\, dx$$

where ρ_I would represent the density of states of the qutrit network.

To define an informational metric space, we would have to consider a graph $G=(V,E)$ where each vertex is a triplet of qutrits, and each edge represents quantum entanglement. This would imply that the metric is defined by $d_I(u,v)$ and not by spatial distance, but by the minimum cost of transformation between states. For example, at a distance similar to those used in information theory:

$$d_I(u,v) = -\log P(u \to v)$$

An informational curvature K_I could now be defined as a function of network connectivity. For example:

$$K_I = -\nabla^2 S$$

where S represents local entropy. Here, we would make a direct approach to the concepts of entropic gravity and emergent geometry.

Reaching this point, we would be enabled to define the "Informational Horn".

Instead of the geometric radius, we introduce $R_I(x)$ assuming it equals $1/x$. Thus, the total capacity becomes:

$$C = \int_1^{\infty} R_I^2(x)\, dx.$$

and thus:

$$C < \infty$$

However, the complexity of the interface will be given by:

$$\Gamma = \int_1^{\infty} R_I(x)\, dx = \infty.$$

Thus, we obtain exactly the same mathematical structure as Gabriel's Horn, now reinterpreted in informational terms.

The horizon ceases to be a physical surface and becomes:

$$\partial G$$

with all information being encoded on that boundary, implying that entropy is represented by:

$$S=\alpha|\partial G|$$

where:

$$\partial G$$

would be the number of boundary states.

This expression immediately recalls the Bekenstein-Hawking formula, but here it arises as a structural hypothesis of TIF.

Now comes the interesting part. Suppose the hierarchical depth of the network is *L*. Because each qutrit possesses three states, the number of states will be approximately defined by:

$$N(L) \sim 3^L$$

Thus, an infinite depth can encode an infinity of configurations although physical capacity remains limited.

$$\lim_{L \to \infty} 3^L = \infty$$

Now, this is precisely the conceptual behavior of the Horn.

We could formulate a new theorem, something that might be called the **Theorem of Asymptotic Informational Compression**, an informational version of Gabriel's Horn, which would be based on the assumption that if a qutrit network satisfies a finite local connectivity, and an unlimited hierarchical depth and has density:

$$\rho(x) \propto x^{-2},$$

then physical capacity converges, while at the same time the complexity of the encoding diverges, and information remains always retrievable through the boundary.

In our opinion, here lies the most interesting conceptual opportunity: the connection to Einstein's equations. If we can show that the emerging metric of the qutrit network satisfies, in a continuous limit, a relationship of the type:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

or an emerging variant based on entropy gradients, then the "Informational Horn" would cease to be just an analogy, and would become a mathematical construction integrated into an emerging gravity theory.

This development would require using tools such as spectral graph theory, discrete geometry (for example, Ollivier–Ricci or Forman curvature [2]), tensor networks, and continuous limits. It would be an ambitious but mathematically plausible research program.

In our assessment, this could be one of the most original extensions of TIF discussed so far. Instead of using Gabriel's Horn as a metaphor, it would be reinterpreted as a universal object of informational compression, whose structure can link geometry, holography, emerging gravity, and cosmology in the same formalism. The challenge is not the initial mathematical consistency—that can be built—but demonstrating that the model quantitatively reproduces known physics and makes new predictions that can be tested.

Both matters are far outside the scope and capabilities of the author.

Meanwhile, perhaps the Cosmos produces a cymatic excitation [3] inspired by Gabriel's Horn. It could generate a sequence of patterns, the cosmic sound vibration, exhibiting self-similarity (a fractal structure, a map of information organization), or preferred discrete scales determined by the organization in triplets (qutrits) instead of a completely continuous distribution, or perhaps even persistent dominant modes that would concentrate energy in specific regions of its geometry. Anything that differed from classical cymatics, in which patterns are determined only by geometry and excitation frequencies.

Notes

[1] The White Holes proposed by Carlo Rovelli are a hypothesis based on Loop Quantum Gravity (LQG) to solve the singularity problem inside black holes.

In simple terms, a Black Hole traps matter and light due to gravitational collapse. In classical General Relativity, this collapse ends in a singularity, where density and curvature tend to infinity. To avoid the singularity, Rovelli proposes that, before the singularity forms, quantum gravity effects become dominant and halt the collapse. Instead of a singularity, a "quantum bounce" occurs, leading the Black Hole to transform very slowly into a White Hole. Then, the previously trapped matter and information can re-emerge because information is not destroyed, but remains stored for an extremely long time interval.

Thus, a White Hole can be seen as the final state of a Black Hole which, after a long quantum evolution, returns the matter and information it had absorbed back to the Cosmos. However, this proposal remains theoretical, although mathematically consistent within certain versions of Loop Quantum Gravity, but there is still no observational confirmation of the existence of White Holes or the transition phenomenon from one to the other.

