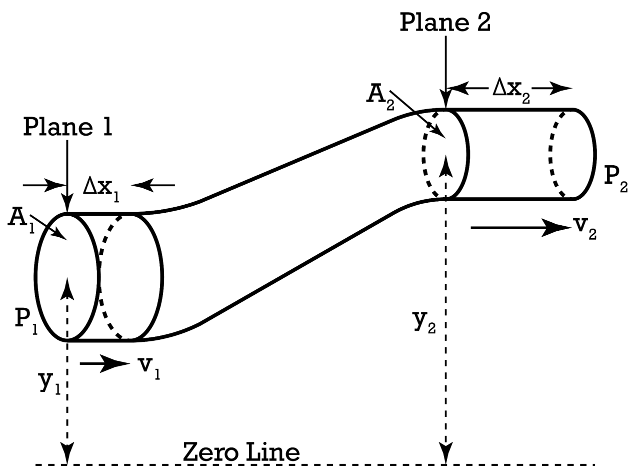
Flipping Physics Lecture Notes:
Bernoulli Derivation
http://www.flippingphysics.com/bernoulli-derivation.html

As a fluid moves such that it's speed and/or height above the Earth's surface changes, the pressure in the fluid also changes. This was first worked out during the 18th century by French physicist Daniel Bernoulli and is now known as Bernoulli's Principle.

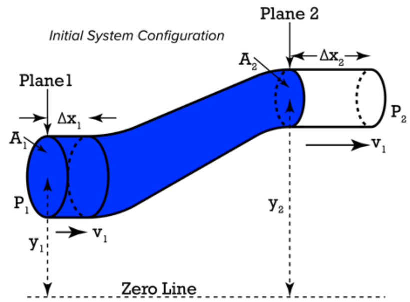
To derive Bernoulli's Principle, let's start by considering the flow of an ideal fluid through a pipe which changes in both diameter and height. Remember in ideal fluid flow the fluid is incompressible and nonviscous, and the fluid flow is laminar and irrotational.



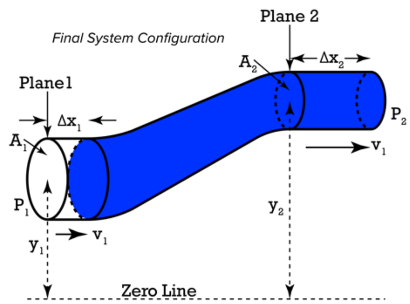
To help you understand the above illustration, realize the fluid is flowing through the pipe from left to right. The two specific locations we are going to refer to are plane 1 and plane 2. The variables are:

- The cross-sectional area of each plane is A.
    - As shown in the figure it is a circular cross-section, however, it could have any shape.
- The distance the fluid flows during change in time, Δt, is Δx.
- The average speed of the fluid flowing through each plane is v.
    - $v_2 > v_1$ because, according to the continuity equation of ideal fluid flow in an enclosed container, $A_1v_1 = A_2v_2$. Therefore, because the cross-sectional area of the pipe is smaller at plane 2, the speed of the fluid flow is larger at plane 2.
- The pressure on each plane is P.
    - P_1 is to the left of plane 1 and pushes to the right on plane 1.
    - P_2 is to the right of plane 2 and pushes to the left on plane 2.
- A horizontal zero line is placed an arbitrary distance below the pipe so we can define the heights of the center of each pipe as y_1 and y_2.

Before we derive Bernoulli's Principle, we need to define our system, and the initial and final configurations of that system. The system is a specific volume of fluid moving through the pipe. The volume of fluid in our system at the initial configuration is shown in blue here:



After change in time, Δt, the volume of fluid in our system is at the final configuration is shown in blue here:



Realize the pipe is completely full of fluid, however, our system is a specific volume of fluid and, because the fluid is flowing to the right, our system moves to the right.

In order to derive Bernoulli's Principle, let's use the Net Work equals Change in Kinetic Energy Theorem: $W_{\text{net}} = \Delta KE$.

