Summer School on Markov Numbers and the Combinatorics of Cluster Algebras

18-29 August 2025, Galatasaray University

Information Booklet

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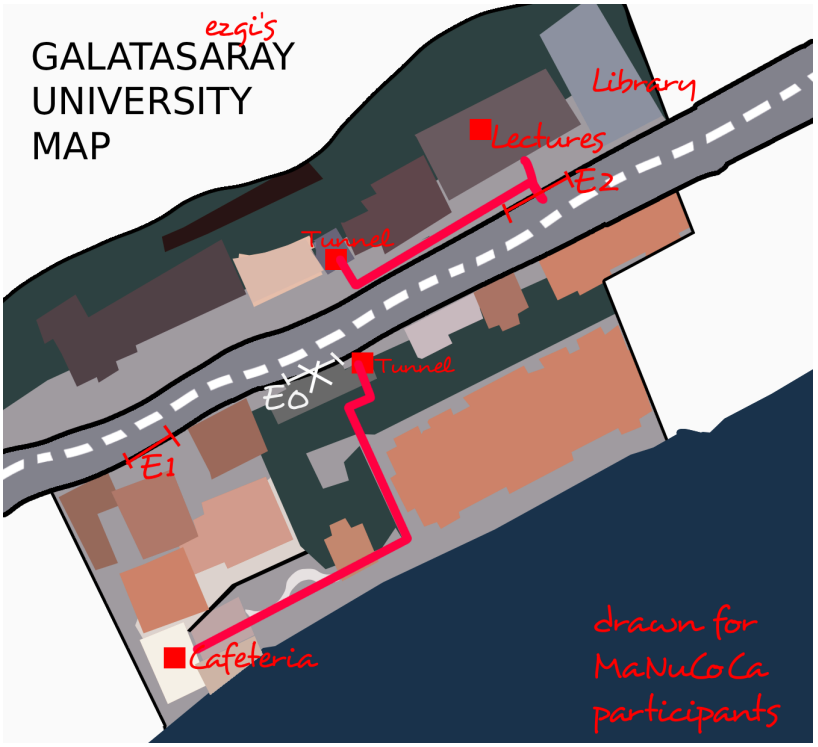
Schedule

Week 1	**Monday**	**Tuesday**	**Wednesday**	**Thursday**	**Friday**
10.00-11.30	Welcome and Introduction to Topics	Emine	Francesca	Emine	Participant Presentations + Activities
11.30-1.00	Emine	Ezgi	Participant Presentations + Activities	Ezgi	Francesca
1.00-2.00	Lunch	Lunch	Lunch	Lunch	Lunch
2.00-3.30	Ezgi	Problem Session in Groups (Emine)	***Excursion:*** Old Istanbul Tour and Conference Dinner	Problem Session in Groups (Ezgi)	Problem Session in Groups (Francesca)
3.30-5.00	Francesca				
5.00-9.00					

Week 2	**Monday**	**Tuesday**	**Wednesday**	**Thursday**	**Friday**
10.00-11.30	Léa	Matthew	***Excursion:*** Yıldız Park Hike and Lunch	Matthew	Problem Session in Groups (Matthew)
11.30-1.00	Matthew	Kağan		Léa	
1.00-2.00	Lunch	Lunch		Lunch	Lunch
2.00-3.30	Léa	Problem Session in Groups (Léa)	Kağan	Problem Session in Groups (Kağan)	Recent Work and Open problems
3.30-5.00	**Q&A:** Life in Academia		Participant Presentations + Activities		

Galatasaray Campus

**Disclaimer: As we do not have an official university map we drew our own. It is an amateur effort and may contain inaccuracies.*



Entering the campus: The main entrance of the campus, marked with E0 is closed. You can enter the sea side of the campus through E1 and the other side through E2. The two sides are connected by a tunnel that goes underneath the road.

Lectures: The lectures will take place in room I107 of the Yiğit Okur building, the first building you see when you enter from E2. For groupwork you can use empty classrooms on the same floor.

