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Learning Outcomes

- Verify that multiplying a matrix by its inverse results in 1.
- Use matrix multiplication to find the inverse of a matrix.
- Find an inverse by augmenting with an identity matrix.

We know that the multiplicative inverse of a real number a is a^{-1} and $a a^{-1} = a^{-1} a = \left(\frac{1}{a}\right)a = 1$. For example, $2^{-1} = \frac{1}{2}$ and $\left(\frac{1}{2}\right)2 = 1$. The **multiplicative inverse of a matrix** is similar in concept, except that the product of matrix A and its inverse A^{-1} equals the **identity matrix**. The identity matrix is a square matrix containing ones down the main diagonal and zeros everywhere else. We identify identity matrices by I_n where n represents the dimension of the matrix. The equations below are the identity matrices for a 2×2 matrix and 3×3 matrix, respectively.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The identity matrix acts as a 1 in matrix algebra. For example, $AI = IA = A$.

A matrix that has a multiplicative inverse has the properties

$$A A^{-1} = I \quad A^{-1} A = I$$

A matrix that has a multiplicative inverse is called an **invertible matrix**. Only a square matrix may have a multiplicative inverse, as the reversibility, $A A^{-1} = A^{-1} A = I$, is a requirement. Not all square matrices have an inverse, but if A is invertible, then A^{-1} is unique. We will look at two methods for finding the inverse of a 2×2 matrix and a third method that can be used on both 2×2 and 3×3 matrices.

A General Note: The Identity Matrix and Multiplicative Inverse

The **identity matrix**, I_n , is a square matrix containing ones down the main diagonal and zeros everywhere else.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If A is an $n \times n$ matrix and B is an $n \times n$ matrix such that $AB = BA = I_n$, then $B = A^{-1}$, the **multiplicative inverse of a matrix** A .

Example: Showing That the Identity Matrix Acts as a 1

Given matrix A , show that $AI = IA = A$.

$$A = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$$

Show Solution

Use matrix multiplication to show that the product of A and the identity is equal to the product of the identity and A .

$$AI = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 \cdot 1 + 4 \cdot 0 & 3 \cdot 0 + 4 \cdot 1 \\ -2 \cdot 1 + 5 \cdot 0 & -2 \cdot 0 + 5 \cdot 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$$

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 1 \cdot 3 + 0 \cdot (-2) & 1 \cdot 4 + 0 \cdot 5 \\ 0 \cdot 3 + 1 \cdot (-2) & 0 \cdot 4 + 1 \cdot 5 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 5 \\ 1 & 4 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 4 \\ -2 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

How To: Given two matrices, show that one is the multiplicative inverse of the other

1. Given matrix A of order $n \times n$ and matrix B of order $n \times n$ multiply AB .
2. If $AB = I$, then find the product BA . If $BA = I$, then $B = A^{-1}$ and $A = B^{-1}$.

Example: Showing That Matrix **A** Is the Multiplicative Inverse of Matrix **B**

Show that the given matrices are multiplicative inverses of each other.

$$A = \begin{bmatrix} 1 & -2 & -9 \\ -9 & 2 & 1 \\ 1 & 5 & 1 \end{bmatrix}, B = \begin{bmatrix} -9 & -5 & 2 \\ 2 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix}$$

Show Solution

Multiply AB and BA . If both products equal the identity, then the two matrices are inverses of each other.

$$AB = \begin{bmatrix} 1 & -2 & -9 \\ -9 & 2 & 1 \\ 1 & 5 & 1 \end{bmatrix} \begin{bmatrix} -9 & -5 & 2 \\ 2 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -9 & -5 & 2 \\ 2 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} -9 & -5 & 2 \\ 2 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -9 \\ -9 & 2 & 1 \\ 1 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -9 \\ -9 & 2 & 1 \\ 1 & 5 & 1 \end{bmatrix}$$

A and B are inverses of each other.

Try It

Show that the following two matrices are inverses of each other.

$$A = \begin{bmatrix} 1 & 4 & -1 \\ -3 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} -3 & 4 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Show Solution

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 4 & -1 \\ -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 4 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1(-3) + 4(1) + (-1)(0) & 1(4) + 4(1) + (-1)(0) & 1(-1) + 4(1) + (-1)(0) \\ -3(-3) + 1(1) + 1(0) & -3(4) + 1(1) + 1(0) & -3(-1) + 1(1) + 1(0) \end{bmatrix} \\ &= \begin{bmatrix} -3 + 4 & 4 + 4 & -1 + 4 \\ 9 + 1 & -12 + 1 & 3 + 1 \end{bmatrix} = \begin{bmatrix} 1 & 8 & 3 \\ 10 & -11 & 4 \end{bmatrix} \\ BA &= \begin{bmatrix} -3 & 4 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 & -1 \\ -3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -3(1) + 4(-3) + 1(0) & -3(4) + 4(1) + 1(0) & -3(-1) + 4(1) + 1(0) \\ 1(1) + 1(-3) + 0(0) & 1(4) + 1(1) + 0(0) & 1(-1) + 1(1) + 0(0) \end{bmatrix} \\ &= \begin{bmatrix} -3 - 12 & -12 + 4 & 3 + 4 \\ 1 - 3 & 4 + 1 & -1 + 1 \end{bmatrix} = \begin{bmatrix} -15 & -8 & 7 \\ -2 & 5 & 0 \end{bmatrix} \end{aligned}$$

Finding the Multiplicative Inverse Using Matrix Multiplication

We can now determine whether two matrices are inverses, but how would we find the inverse of a given matrix? Since we know that the product of a matrix and its inverse is the identity matrix, we can find the inverse of a matrix by setting up an equation using **matrix multiplication**.