There are three forces doing work on this system;

- W_1: Work done on plane 1 of our system caused by the pressure of the water which is located to the left of plane 1 pushing to the right on plane 1.
- W_2: Work done on plane 2 of our system caused by the pressure of the water which is located to the right of plane 2 pushing to the left on plane 2.
- W_{Fg}: Work done by the force of gravity on the fluid as the fluid moves through the pipe and changes its vertical height relative to the zero line.

$$W_{\text{net}} = \Delta KE \Rightarrow W_1 + W_2 + W_{F_g} = KE_f - KE_i$$

Let's determine the equation for each work separately and then substitute them into the above equation.

The work done on plane 1:

$$W = Fd\cos\theta \Rightarrow W_1 = F_1 \Delta x_1 \cos(0°) \Rightarrow W_1 = P_1 A_1 \Delta x_1$$

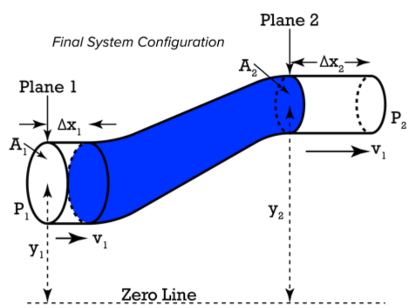
$$\& \; P = \frac{F_\perp}{A} \Rightarrow F_\perp = PA \Rightarrow F_1 = P_1 A_1$$

The work done plane 2:

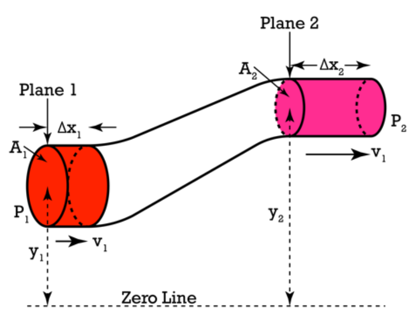
$$\Rightarrow W_2 = F_2 \Delta x_2 \cos(180°) \Rightarrow W_2 = -P_2 A_2 \Delta x_2$$

The work done by the force of gravity:

First off, realize the volume of the fluid in the curved section of the pipe between the volume of fluid which passes through plane 1 and the volume of fluid which passes through plane 2 does not change. Again, this volume of water, shown in blue below, does not change between the initial and final configurations of the system:



Therefore, the net effect is that the change in the system is that the volume of the fluid which passes through plane 1 (in red) becomes the volume of fluid which passes through plane 2 (in pink).



As shown in a previous lesson, because the force of gravity is a conservative force, the work done by the force of gravity is independent of path taken. In other words, the work done by the force of gravity on the volume of fluid in our system is the work done by the force of gravity as the volume of water which passes through plane 1 moves up a vertical distance, Δy.

$$\Rightarrow W_{F_g} = F_g \Delta y \cos\theta = mg(y_2 - y_1)\cos(180°) = -mg(y_2 - y_1)$$

And we can substitute equations into the Net Work equals Change in Kinetic Energy equation:

$$\Rightarrow W_{\text{net}} = \Delta KE \Rightarrow P_1 A_1 \Delta x_1 + (-P_2 A_2 \Delta x_2) + [-mg(y_2 - y_1)] = KE_f - KE_i$$

$$\Rightarrow P_1 A_1 \Delta x_1 - P_2 A_2 \Delta x_2 - mgy_2 + mgy_1 = KE_f - KE_i$$

And now let's look at the right-hand side of the equation; the change in kinetic energy of the system. Realize the mass of water in the middle of the pipe does not change its kinetic energy because its mass and velocity are the same in the initial and final configurations of the system. In other words, the final kinetic energy of the system is the kinetic energy of the mass of the fluid which passes through plane 2, and the initial kinetic energy of the system is the kinetic energy of the mass of the fluid which passes through plane 1.

