Vector Calculus MAT226 Fall 2021
Professor Sormani
Lesson 16: Tangent Planes 13.7

Please be sure to mark down the date and time that you start this lesson. Carefully take notes on pencil and paper while watching the lesson videos. Pause the lesson to try classwork before watching the video going over that classwork. If you work with any classmates, be sure to write their names on the problems you completed together. Please wear masks when meeting with classmates even if you meet off campus.

You will cut and paste the photos of your notes and completed classwork and a selfie taken holding up the first page of your work in a googledoc entitled:

MAT226F21-lesson16-lastname-firstname

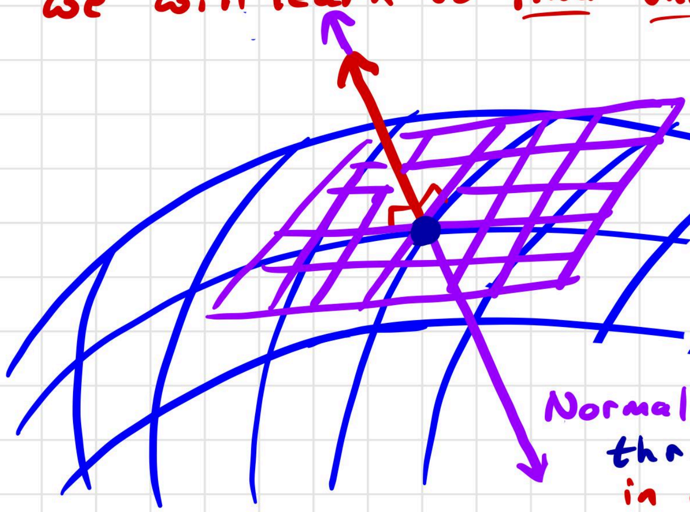
and share editing of that document with me sormanic@gmail.com and with our graders. If you have a question, type QUESTION in your googledoc next to the point in your notes that has a question and email me with the subject MAT226 QUESTION. I will answer your question by inserting a photo into your googledoc or making an extra video.

Watch Playlist 226F21-16-1to11

Tangent Planes and Normal Lines

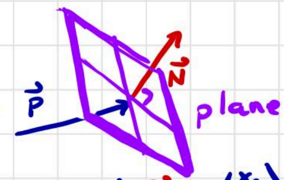
Given a surface $S \subset \mathbb{R}^3$ and $\vec{p} \in S$

we will learn to find the Normal $\vec{N} \perp$ to S at p



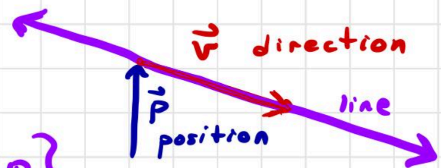
Review of Planes and Lines

A plane $\{\vec{x} \mid \vec{x}\cdot\vec{N} = \vec{P}\cdot\vec{N}\}$



$= \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \,\middle|\, \begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}a\\b\\c\end{pmatrix} = d\right\}$ where $d = \vec{P}\cdot\vec{N} = \begin{pmatrix}x_0\\y_0\\z_0\end{pmatrix}\cdot\begin{pmatrix}a\\b\\c\end{pmatrix}$

$= \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \,\middle|\, ax+by+cz=d\right\}$ where $d = ax_0 + by_0 + cz_0$



A line $\{\vec{x} = \vec{P} + t\vec{v} \mid t \in \mathbb{R}\}$

$= \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x_0\\y_0\\z_0\end{pmatrix} + t\begin{pmatrix}a\\b\\c\end{pmatrix} \,\middle|\, t\in\mathbb{R}\right\}$

where

$= \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x_0+ta\\y_0+tb\\z_0+tc\end{pmatrix} \,\middle|\, t\in\mathbb{R}\right\}$ $\quad \vec{P} = \begin{pmatrix}x_0\\y_0\\z_0\end{pmatrix}$ $\vec{v} = \begin{pmatrix}a\\b\\c\end{pmatrix}$