Example: Finding the Multiplicative Inverse Using Matrix Multiplication

Use matrix multiplication to find the inverse of the given matrix.

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -3 \end{bmatrix}$$

Show Solution

For this method, we multiply A by a matrix containing unknown constants and set it equal to the identity.

$$\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Find the product of the two matrices on the left side of the equal sign.

$$\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1a - 2c & 1b - 2d \\ 2a - 3c & 2b - 3d \end{bmatrix}$$

Next, set up a system of equations with the entry in row 1, column 1 of the new matrix equal to the first entry of the identity, 1. Set the entry in row 2, column 1 of the new matrix equal to the corresponding entry of the identity, which is 0.

$$\begin{cases} 1a - 2c = 1 \\ 2a - 3c = 0 \end{cases}$$

Using row operations, multiply and add as follows: $(-2)(R_1) + (R_2)$ to (R_2) . Add the equations, and solve for c .

$$\begin{cases} 1a - 2c = 1 \\ 0 + 1c = -2 \end{cases}$$

Back-substitute to solve for a .

$$\begin{array}{r} \hfill a - 2\left(-2\right)=1 \\\hfill a+4=1 \\\hfill a=-3 \end{array}$$

Write another system of equations setting the entry in row 1, column 2 of the new matrix equal to the corresponding entry of the identity, 0. Set the entry in row 2, column 2 equal to the corresponding entry of the identity.

$$\begin{array}{rr} \hfill 1b - 2d=0 & \hfill \{R\}_1 \\ \hfill 2b - 3d=1 & \hfill \{R\}_2 \end{array}$$

Using row operations, multiply and add as follows:

$\left(-2\right)\{R\}_1 + \{R\}_2 = \{R\}_2$. Add the two equations and solve for d .

$$\begin{array}{r} \hfill 1b - 2d=0 \\\hfill \frac{0+1d=1}{d=1} \\\hfill \end{array}$$

Once more, back-substitute and solve for b .

$$\begin{array}{r} \hfill b - 2\left(1\right)=0 \\\hfill b - 2=0 \\\hfill b=2 \end{array}$$

$${A}^{-1}=\left[\begin{array}{rrr} \hfill -3 & \hfill & \hfill 2 \\ \hfill & \hfill -2 & \hfill & \hfill 1 \end{array}\right]$$

Finding the Multiplicative Inverse by Augmenting with the Identity

Another way to find the **multiplicative inverse** is by augmenting with the identity. When matrix A is transformed into I , the augmented matrix I transforms into ${A}^{-1}$.

For example, given

$$A=\left[\begin{array}{rrr} \hfill 2 & \hfill & \hfill 1 \\ \hfill & \hfill 5 & \hfill & \hfill 3 \end{array}\right]$$

augment A with the identity

$$\left[\begin{array}{rr} \hfill 2 & \hfill 1 & \hfill 5 & \hfill 3 \\ \hfill 1 & \hfill 0 & \hfill 0 & \hfill 1 \end{array}\right]$$

Perform **row operations** with the goal of turning A into the identity.

1. Switch row 1 and row 2.

$$\left[\begin{array}{rr} \hfill 5 & \hfill 3 & \hfill 2 & \hfill 1 \\ \hfill 0 & \hfill 1 & \hfill 1 & \hfill 0 \end{array}\right]$$

2. Multiply row 2 by -2 and add to row 1.

$$\left[\begin{array}{rr} \hfill 1 & \hfill 1 & \hfill 2 & \hfill 1 \\ \hfill -2 & \hfill 1 & \hfill 1 & \hfill 0 \end{array}\right]$$

3. Multiply row 1 by -2 and add to row 2.

$$\left[\begin{array}{rr} \hfill 1 & \hfill 1 & \hfill 0 & \hfill -1 \\ \hfill -2 & \hfill 1 & \hfill 5 & \hfill -2 \end{array}\right]$$

4. Add row 2 to row 1.

$$\left[\begin{array}{rr} \hfill 1 & \hfill 0 & \hfill 0 & \hfill -1 \\ \hfill 3 & \hfill -1 & \hfill 5 & \hfill -2 \end{array}\right]$$

5. Multiply row 2 by -1 .

$$\left[\begin{array}{rr} \hfill 1 & \hfill 0 & \hfill 0 & \hfill 1 \\ \hfill 3 & \hfill -1 & \hfill -5 & \hfill 2 \end{array}\right]$$

The matrix we have found is A^{-1} .

$$A^{-1} = \left[\begin{array}{rr} \hfill 3 & \hfill -1 & \hfill -5 & \hfill 2 \\ \hfill 1 & \hfill 0 & \hfill 0 & \hfill 1 \end{array}\right]$$

Finding the Multiplicative Inverse of 2×2 Matrices Using a Formula

When we need to find the **multiplicative inverse** of a 2×2 matrix, we can use a special formula instead of using matrix multiplication or augmenting with the identity.

