Complex Numbers

TATS: When is the quadratic formula limited in finding the solutions to quadratics?

Learning Outcome: I can utilize knowledge of new numbers, complex numbers, to solve all quadratic equations.

Explore:

1. The quadratic formula provides a tool for solving any quadratic equation by algebraic reasoning. But we have also learned other methods to solve quadratics.
    a. How else could you solve $3x^2 + 7x - 12 = 0$ other than using the quadratic formula?
    b. What are the solutions to that quadratic equation? How did you solve?

2. Use the quadratic formula to solve each of the following equations. Report your answers in the exact form, using radicals where necessary rather than decimal approximations. Be sure to check your answers.

    a. $3x^2 + 9x + 6 = 0$
    b. $3x^2 - 6 = -3x$
    c. $2x^2 + x - 10 = 0$
    d. $2x^2 + 5x - 1 = 0$
    e. $x^2 + 2x + 1 = 0$
    f. $x^2 - 6x + 13 = 0$

3. The solutions for quadratic equations in Problem 3 included several kinds of numbers. Some could be expressed as integers. But others could only be expressed as fractions or as irrational numbers involving radicals. One of the quadratic equations appears to have no solutions. At what point in the use of the quadratic formula do you learn whether the equation has **two distinct solutions, only one solution, or no real number solutions**?

Complex Numbers: In work in Problems 1 and 2, you discovered that some quadratic equations do not have real number solutions. For example, when you tried to solve $x^2 - 6x + 13 = 0$, the quadratic formula gives

$$x = 3 + \frac{\sqrt{-16}}{2} \quad \text{and} \quad x = 3 - \frac{\sqrt{-16}}{2}.$$

4. Sketch a graph of the function $f(x) = x^2 - 6x + 13$. Explain how it shows there are no real number solutions for the equation $x^2 - 6x + 13 = 0$.

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Course 3 Unit 5 Lesson 2 Investigation 2

The obstacle to solving $x^2 - 6x + 13 = 0$ appears with the radical $\sqrt{-16}$. You prior experience with square roots tells you that *no* real numbers has -16 as its square. For thousands of years, mathematicians seemed to accept as a fact that equations like this simply have no solutions. In general, a quadratic equation in the form of $ax^2 + bx + c = 0$ has no real number solutions when the value of $b^2 - 4ac$, called the **discriminant** of the quadratic, is a negative number.

In the middle of the 16th century, Italian scholar Girolamo Cardano suggested all that we needed was a new kind of number. Mathematicians explored Cardano's idea for several more centuries until **complex numbers** became an accepted and well-understood tool for both pure and applied mathematics.

The obstacle to solving $x^2 - 6x + 13 = 0$ was removed by reasoning like this:

It makes sense that $\sqrt{-16}$ should equal $\sqrt{16(-1)}$.

But $\sqrt{16(-1)}$ should equal $\sqrt{16}\sqrt{-1}$.

So, $3 + \frac{\sqrt{-16}}{2}$ should equal $3 + \frac{4\sqrt{-1}}{2}$.

Or, the solutions for $x^2 - 6x + 13 = 0$ should be $x = 3 + 2\sqrt{-1}$ and $x = 3 - 2\sqrt{-1}$.

This kind of observation led mathematicians to develop what is now called the **complex number** system whose elements are in the form of $a + b\sqrt{-1}$ with a and b real numbers. Because $\sqrt{-1}$ was long considered an impossible or **imaginary number**, expressions of complex numbers are commonly written with $\sqrt{-1}$ replaced by the letter "*I*". So, $3 + 2\sqrt{-1}$ is written as $3 + 2i$.

5. Solve these quadratic equations. Express those solutions in the form of $a + bi$, where a and b are both real numbers.

   a. $x^2 - 10x + 29 = 0$
   b. $4x^2 + 16 = 0$
   c. $x^2 - 4x + 13 = 0$
   d. $3x^2 - 18x + 30 = 0$

CYU: Use quadratic formula or other reasoning to solve each of the following equations. If the solutions are real numbers, identify them as integer, non-integer rational, or irrational numbers. Write nonreal complex number solutions in *standard form a + bi.*

a. $x^2 - 6x - 7 = 0$
b. $5x^2 - 6x + 2 = 0$
c. $6x^2 - 11x - 10 = 0$

d. $5(x - 3)^2 + 6 = 11$

Extra Practice:

1. How do you decide on an approach to solving a quadratic equation? What conditions influence your strategy choice in various situations?

2. One solution of a quadratic equation $ax^2 + bx + c = 0$ is $2 + 3i$. What is the other solution? Explain your reasoning.

3. Sketch graphs of these quadratic functions. Explain how the collection of graphs can show that quadratic equations can have: 1) two real number solutions, 2) one real number solution, or 3) no real number solutions. Give the solutions.

   a. $f(x) = x^2 - 4x - 5$
   b. $g(x) = x^2 - 4x + 4$
   c. $h(x) = x^2 - 4x + 6$

4. Solve each of these quadratic equations. Write nonreal complex number solutions in standard form $a + bi$.

   a. $2x^2 + 3x - 5 = 0$
   b. $2x^2 + x - 3 = 0$
   c. $3x^2 + x = 10$
   d. $5x + x^2 + 10 = 0$
   e. $x^2 + 9x - 10 = -24$
   f. $3x^2 + 10 = 25$