parametric

$x = x_0 + ta$

$y = y_0 + tb$

$z = z_0 + tc$

symmetric

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

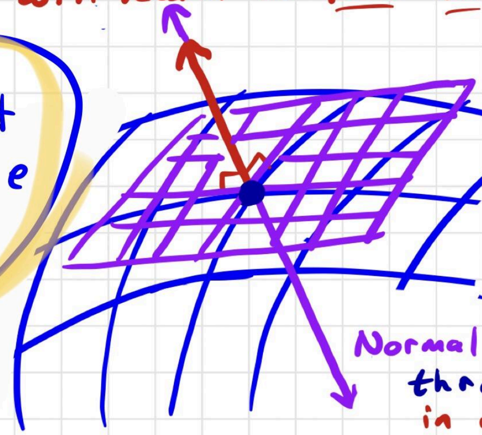
when $a\neq 0$ $b\neq 0$ $c\neq 0$

Tangent Planes and Normal Lines

Given a surface $S \subset \mathbb{R}^3$ and $\vec{p} \in S$

we will learn to find the Normal $\vec{N} \perp$ to S at p

we next describe S



Review

Functions of Several Variables

Functions of Several Variables

$z = f(x, y)$ is a function of two variables

$f: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$

(x, y: two inputs "independent variables"; z: 1 output "dependent variable")

$f(x_1, x_2, x_3 \ldots x_m)$

$f: \mathbb{R}^m \longrightarrow \mathbb{R}$

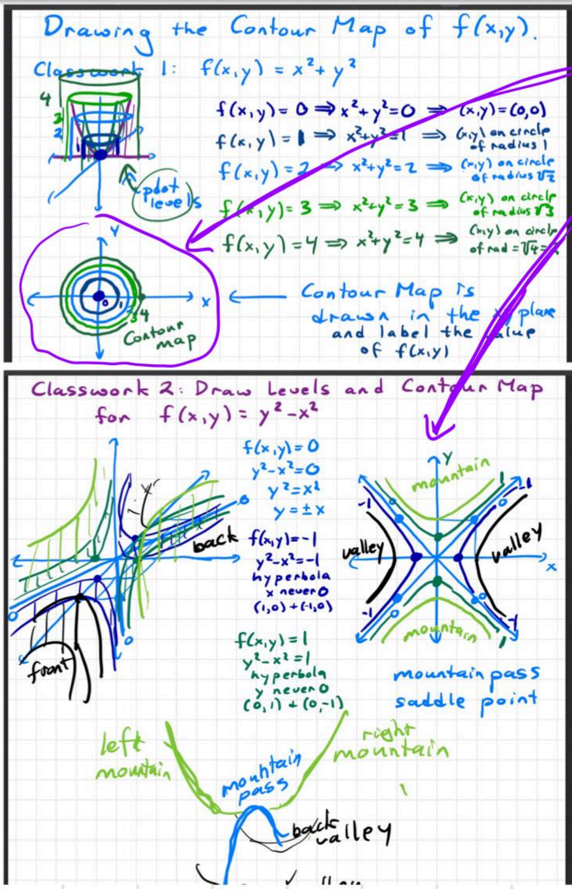
(m inputs "independent variables"; 1 output "dependent variable")

Today we consider $f: \mathbb{R}^3 \to \mathbb{R}$

$f(x, y, z)$ cannot be plotted but can still be studied

Sometimes $f: D \subset \mathbb{R}^3 \longrightarrow \mathbb{R}$

(D: domain)



Contour Maps are graphs of level sets of $f: D \subset \mathbb{R}^2 \to \mathbb{R}$

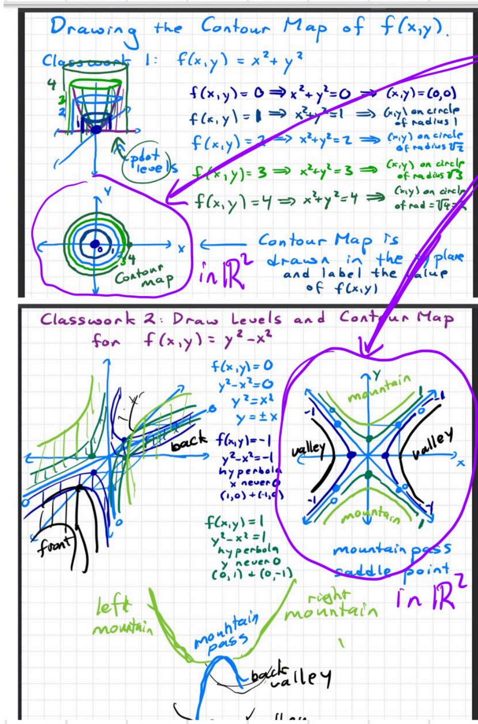
So we can also use contour maps to study

$$f: D \subset \mathbb{R}^3 \to \mathbb{R}$$

Classwork:
Plot the contour map for

$$f(x,y,z) = x^2 + y^2 + z^2$$

plot levels $f(x,y,z) = 0$
$f(x,y,z) = 1$
$f(x,y,z) = 4$



Contour Maps are graphs of level sets of $f: D \subset \mathbb{R}^2 \to \mathbb{R}$ in the domain $D \subset \mathbb{R}^2$