If A is a 2×2 matrix, such as

$$A = \begin{bmatrix} a & b & c \\ d \end{bmatrix}$$

the multiplicative inverse of A is given by the formula

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

where $ad-bc \neq 0$. If $ad-bc=0$, then A has no inverse.

Example: Using the Formula to Find the Multiplicative Inverse of Matrix **A**

Use the formula to find the multiplicative inverse of

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$$

Show Solution

Using the formula, we have

$$\begin{aligned} A^{-1} &= \frac{1}{\begin{vmatrix} 1 & -2 \\ 2 & -3 \end{vmatrix}} \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} \\ &= \frac{1}{-3+4} \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} \end{aligned}$$

Analysis of the Solution

We can check that our formula works by using one of the other methods to calculate the inverse. Let's augment A with the identity.

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{array} \right]$$

Perform **row operations** with the goal of turning A into the identity.

1. Multiply row 1 by -2 and add to row 2.

$$\left[\begin{array}{cc} 1 & -2 \\ 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]$$

2. Multiply row 1 by 2 and add to row 1.

$$\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc} -3 & 2 \\ 0 & 1 \end{array}\right]$$

So, we have verified our original solution.

$$A^{-1} = \left[\begin{array}{cc} -3 & 2 \\ -2 & 1 \end{array}\right]$$

Try It

Use the formula to find the inverse of matrix A . Verify your answer by augmenting with the identity matrix.

$$A = \left[\begin{array}{cc} 1 & -1 \\ 2 & 3 \end{array}\right]$$

Show Solution

$$A^{-1} = \left[\begin{array}{cc} \frac{3}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{1}{5} \end{array}\right]$$

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Example: Finding the Inverse of the Matrix, If It Exists

Find the inverse, if it exists, of the given matrix.

$$A = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$$

Show Solution

We will use the method of augmenting with the identity.

$$\left[\begin{array}{cc|cc} 3 & 6 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array} \right]$$

1. Switch row 1 and row 2.

$$\left[\begin{array}{cc|cc} 1 & 2 & 3 & 6 \\ 0 & 1 & 1 & 0 \end{array} \right]$$

2. Multiply row 1 by -3 and add it to row 2.

$$\left[\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right]$$

3. There is nothing further we can do. The zeros in row 2 indicate that this matrix has no inverse.

Finding the Multiplicative Inverse of 3×3 Matrices

Unfortunately, we do not have a formula similar to the one for a 2×2 matrix to find the inverse of a 3×3 matrix. Instead, we will augment the original matrix with the identity matrix and use **row operations** to obtain the inverse.

Given a 3×3 matrix

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}$$

augment A with the identity matrix

$$[A] = \left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 3 & 3 & 1 & 2 & 4 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right]$$

To begin, we write the **augmented matrix** with the identity on the right and $[A]$ on the left. Performing elementary **row operations** so that the **identity matrix** appears on the left, we will obtain the **inverse matrix** on the right. We will find the inverse of this matrix in the next example.

How To: Given a 3×3 matrix, find the inverse

1. Write the original matrix augmented with the identity matrix on the right.
2. Use elementary row operations so that the identity appears on the left.
3. What is obtained on the right is the inverse of the original matrix.
4. Use matrix multiplication to show that $A A^{-1} = I$ and $A^{-1} A = I$.

Example: Finding the Inverse of a 3×3 Matrix

Given the 3×3 matrix $[A]$, find the inverse.

$$[A] = \left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 3 & 3 & 1 & 2 & 4 & 1 \end{array} \right]$$

Show Solution

Augment $[A]$ with the identity matrix, and then begin row operations until the identity matrix replaces $[A]$. The matrix on the right will be the inverse of $[A]$.

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 3 & 3 & 1 & 2 & 4 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\text{Interchange } R_2 \text{ and } R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 3 & 3 & 1 & 2 & 3 & 1 & 2 & 4 & 1 \end{array} \right]$$

$$-R_2 + R_1 = R_1 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 3 & 1 & 2 & 4 & 1 \\ & & & -1 & -1 & 0 & -1 & -1 & 0 \\ & & & 0 & 0 & 1 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus,

$$A^{-1}B = I$$

Analysis of the Solution

To prove that $B = A^{-1}$, let's multiply the two matrices together to see if the product equals the identity, if $AA^{-1} = I$ and $A^{-1}A = I$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \\ 6 \end{bmatrix}$$

Try It

Find the inverse of the 3×3 matrix.

$$A = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix}$$

Show Solution

$$A^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}$$

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