Lunch: Lunch will take place in the Cafeteria on the sea side part of the campus. The easiest way to get there from the lecture building is using the tunnel.

Library: Library is open for participants. It is air-conditioned, has study areas and a nice mathematics collection.

Abstracts

Notes and exercises will also be shared online at this folder.

Cluster Structures

Emine Yıldırım, University of Leeds

Cluster algebras are in the nexus of many mathematical and physics phenomena such as total positivity, configurations spaces, algebraic varieties, representation theory of algebras, integrable systems in physics and many more. Briefly, cluster algebras are a class of commutative algebras defined recursively forming clusters and certain exchange relations. Thus, they are very combinatorial in nature. We will present the definition of cluster algebras and investigate some of their connections to other fields in the series of talks.

We'll follow Schiffler's lecture notes on cluster algebras from a previous CIMPA event: https://schiffler.math.uconn.edu/wp-content/uploads/sites/914/2019/03/LNCIMPA.pdf

Computer resources for cluster algebras:

Sagemath - Cluster Algebras (You can use a SageMath Cell server for a single computation, or use cocalc website for larger projects. You can also install SageMath on your computer, it is free and open-source.)

A compendum on the cluster algebra and quiver package in sage - Musiker, Stump

Quiver mutation in JavaScript - Keller

GAP - QPA (Quivers and Path Algebras) package

Combinatorics of Markov Numbers

Ezgi Kantarcı Oğuz, Galatasaray University

We will start by introducing the Markov Diophantine equation. We will see how to generate new solutions from known ones via Vieta jumps and show all solution triples occur in the Markov tree exactly once. We will introduce Frobenius's Uniqueness conjecture and discuss partial process. Then we will move on to different combinatorial models: Christoffel words, Cohn matrices, snake graphs, lattice paths and posets. We will also look at connections to cluster algebras.

References

Main Reference:

[AIG100] Martin Aigner, Markov's Theorem and 100 Years of the Uniqueness Conjecture

Lecture 1:

📖[AIG100] Part II - 3.1 Markov Triples

📑Phillipp Lampe, Diophantine equations via cluster transformations (Part 2.1)

📑Xueyuan Peng and Jie Zhang, Cluster algebras and Markoff numbers

Lecture 2:

📖[AIG100] Part I - 1 Approximation of Irrational Numbers

📖[AIG100] Part I - 2 Markov's Theorem and the Uniqueness Conjecture

📑James Enouen, What are the Markov and Lagrange Spectra?

📖Niven, Zuckerman and Montgomery, An Introduction to the theory of Numbers.
Part 7: Simple Continued Fractions

Lecture 3:

📖[AIG100] Part II - 3.2 Farey Tree

📖[AIG100] Part IV - 7 Christoffel Words

📖Niven, Zuckerman and Montgomery, An Introduction to the theory of Numbers.
Part 6: Farey Fractions and Irrational Numbers

📑Ester Banaian and Archan Sen, A Generalization of Markov Numbers
2.1 Labeling Markov Numbers with Rational Numbers

📑Ezgi Kantarcı Oğuz, Oriented posets, rank matrices and q-deformed Markov numbers

📑Ezgi Kantarcı Oğuz and Emine Yıldırım, Cluster expansions: T-walks, labeled posets and matrix calculations

Extra References:

Website about resources on Markov Numbers:

https://www.maths.dur.ac.uk/users/anna.felikson/Projects/markov/markov-res.html

On q-Deformations of Markov Numbers:

- Kogiso, q-Deformations and t-deformations of Markov triples
- Leclere and Morier-Genoud, q-deformations of the modular group and of the real quadratic irrational numbers

On generalizations of Markov Numbers:

- Yasuaki Gyoda, Positive integer solutions to $(x+y)^2+(y+z)^2+(z+x)^2=12xyz$- Gyoda
- Gyoda and Matsushita, Generalization of Markov Diophantine equation via generalized cluster algebra-
- Gyoda and Maruyama, Uniqueness theorem of generalized Markov numbers that are prime powers-Gyoda, MaruyamaCluster algebraic interpretation of generalized Markov numbers and their matrixizations
- Banaian and Gyoda, Cluster algebraic interpretation of generalized Markov numbers and their matrixizations