So we can also use contour maps to study $f: D \subset \mathbb{R}^3 \to \mathbb{R}$

this contour map will lie in $\mathbb{R}^3$

Classwork:
Plot the contour map for

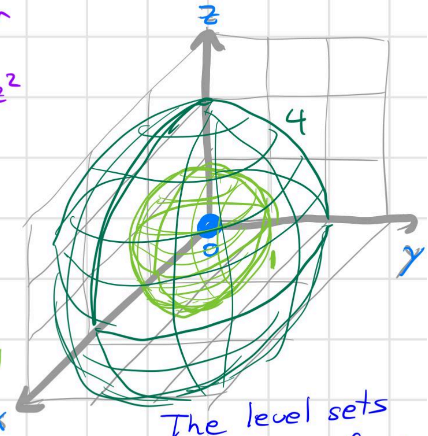
$f(x,y,z) = x^2+y^2+z^2$

plot levels

$f(x,y,z) = 0$
$x^2+y^2+z^2 = 0$
$x=0 \quad y=0 \quad z=0$

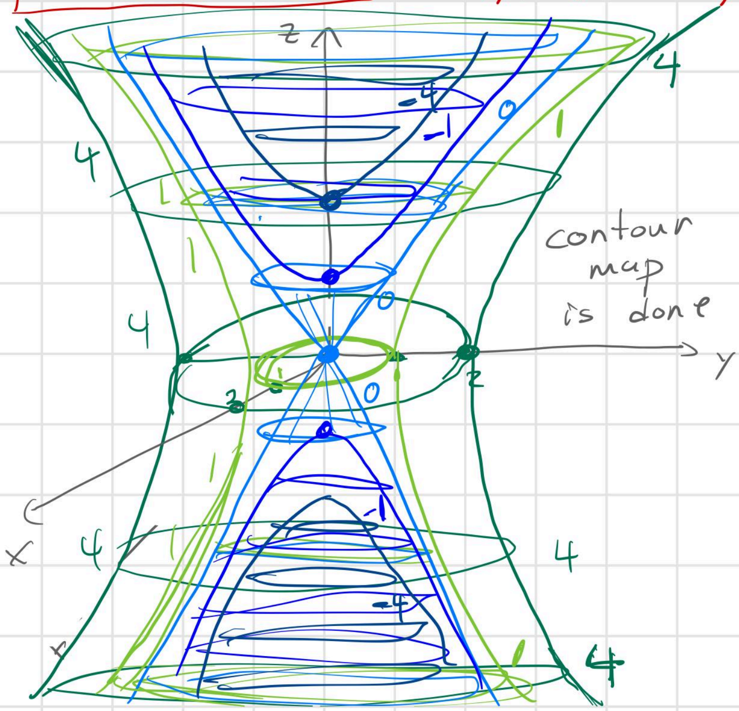
$f(x,y,z) = 1$
$x^2+y^2+z^2 = 1$
sphere of radius 1

$f(x,y,z) = 4$
$x^2+y^2+z^2 = 4$
sphere of radius 2



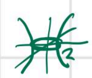
The level sets can be surfaces or points or curves.

Classwork 2 $f(x,y,z) = x^2+y^2-z^2$ plot levels −4, −1, 0, 1, 4



contour map is done

$f(x,y,z) = 4$
$x^2+y^2-z^2 = 4$

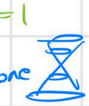


hyperboloid $z=0$
$x^2+y^2=4$

$f(x,y,z) = 1$
$x^2+y^2-z^2 = 1$

hyperboloid $z=0$
$x^2+y^2=1$

$f(x,y,z) = 0$
$x^2+y^2-z^2 = 0$

cone 

$z=1$ $x^2+y^2=1$
$z=0$ $x^2+y^2=0$ $(0,0,0)$

$f(x,y,z) = -1$
$x^2+y^2-z^2 = -1$

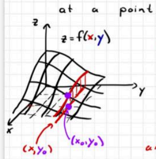
$x=0$ $y=0$ $z=\pm 1$ two sheeted hyperboloid