This idea is particularly interesting because it can be reinterpreted as a process of information compression and subsequent decompression. The Black Hole would correspond to a phase of maximum encoding of information in the fundamental network, while the White Hole would represent the phase where this information is reorganized and re-emerges in the form of space, time, and matter. Our interpretation with the TIF model goes beyond Rovelli's original proposal and constitutes a hypothesis that would require mathematical development and confrontation with observations.

[2] Ollivier–Ricci and Forman curvatures are generalizations of the notion of curvature in differential geometry to discrete graphs and networks. Instead of measuring the curvature of a continuous surface, they measure the "curvature" of the connection structure between the nodes of a network.

In TIF, where spacetime emerges from a network of qutrits, these tools can be particularly relevant. In TIF, each node is a triplet of qutrits, and each edge is an entanglement or information channel.

Thus, Ollivier–Ricci curvature can quantify the ease with which information propagates, being interpreted as a measure of "informational gravity," while Forman curvature can measure the local robustness or stability of the network, identifying regions where the emerging geometry is more or less stable. In other words, Ollivier–Ricci curvature describes how information or probability propagates dynamically in the network, measuring how information flows between neighboring regions of the network, while Forman curvature describes the combinatoric structure of the network, being a more static and local notion, and can be seen as a measure of the structural rigidity of the network.

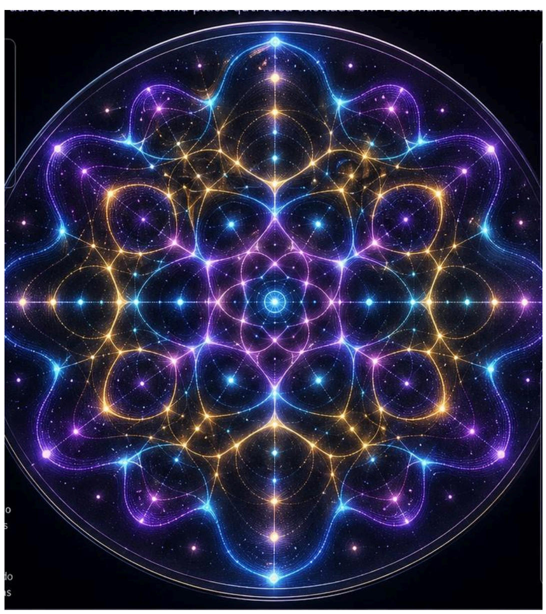
Thus, instead of starting from a continuous manifold with a metric gμν, as General Relativity does, TIF could start with a discrete network of qutrits and calculate these curvatures. In the limit of increasingly fine networks, it would be possible to investigate whether these quantities converge to the Ricci curvature of continuous geometry. This is precisely one of the current lines of research in discrete geometry and emerging gravity, which makes these tools particularly suitable for mathematically formalizing a theory like TIF.

[3] Cymatics is the study of geometric patterns produced when vibrational waves propagate through a physical medium, such as sand, water, viscous liquids, or metal plates. When a surface is excited by a specific frequency, standing waves are formed. The particles of the medium move from regions of greater vibration to those of lesser vibration, organizing themselves into regular patterns called Chladni figures. It shows how vibration can generate order from apparent chaos.

From a perspective inspired by the TIF model, cymatics could be seen as an analogy for the emergence of structure from a more fundamental substrate, where the waves would correspond to informational processes, and the patterns to stable states of the network, with an emerging geometry of resonance relationships.

Thus, cymatics would provide an intuitive image of how simple interaction rules can reveal complex and organized structures; that is, the quantum excitations of spin triplets would generate informational phonons. If TIF describes the Cosmos as a vibrational network of qutrits, it would be natural to look for an analogue of phonons,

