Uncertainties and Error Analysis Tutorial

I. Reporting Measurements

When scientists make a measurement or calculate some quantity from their data, they compare them to some accepted standards. The length of a book is compared to marking of a ruler; the time for a ball to fall some distance is compared to the seconds ticked off on the stopwatch, etc.
Scientists reporting their results usually specify a range of values that they expect this "true value" to fall within. If the height of a basketball player were reported as 1.94 m, this value would mean that the measurement fell between 1.90 m and 2.00 m. That 4 cm is the best guess of the final value.

The rule is: **You should always measure as many digits as you can with certainty and then estimate one more digit.**

The most common way to show the range of values is:

Measurement = Best Estimate ± Uncertainty
or

$$X = X_{best} \pm \delta X$$

The uncertainty is the experimenter's best estimate of how far an experimental quantity might be from the "true value." (The art of estimating this uncertainty is what error analysis is all about).
According to "An Introduction to Error Analysis" by John R. Taylor (pg. 15), there are two rules we have to follow when communicating a measurement:
Rule 1 - Stating Uncertainties: Experimental uncertainties should almost always be rounded to one significant figure.

Wrong: 52.3 ± 4.1 cm
Correct: 52 ± 4 cm

Exception: If the leading digit in the uncertainty is a 1, then keeping two significant figures in it is more appropriate. We will not apply this exception for leading digit of 2. Examples:

Wrong: 9.81 ± 0.1 m/s^2
Correct: 9.81 ± 0.14 m/s^2
Wrong: 3.262 ± 0.127 cm
Correct: 3.26 ± 0.13 cm

Rule 2 - Stating Answers: The last significant figure in any stated answer should usually be of the same order of magnitude (in the same decimal position) as the uncertainty.

Wrong: 1.237 ± 0.2 s
Correct: 1.2 ± 0.2 s

II. Uncertainty in a single measurement

Finding the uncertainty in a measurement is not a trivial matter. Many digital instruments will have an uncertainty value given by the manufacturer, but for other tools you will need to make an *uncertainty estimate.*

In many circumstances, a single measurement of a quantity is often sufficient for the purposes of the measurement being taken. How can you estimate the uncertainty in a single measurement? The uncertainty of a single measurement is limited by the precision and accuracy of the measuring instrument, along with any other factors that might affect the ability of the experimenter to make the measurement. Ask yourself the question “How far off from my measured value could the ‘true value’ reasonably be?” and consider possible sources of uncertainty.

A useful rule of thumb for establishing an order of magnitude is that the uncertainty of a measurement tool **is around half of the smallest increment that tool can measure.** Your uncertainty will almost always be larger than this rule of thumb value because the tool itself is rarely the only significant source of uncertainty.

III. Uncertainty in multiple measurements of the same quantity

If on the other hand, when making repeated measurements, the best estimate of a parameter is determined by computing the average value from the multiple trials, the uncertainty associated with the average value (or mean) of your measurements can be determined from the *standard deviation,* σ. For *n* measurements of a quantity *x*, the standard deviation can be expressed as:

$$\sigma_x = \sqrt{\frac{\sum(x-\bar{x})^2}{n-1}}$$

Though you are welcome to calculate this for each individual value, Excel or Google Sheets will do standard deviations for you using the *=STDEV()* function.

If $x_{best} = \bar{x}$, then the *standard deviation of the mean (SDOM),* $\sigma_{\bar{x}}$, can be expressed as:

$$\delta x = \sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{n}}$$

You can enter this formula in MS Excel to simplify your calculations.

IV. Combining uncertainties in calculations

When calculating a value from several measured quantities, your calculated value will also have an uncertainty. You can find the uncertainty of your calculated value by using the following rules. If both addition/subtraction and multiplication/division are included, calculate the uncertainty in several steps.

Addition/Subtraction:
If several quantities $x, \ldots, w$ are measured with uncertainties $\delta x, \ldots, \delta w$, and the measured values are used to compute

$$q = x + \ldots + z - (u + \ldots + w),$$

then the uncertainty in the computed value of q is the sum,

$$\delta q = \sqrt{\left(\delta x^2\right) + \ldots + \left(\delta z^2\right) + \left(\delta u^2\right) + \ldots + \left(\delta w^2\right)}$$

of all the original uncertainties.

Multiplication/Division:
If several quantities $x, \ldots, w$ are measured with uncertainties $\delta x, \ldots, \delta w$, and the measured values are used to compute

$$q = \frac{x \cdot \ldots \cdot z}{u \cdot \ldots \cdot w}$$

then the *fractional uncertainty* in the computed value of q is the quadratic sum,

$$\frac{\delta q}{q} = \sqrt{\left(\frac{\delta x}{x}\right)^2 + \ldots + \left(\frac{\delta z}{z}\right)^2 + \left(\frac{\delta u}{u}\right)^2 + \ldots + \left(\frac{\delta w}{w}\right)^2}$$

of the *fractional uncertainties* in $x, \ldots, w$.

Power:
If a quantity x *is* measured with uncertainties δx and the measured value is used to compute

$$q = x^n$$

then the uncertainty in the computed value of q is

$$\frac{\delta q}{q} = n\frac{\delta x}{x}$$

Example:

$$q = r^2$$

$$\frac{\delta(r^2)}{r^2} = 2\frac{\delta(r)}{r}$$

$$\delta(r^2) = 2 \cdot r \cdot \delta(r)$$

<u>Exact Numbers</u>:
Constants like 2, π, *g*, etc. are treated as *exact*, meaning they have no meaningful uncertainty. When you do math with a number that has no uncertainty, use the rules above but give the value an uncertainty of zero. For example:

If $q = m \cdot x$ where m is a constant, then

$$\frac{\delta q}{q} = \frac{\delta x}{x}$$

V. Comparing values quantitatively

Percent error is used when the theoretical value of a physical quantity is known, and you've obtained an experimental value. Use a *percent error* to evaluate how accurate your experimental value is. The formula used to calculate the *percent error* is below.

$$percent\ error = \left|\frac{experimental\ value - theoretical\ value}{theoretical\ value}\right| \times 100\%$$

Percent difference is used when the quantity measured does *not* have a theoretical value. In this case, you are comparing two measured values to one another. Terms x_1 and x_2 in the formula below represent any two measured values of the same physical quantity.

$$percent\ difference = \frac{\left|x_1 - x_2\right|}{\left(\frac{x_1 + x_2}{2}\right)} \times 100\%$$