$f(x,y,z) = -4$
$x^2+y^2-z^2 = -4$
two sheeted
$x=0, y=0 \quad z=\pm 2$

Partial Derivatives

of a function $f(x,y)$ $\quad f:\mathbb{R}^2 \to \mathbb{R}$

at a point (x_0,y_0)



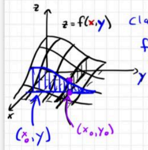
$$f_x(x_0,y_0) = \lim_{x\to x_0} \frac{f(x,y_0)-f(x_0,y_0)}{x-x_0}$$

↑ slope of the red curve

a "the partial derivative with respect to x"

$$= \lim_{\Delta x\to 0} \frac{f(x_0+\Delta x, y_0)-f(x_0,y_0)}{\Delta x}$$

another notation for this is $\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}$ not $\frac{df}{dx}$



Classwork: try to define

$$f_y(x_0,y_0) = \lim_{y\to y_0} \frac{f(x_0,y)-f(x_0,y_0)}{y-y_0}$$

slope of the blue curve

"the partial derivative with respect to y"

$$= \lim_{\Delta y\to 0} \frac{f(x_0,y_0+\Delta y)-f(x_0,y_0)}{\Delta y}$$

∂ another notation is $\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}$ not $\frac{df}{dy}$

Partial Derivative of $f:\mathbb{R}^3\to\mathbb{R}$ at a point (x_0,y_0,z_0)

Now we cannot plot but can imitate the defn.

$$f_x(x_0,y_0,z_0) = \lim_{x\to x_0} \frac{f(x,y_0,z_0)-f(x_0,y_0,z_0)}{x-x_0}$$

$$= \lim_{\Delta x\to 0} \frac{f(x_0+\Delta x,y_0,z_0)-f(x_0,y_0,z_0)}{\Delta x}$$

$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0,z_0)}$ and we differentiate x only treating y & z as constants

← fill in these explanations too,

Classwork 3 Define $f_y(x_0,y_0,z_0)$, $f_z(x_0,y_0,z_0)$

$$f_y(x_0,y_0,z_0) = \lim_{y\to y_0} \frac{f(x_0,y,z_0)-f(x_0,y_0,z_0)}{y-y_0}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0,z_0)}$$

$$f_z(x_0,y_0,z_0) = \lim_{z\to z_0} \frac{f(x_0,y_0,z)-f(x_0,y_0,z_0)}{z-z_0}$$

$$\left.\frac{\partial f}{\partial z}\right|_{(x_0,y_0,z_0)}$$

Directional Derivative for $f:\mathbb{R}^3\to\mathbb{R}$

Let $\hat{u}$ be a unit vector $\hat{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$

the $D_{\hat{u}} f(\vec{x}) = \left.\frac{d}{dt} f(\vec{x}+t\vec{u})\right|_{t=0}$

$$D_{\hat{u}} f(x_1,x_2) = \left.\frac{d}{dt} f(x_1+tu_1,\ x_2+tu_2,\ x_3+tu_3)\right|_{t=0}$$

Special Case

If f is differentiable at (x_0,y_0) then

Thm: $D_{\hat{u}} f(x_0,y_0,z_0) = \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0,z_0)} u_1 + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0,z_0)} u_2 + \left.\frac{\partial f}{\partial z}\right|_{(x_0,y_0,z_0)} u_3$

Proof: (use Chain Rule)

① $D_{\hat{u}} f(x_0,y_0) = \left.\frac{d}{dt} f(x_0+tu_1, y_0+tu_2, z_0+tu_3)\right|_{t=0}$ by defn of $D_{\hat{u}}f$

② $= \frac{d}{dt} f(x(t), y(t), z(t))$ where $x(t) = x_0 + t\,u_1$, $y(t) = y_0 + t\,u_2$, $z(t) = z_0 + t\,u_3$

③ $= \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0,z_0)} \left.\frac{dx}{dt}\right|_0 + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0,z_0)} \cdot \left.\frac{dy}{dt}\right|_0 + \left.\frac{\partial f}{\partial z}\right|_{(x_0,y_0,z_0)} \cdot \left.\frac{dz}{dt}\right|_0$ update the chain rule!

