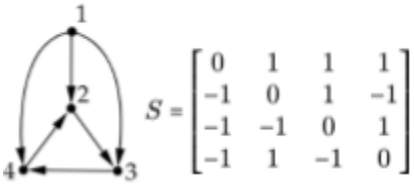
Tournaments

Comps Fall 2024 - Winter 2025 — MurphyKate Montee

This comps project lies somewhere in the union of graph theory, linear algebra, and combinatorics. The main topic of investigation is a **tournament**: for our purposes, a *tournament on n teams* is a directed complete graph with n vertices. In other words, a tournament is a way to record win-loss records for a round-robin tournament with n teams, where ties are not allowed.



A tournament on 4 teams and the corresponding Seidel matrix.

We can encode any tournament with a *Seidel matrix* S where $S_{ij} = 1$ if team i beats team j, $S_{ij} = -1$ if team j beats team i, and $S_{ii} = 0$. The benefit of doing this is that we can now use all of our linear algebra tools to analyze the tournament!

For example, you might want to take the determinant of the Seidel matrix. Here is some data on those determinants. Do you spot any patterns? The first part of this comps project will be devoted to understanding and verifying these patterns. Along the way we will prove some general results about determinants (did you know that in some cases, you can actually compute the determinant of a *sum* of matrices??) and skew-symmetric matrices. We will also make regular use of SageMath, an open-source mathematics software system based on Python (no coding experience required!). We'll also touch on some long-standing open problems in mathematics, including the Hadamard Conjecture.

Number of Teams	All possible values of the determinant of the Seidel matrix of a tournament on teams
1	0
2	1
3	0
4	1, 9
5	0
6	1, 9, 25, 49, 81
7	0
8	1, 9, 25, 49, 81, ..., 1089, 1225, 2401

Structure of this comps: This will be a "large group" comps, which means that in the Fall we will have a seminar-style course devoted to understanding the basics of tournaments. In the second term we will break into smaller groups and each group will focus on their own questions. This may involve reading published papers, doing computations, and/or doing your own new research.

Prerequisites: Linear Algebra and Math Structures. Any experience with abstract algebra, graph theory, or combinatorics will be helpful, but not required.

Further Reading: Want to know more? Check out these papers:

- *Skew-adjacency matrices of tournaments with bounded principal minors.* Boussaïri, Essahir, Lakhlifi, Mahzoum. 2022.
- *On skew-Hadamard matrices.* Koukovinos, Stylianou. 2006.
- *The skew energy of tournaments.* Ito. 2016.
- *Tournament games and positive tournaments.* Fisher, Ryan. 1995.