Analysis on what percentage of the Markov Conjecture is solved:

- Brandon John Metz, Comparison of Recent Results on the Unicity Conjecture of the Markoff Equation

Representation theory of quivers

Francesca Fedele, University of Leeds

Any finite dimensional algebra over an algebraically closed field k corresponds to some directed multigraph, called a quiver, with relations. Conversely, starting with a quiver, one can construct an associative (unitary) k-algebra, that is finite dimensional under certain assumptions. This approach to studying the representation theory of algebras is very powerful as it gives ways to visualise algebraic structures using combinatorial methods.

In this series, we will explore this connection and the link between the representations of quivers and the modules of the corresponding algebras. We will also relate these constructions to some of the topics discussed in the other series of talks, such as mutation of quivers in cluster algebras.

Francesca's Lecture Notes

Helpful References:

I. Assem, D. Simson and A. Skowronski (2006) "Elements of the Representation Theory of Associative Algebras: Techniques of Representation Theory", Cambridge University Press.
R. Schiffler (2014) "Quiver representations", Cham: Springer.

Combinatorics of q-deformations

Kağan Kurşungöz, Sabancı University

A q-deformation of a formula is a way to introduce a new variable to the formula where we can recover the original by setting q to a predetermined value. In this series of lectures, we will start with introducing the classical q-deformation for integers. We will discuss their properties and applications in counting. We will also look at cases where the power of q denotes a particular statistic, such as the area under a lattice path.

Helpful Reference: Gasper, G. and Rahman, M., 2004. Basic hypergeometric series

Quantum Groups

Léa Bittmann, Université de Strasbourg

Quantum groups were introduced in the 70s to build non trivial solutions to the Yang-Baxter equation (also called the braid relation - the relation between consecutive permutations in the symmetric group). These objects have since then grown into their own subject of research, with many applications in different area of mathematics. One of these applications is directly linked to the original construction: because representations of quantum groups satisfy the braid relation, they can be represented using tangle diagrams. As a further application, they can be used to compute knot invariants, and so help answer the question: can this knot be unknotted?

We'll closely follow these references:

Kassel, Christian, Rosso, Marc, Turaev, Vladimir, Quantum groups and knot invariants, Société Mathématique de France, 1997, in open access at https://smf.emath.fr/publications/quantum-groups-and-knot-invariants .

Kassel, Christian, Quantum groups, Grad. Texts in Math., 1995.

Frieze Patterns

Matthew Pressland, Université de Caen Normandie

Frieze patterns were introduced and developed by Coxeter and Conway in the 80s, who presented them as a kind of combinatorial game, whereby the player attempts to fill a grid with positive integers obeying certain rules. It turns out, however, that frieze patterns appear naturally in a range of mathematical problems from a number of different areas. For example, in geometry, a frieze pattern represents a positive integer valued point in a decorated Teichmüller space, or in the totally positive Grassmannian. In representation theory, the entries in a frieze pattern count submodules of quiver representations. In this series of lectures, we will look at some of these different interpretations of frieze patterns and the connections between them.

Matthew's notes from a previous course on friezes: https://arxiv.org/abs/2010.14302

Opening Talk: Codenominator and quantizations of real numbers

Muhammed Uludağ, Galatasaray University

We will introduce the codenominator function F that extends the Fibonacci sequence to the index set of positive rational numbers, and show that one can express Dyer's outer automorphism of PGL(2, Z) in terms of F. This automorphism can be viewed as an automorphism group of the trivalent tree. The real equivariant modular function Jimm (J) on the real line is defined via the codenominator. J relates the Stern-Brocot tree to the Bird tree. Jimm induces an involution of the moduli space of rank-2 pseudolattices and is related to the arithmetic of real quadratic irrationals. Finally, we will explain how J is related to the quantizations of real numbers recently found by Morier-Genoud and Ovsienko and introduce a new conjugate pair of quantizations discovered recently by Mustafa Topkara.