④ $= \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0,z_0)} u_1 + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0,z_0)} u_2 + \left.\frac{\partial f}{\partial z}\right|_{(x_0,y_0,z_0)} \cdot u_3$ by $x'(t) = 0 + 1\cos\theta$, $y'(t) = 0 + 1\sin\theta$ because θ = const

QED

Classwork 4.

classwork update this proof for $f:\mathbb{R}^3\to\mathbb{R}$

recall $f_x = \frac{\partial f}{\partial x}$, $f_y = \frac{\partial f}{\partial y}$, $f_z = \frac{\partial f}{\partial z}$

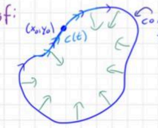
So by This theorem

$$D_{\hat{u}} f(x_0,y_0,z_0) = \nabla f\big|_{(x_0,y_0,z_0)} \cdot \vec{u} \quad \text{where } \nabla f = \begin{pmatrix} f_x(x_0,y_0,z_0) \\ f_y(x_0,y_0,z_0) \\ f_z(x_0,y_0,z_0) \end{pmatrix}$$

Thm The gradient field is $\perp$ to the contours.

$\nabla f_{(x_0,y_0)} \perp C'(t_0)$ if $C(t_0) = (x_0, y_0)$ and $C(t) \subset$ Contour for $f(x_0,y_0)$

Proof:



$f(x,y) = f(x_0,y_0)$

$C(t) = (x(t), y(t))$ inside the contour

$\underbrace{f(x(t), y(t))}_{\text{constant height}} = \underbrace{f(x_0,y_0)}_{\text{constant}}$

① $\frac{d}{dt} f(x(t), y(t)) = 0$ because we are on a level the height is not changing

② $\underbrace{f_x(x(t_0),y(t_0)) \cdot x'(t_0)}_{\text{change in x}} + \underbrace{f_y(x(t_0),y(t_0))\, y'(t_0)}_{\text{change in y}} = 0$ by chain rule

③ $\begin{pmatrix} f_x(x_0,y_0) \\ f_y(x_0,y_0) \end{pmatrix} \cdot \begin{pmatrix} x'(t_0) \\ y'(t_0) \end{pmatrix} = 0$ by dot product

④ $\nabla f_{(x_0,y_0)} \cdot C'(t_0) = 0$ by defn ∇f and defn C'

⑤ $\nabla f \perp$ Contour QED

Thm ∇f points in the direction of steepest increase

Let $\hat{u} = \frac{\nabla f}{|\nabla f|}$ then $D_{\hat{u}} f \geq D_{\hat{\omega}} f$ for any other direction $\vec{\omega}$

Proof ① $D_{\hat{u}} f = \hat{u} \cdot \nabla f$ ① Thm from today

$D_{\hat{\omega}} f = \hat{\omega} \cdot \nabla f$

② $\hat{u} \cdot \nabla f = |\hat{u}|\,|\nabla f| \cos(\text{angle between them})$ ② $\vec{v} \cdot \vec{\omega} = |\vec{v}||\vec{\omega}|\cos\theta$

also $\hat{\omega} \cdot \nabla f = |\hat{\omega}|\,|\nabla f| \cos(\text{angle for these two})$

③ If $\hat{u} = \frac{\nabla f}{|\nabla f|}$ then angle between $\hat{u}$ and ∇f is 0 in same direction

So $\hat{u} \cdot \nabla f = |\hat{u}||\nabla f| \cos(0) = 1 \cdot |\nabla f| \cdot 1 = |\nabla f|$

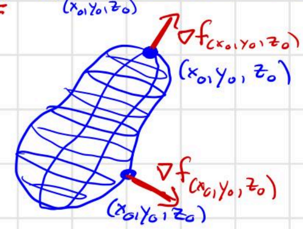
But $\hat{\omega} \cdot \nabla f = |\hat{\omega}||\nabla f| \cos(\text{angle between them}) \leq 1 \cdot |\nabla f| \cdot 1 = |\nabla f|$

④ $\hat{\omega} \cdot \nabla f \leq |\nabla f| = \hat{u} \cdot \nabla f$ by step 3

⑤ $D_{\hat{\omega}} f \leq D_{\hat{u}} f$ by step 1+4

QED

Thm $\nabla f_{(x_0,y_0,z_0)} \perp$ to level set $f(x,y,z) = f(x_0,y_0,z_0)$



So we will use ∇f to define the normal! $\vec{N} = \nabla f$

Given a surface S

write $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| f(x,y,z) = C \right\}$

describe S as a level set of a function.

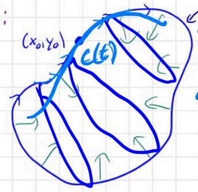
then $\vec{N} = \nabla f$ at (x_0,y_0,z_0)

Classwork 5:
fix the proof of this thm. to work for $f: \mathbb{R}^3 \to \mathbb{R}$

Thm The gradient field is ⊥ to the contours.