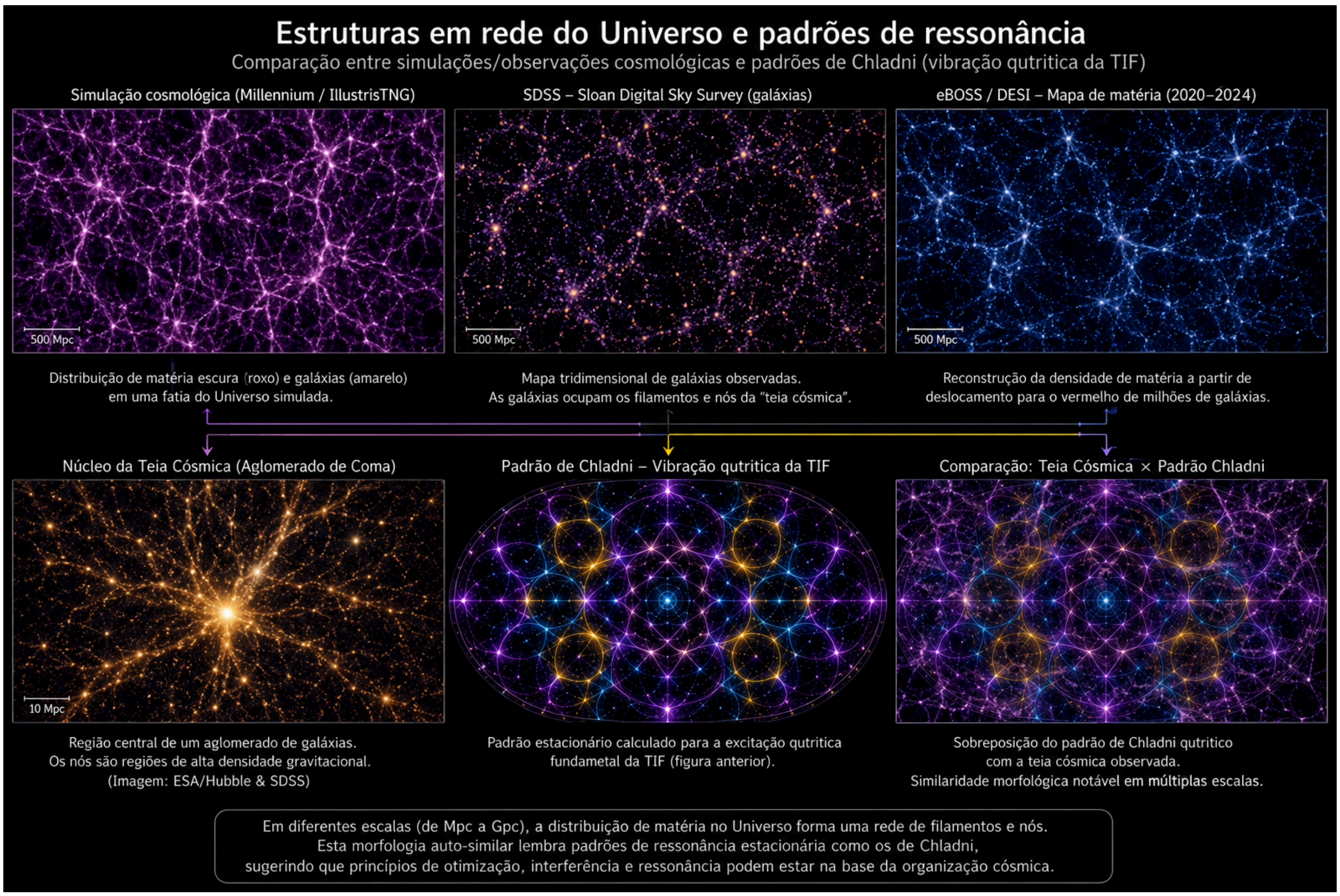
which would ultimately be quantized excitations of this fundamental network. These "informational phonons" would not eventually carry only energy, but also logical structure and quantum correlations. The geometry of spacetime and particles could emerge as collective modes of vibration, analogous to the emergence of phonons in a crystal.



Possible Chladni Figure for TIF (AI concept)

An interesting line of research because it could establish a bridge between an experimentally consolidated concept of solid-state physics (phonons) and the still speculative idea of a discrete substrate of information. If this bridge could be formulated mathematically, phonons would cease, in this case, to be just an analogy and become a model to understand how macroscopic properties can also emerge from collective excitations in a discrete quantum fundamental system.

The following figure, prepared by the AI, compares different simulations of the network structures, filaments, and nodes of the currently known Universe based on cosmological observational data, namely Millennium, SDSS – Sloan Digital Sky Survey, eBOSS/DESI, and Coma Cluster at 10Mpc from ESA/Hubble & SDSS. It then compares this with the previous image of a possible Chladni Figure produced by the same AI, concluding that there is a "remarkable similarity at multiple scales" from Mpc to Gpc.



Network structures of the Universe and resonance patterns *Comparison between cosmological simulations/observations and Chladni patterns (TIF's qutrit vibration)*

- **Cosmological Simulation (Millennium / IllustrisTNG):** Distribution of dark matter (purple) and galaxies (yellow) in a slice of the simulated Universe.
- **SDSS – Sloan Digital Sky Survey (galaxies):** 3D map of observed galaxies. Galaxies occupy the filaments and nodes of the "cosmic web".
- **eBOSS / DESI – Matter Map (2020-2024):** Reconstruction of matter density from redshift data of millions of galaxies.
- **Core of the Cosmic Web (Coma Cluster):** Central region of a galaxy cluster. Nodes are regions of high gravitational density (Image: ESA/Hubble & SDSS).
- **Chladni Pattern – TIF Qutrit Vibration:** Stationary pattern calculated for the fundamental qutrit excitation of TIF (previous figure).

- **Comparison: Cosmic Web vs. Chladni Pattern:** Superposition of the qutrit Chladni pattern with the observed cosmic web. Remarkable morphological similarity at multiple scales.

At different scales (from Mpc to Gpc), the distribution of matter in the Universe forms a network of filaments and nodes. This self-similar morphology resembles stationary resonance patterns like those of Chladni, suggesting that principles of optimization, interference, and resonance may lie at the base of cosmic organization.

If the AI's conclusion is verified by expanding cosmological observations now strengthened by the Vera Rubin Observatory and James Webb, we will have major paradigm shifts in many areas of knowledge.

We will keep our hopes high.