$$KE_f = \frac{1}{2} m_2 {v_2}^2 \ \& \ KE_i = \frac{1}{2} m_1 {v_1}^2$$

We can substitute these into the Net Work equals Change in Kinetic Energy equation:

$$\Rightarrow P_1 A_1 \Delta x_1 - P_2 A_2 \Delta x_2 - mgy_2 + mgy_1 = \frac{1}{2} m_2 {v_2}^2 - \frac{1}{2} m_1 {v_1}^2$$

And rearrange to put all the 1's on the left-hand side and the 2's on the right-hand side:

$$\Rightarrow P_1 A_1 \Delta x_1 + \frac{1}{2} m_1 {v_1}^2 + mgy_1 = P_2 A_2 \Delta x_2 + \frac{1}{2} m_2 {v_2}^2 + mgy_2$$

Now, let's use the equations for volumetric mass density and volume:

$$\rho = \frac{m}{V} \Rightarrow m = \rho V \ \& \ V = A \Delta x \Rightarrow m = \rho A \Delta x$$

And substitute that equation for mass into our equation:

$$\Rightarrow P_1 A_1 \Delta x_1 + \frac{1}{2} (\rho A_1 \Delta x_1) {v_1}^2 + (\rho A_1 \Delta x_1) g y_1 = P_2 A_2 \Delta x_2 + \frac{1}{2} (\rho A_2 \Delta x_2) {v_2}^2 + (\rho A_2 \Delta x_2) g y_2$$

Remembering $v_{\text{avg}} = \frac{\Delta x}{\Delta t}$ and dividing that entire equation by change in time, gives us:

$$\Rightarrow P_1 \cancel{A_1 v_1} + \frac{1}{2} (\rho \cancel{A_1 v_1}) {v_1}^2 + (\rho \cancel{A_1 v_1}) g y_1 = P_2 \cancel{A_2 v_2} + \frac{1}{2} (\rho \cancel{A_2 v_2}) {v_2}^2 + (\rho \cancel{A_2 v_2}) g y_2$$

And remember the continuity equation of ideal fluid flow: $A_1 v_1 = A_2 v_2$
The volumetric flow rate remains constant during ideal fluid flow through an enclosed pipe. Everybody brough volumetric flow rate to the party!

$$\Rightarrow P_1 + \frac{1}{2}\rho {v_1}^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho {v_2}^2 + \rho g y_2$$

Or a more general statement of Bernoulli's Principle:

$$\Rightarrow P + \frac{1}{2}\rho v^2 + \rho g y = \text{constant}$$

Recognize that, in order to use Bernoulli's Principle, you need to identify the initial and final points, and the horizontal zero line, just like mechanical energy equations.

Notice if the fluid is at rest in our example:

$$v_1 = v_2 = 0 \Rightarrow P_1 + \rho g y_1 = P_2 + \rho g y_2 \Rightarrow P_2 - P_1 = \rho g y_1 - \rho g y_2$$

$$\Rightarrow \Delta P = -\rho g \left(y_2 - y_1\right)$$

We get the equation for the change in gauge pressure of the fluid.
In other words, because y_2 is more than y_1 in our example, $\Delta y > 0$, the change in pressure of the fluid will be negative – as you move upward in a fluid the pressure decreases.

Notice if we change our example to have no change in y then: $y_1 = y_2 = y$

$$\Rightarrow P_1 + \frac{1}{2}\rho {v_1}^2 + \rho g y = P_2 + \frac{1}{2}\rho {v_2}^2 + \rho g y$$

$$\Rightarrow P_1 + \frac{1}{2}\rho {v_1}^2 = P_2 + \frac{1}{2}\rho {v_2}^2 \Rightarrow P_2 - P_1 = \frac{1}{2}\rho {v_1}^2 - \frac{1}{2}\rho {v_2}^2$$

$$\Rightarrow \Delta P = \frac{1}{2}\rho \left({v_2}^2 - {v_1}^2\right)$$

This shows that it is changes in pressure which cause fluids to change speed. In other words, fluid flow is caused by differences in pressure.