$\nabla f_{(x_0,y_0)} \perp C'(t_0)$ if $C(t_0)=(x_0,y_0,z_0)$ and $C(t) \subset$ Contour for $f(x_0,y_0,z_0)$

Proof:



contour $f(x,y,z) = f(x_0,y_0,z_0) = k$ surface

any $C(t) = (x(t), y(t), z(t))$ inside the contour

$f(x(t), y(t), z(t)) = k$ (constant, constant)

① $\frac{d}{dt} f(x(t), y(t), z(t)) = 0$ because $\frac{d}{dt} k = 0$ (a level) the height is not changing

add more terms → ② $f_x(x(t_0),y(t_0),z) \cdot x'(t_0) + f_y(x(t_0),y(t_0),z)\, y'(t_0) + f_z(\ldots)\, z'(t_0) = 0$ by chain rule (change in x)

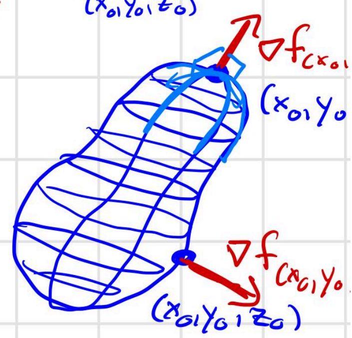
③ $\begin{pmatrix} f_x(x_0,y_0,z_0) \\ f_y(x_0,y_0,z_0) \\ f_z(x_0,y_0,z_0) \end{pmatrix} \cdot \begin{pmatrix} x'(t_0) \\ y'(t_0) \\ z'(t_0) \end{pmatrix} = 0$ by dot product

④ $\nabla f_{(x_0,y_0,z_0)} \cdot C'(t_0) = 0$ by defn ∇f and defn C' — for all curves in the contour

⑤ $\nabla f \perp$ Contour level set QED

171 of 171

Thm $\nabla f_{(x_0,y_0,z_0)} \perp$ to l



So we will …
the n…

Given a surf…
write $S = \{(\ldots$
descr…
set

Thm ∇f points in the direction of steepest increase

Let $\hat{u} = \frac{\nabla f}{|\nabla f|}$ then $D_{\hat{u}} f \geq D_{\hat{\omega}} f$ for any other direction $\vec{\omega}$

Proof ① $D_{\hat{u}} f = \hat{u} \cdot \nabla f$ ① Thm from today

$D_{\hat{\omega}} f = \hat{\omega} \cdot \nabla f$

② $\hat{u} \cdot \nabla f = |\hat{u}|\,|\nabla f| \cos(\text{angle between them})$ ② $\vec{v} \cdot \vec{w} = |\vec{v}|\,|\vec{w}| \cos\theta$

also $\hat{\omega} \cdot \nabla f = |\hat{\omega}|\,|\nabla f| \cos(\text{angle for these two})$

③ If $\hat{u} = \frac{\nabla f}{|\nabla f|}$ then angle between $\hat{u}$ and ∇f is 0 in same direction

So $\hat{u} \cdot \nabla f = |\hat{u}|\,|\nabla f| \cos(0) = 1 \cdot |\nabla f| \cdot 1 = |\nabla f|$

But $\hat{\omega} \cdot \nabla f = |\hat{\omega}|\,|\nabla f| \cos(\text{angle between them}) \leq 1 \cdot |\nabla f| \cdot 1 = |\nabla f|$

④ $\hat{\omega} \cdot \nabla f \leq |\nabla f| = \hat{u} \cdot \nabla f$ by step 3