Contributed Talks

Cluster algebras and their connections with discrete integrable systems

Saeideh Noori, Institute for Advanced Studies in Basic Sciences, Iran

Discrete integrable systems can be studied from the perspective of cluster algebras. To this end, particular reductions can be applied to the discrete independent variables of the discrete system so that the resulted system can be represented as a cluster algebra. Associating a Poisson bracket to this cluster algebra, its Liouville integrability can be evaluated. In this talk, I will present examples of such a system.

Quantized symmetries of the real projective line

Perrine Jouteur, Université de Reims Champagne-Ardenne

Quantum analogues of real numbers were introduced recently by Morier-Genoud and Ovsienko, using a q-deformed action of the modular group on the projective line. I will present their construction and give additional symmetries of this quantized real projective line.

Generalized Markov Numbers

Hüsnanur Gündoğdu, Galatasaray University

What happens if we modify the Markov Diophantine equation? Can we still obtain a similar structure? In this talk, we will discuss the generalization of the Markov equation and try to extend related concepts.

From playing Hex to folding a sheet of paper.

Lucas Toury, Université de Strasbourg

Hex is a board game whose rules are disconcertingly simple, but there is a fundamental property of this game: in the end, there is always a winner. As surprising as it may seem, this impossibility of a draw has profound topological consequences. In this talk, we propose to demonstrate Brouwer's fixed point theorem in dimension 2 via the game of Hex. It is this same theorem which states that when you crumple a sheet of paper and crush it in its former place, at least one point on the sheet returns to its original position.

Visiting İstanbul

Public Transportation: İstanbul has a large public transportation network with buses, metro lines and ferries. Public transport can be paid via credit card (contactless payment) or the public transportation card "İstanbulkart". Using the public transportation card is a better option as it is slightly cheaper and you get transit discounts. You can buy one from the machines and fill it up using cash or credit card.

Finding an empty cab can be difficult on the street, and we do not have Uber. The best way to call a cab is to use the BiTaksi app on your phone.

Things to try: Brewed Turkish tea is a staple of Turkish life: households, work places, restaurants... There is complimentary Turkish tea after most meals, and you can even be offered tea in shops. Turkish coffee is also a part of daily life, people tend to drink one a day as it is strong and bitter. (Sahlep and boza are other specialty drinks but they tend to be served in winter). Turkish delight and roasted chickpeas (leblebi) are good things to try.

Foodwise, we have the east-southeast Anatolian cuisine with lots of meat kebabs, the Aegean side has a lot of olive oil based and vegetarian options, the Black Sea cuisine with a lot of fish based (anchovis) dishes, middle Anatolian cuisine based on grains and vegetables, etc.

Money Exchange: There are money exchange spots in both Beşiktaş and Ortaköy that you can visit. 1 Euro is worth about 47 TRY at the moment, you can check the current exchange rates from your phone. Popular tourist destinations may also accept foreign currency, but the exchange rates may be unfavorable. If a shop or restaurant lists prices in Euro, that is generally a red flag that they are overpriced.

Safety: Galatasaray University lies between Beşiktaş and Ortaköy, two hubbubs of activity. It tends to be crowded at all hours. Coffee places and shops open early and stay open till about 10 pm. Many restaurants and bars stay open into the night. As it never gets deserted, there are always people to help if you need help and police presence is nearby, so serious issues are unlikely. Be vary of pickpockets and take care of your belongings.

In popular tourist destination spots, there are also aggressive salespeople who might try to sell you overpriced things on the street or bring you to their shop. Try to limit your shopping to reputable looking shops.

About MaNuCoCa

*The **Summer School on Markov Numbers and the Combinatorics of Cluster Algebras** is hosted by Galatasaray University in collaboration with Université de Strasbourg.*

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Useful Links

School Website: https://sites.google.com/view/manucoca/
Lecture Notes, References and Exercises: *Lecture Notes and Question Sheets*
Conference Info Booklet: *About MaNuCoCa.docx*

General Emergency Number for Turkey: 112