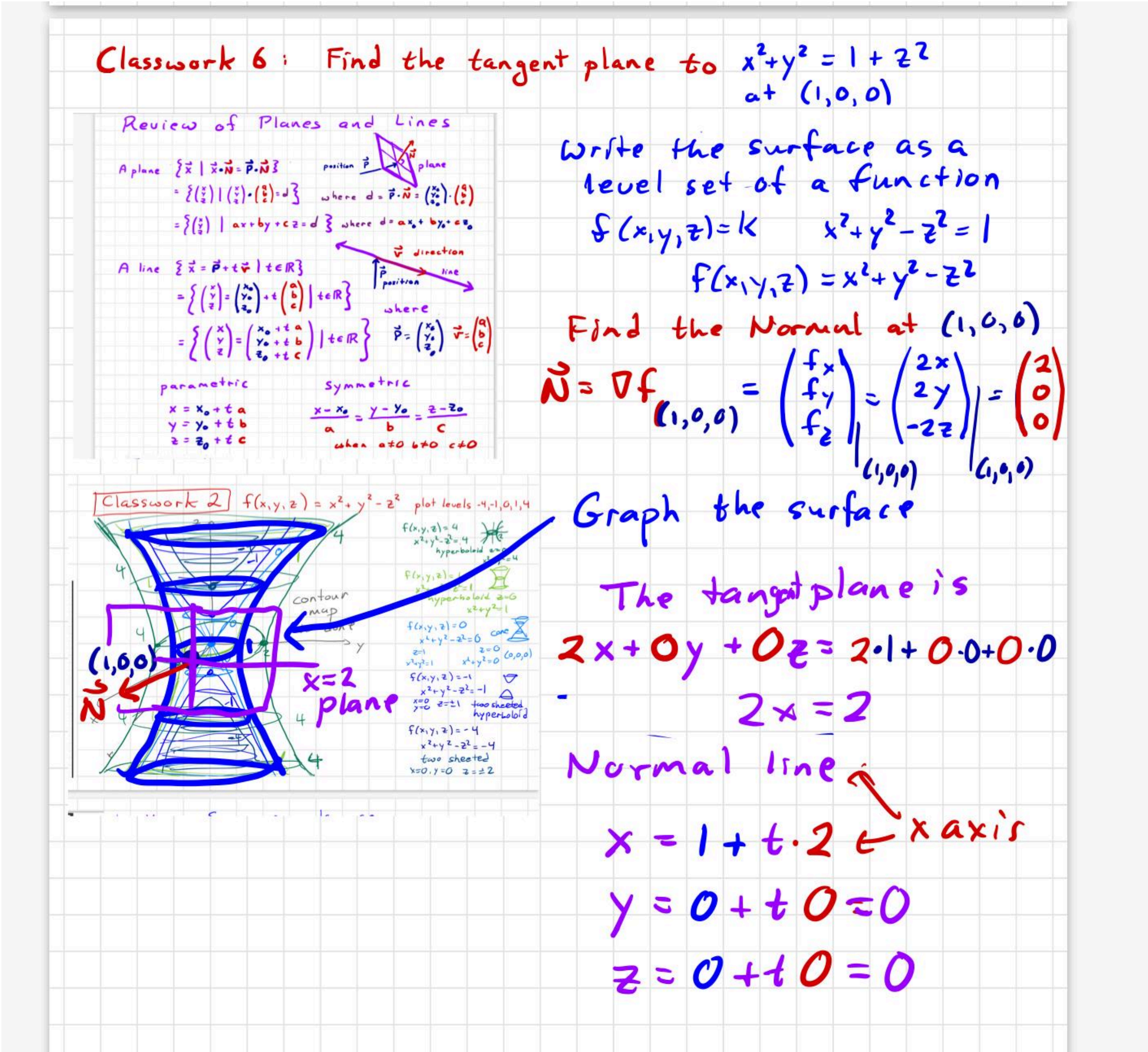
⑤ $D_{\hat{\omega}} f \leq D_{\hat{u}} f$ by step 1 + 4

QED

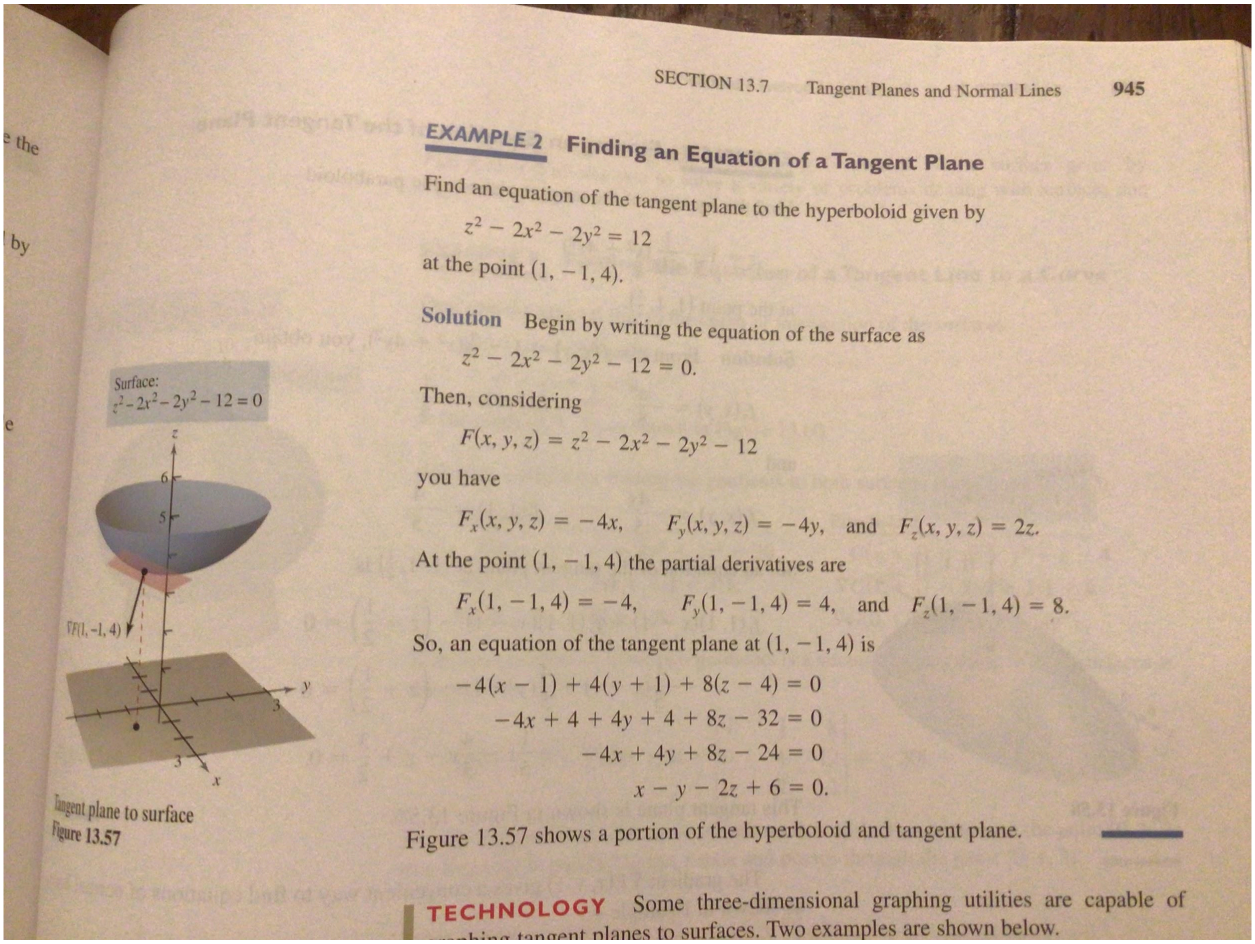
This does not need any changes its all just dot products and vectors

Class…
f…

Clas…
f…



Read the examples in the book. In particular read this example:



EXAMPLE 2 Finding an Equation of a Tangent Plane

Find an equation of the tangent plane to the hyperboloid given by

$$z^2 - 2x^2 - 2y^2 = 12$$

at the point $(1, -1, 4)$.

Solution Begin by writing the equation of the surface as

$$z^2 - 2x^2 - 2y^2 - 12 = 0.$$

Then, considering

$$F(x, y, z) = z^2 - 2x^2 - 2y^2 - 12$$

you have

$$F_x(x, y, z) = -4x, \quad F_y(x, y, z) = -4y, \quad \text{and} \quad F_z(x, y, z) = 2z.$$

At the point $(1, -1, 4)$ the partial derivatives are

$$F_x(1, -1, 4) = -4, \quad F_y(1, -1, 4) = 4, \quad \text{and} \quad F_z(1, -1, 4) = 8.$$

So, an equation of the tangent plane at $(1, -1, 4)$ is

$$-4(x - 1) + 4(y + 1) + 8(z - 4) = 0$$
$$-4x + 4 + 4y + 4 + 8z - 32 = 0$$
$$-4x + 4y + 8z - 24 = 0$$
$$x - y - 2z + 6 = 0.$$

Figure 13.57 shows a portion of the hyperboloid and tangent plane.

Tangent plane to surface
Figure 13.57

TECHNOLOGY Some three-dimensional graphing utilities are capable of graphing tangent planes to surfaces. Two examples are shown below.

Please check that you watched all the videos in [Playlist 226F21-16-1to11] before doing the homework.

Homework is required for this lesson. Tangent plane problems are on the exam so definitely do 1-2 of them.

HW (do before next lesson): 13.6/23, 25, 27, 31, normal to level, topography, heat seeking, meteorology 13.7/ 5, 7, 9, 17, 19, 21