Analysis Lesson 22
[MAT320/MAT640 Analysis]
with Professor Sormani
Spring 2022

Cauchy Sequences, Series, and Induction

Your work for today's lesson will go in a googledoc you create entitled MAT320S22-Lesson22-Lastname-Firstname with your last name and your first name. The googledoc will be shared with the professor sormanic@gmail.com as an editor. Put any questions you have inside your doc and email me to let me know it is there. Be sure to complete one page of HW on paper and take a selfie holding up a few pages.

This lesson has two parts and five homework problems.

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Part 1 Cauchy Sequences in Metric Spaces
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Watch the [Cauchy Playlist] and do HW1-HW5. If you are far behind schedule at least do HW2 and learn the theorems and definitions taught in this part.

Converging Points in Metric Spaces

Defn

A metric space is a collection of points, X, with a distance $d: X \times X \to \mathbb{R}$

- $d(x,y) \geq 0 \;\; \forall x,y \in X$ and $d(x,y) = 0 \iff x = y$
- $d(x,y) = d(y,x) \;\; \forall x,y \in X$
- $d(x,y) \leq d(x,z) + d(z,y) \;\; \forall x,y,z \in X$

Defn

A sequence of points $p_i \in X$ converge to $p \in X$

$p_i \to p$ if $\forall \varepsilon > 0 \; \exists N_\varepsilon$ s.t $\forall i \geq N_\varepsilon \;\; d(p_i, p) < \varepsilon$

Theorem: If $p_i \to p$ then $\{p_i \mid i \in \mathbb{N}\}$ is Cauchy.

Given: $p_i \to p \;\; \forall \varepsilon > 0 \; \exists N_\varepsilon$ s.t $\forall i \geq N_\varepsilon \;\; d(p_i, p) < \varepsilon$

Show: p_i Cauchy: $\forall \varepsilon > 0 \; \exists N_\varepsilon^c$ s.t $\forall i,j \geq N_\varepsilon \;\; d(p_i, p_j) < \varepsilon$

Proof: ① Given any $\varepsilon > 0$ Choose $N_\varepsilon^c = \boxed{}$

② Whenever $i,j \geq N_\varepsilon$

Structure

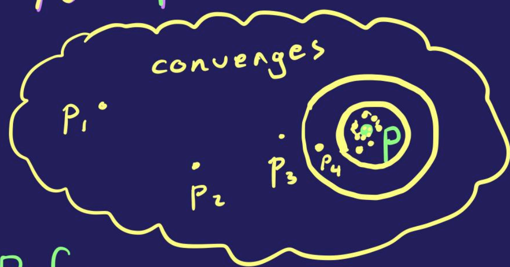
Next will fill in proof

final $d(p_i, p_j) < \varepsilon$

Defn

A sequence of points $p_i \in X$ converge to $p \in X$

$p_i \to p$ if $\forall \varepsilon > 0 \ \exists N_\varepsilon$ s.t $\forall i \geq N_\varepsilon \ d(p_i, p) < \varepsilon$



Defn

A sequence of points $p_i \in X$ is Cauchy

if $\forall \varepsilon > 0 \ \exists N_\varepsilon$ s.t $\forall i,j \geq N_\varepsilon \ d(p_i, p_j) < \varepsilon$

For Cauchy we do not mention any limit

Theorem: If $p_i \to p$ then $\{p_i \mid i \in \mathbb{N}\}$ is Cauchy.

Given: $p_i \to p$ $\forall \varepsilon > 0 \ \exists N_\varepsilon$ s.t $\forall i \geq N_\varepsilon \ d(p_i, p) < \varepsilon$

Show: p_i Cauchy: $\forall \varepsilon > 0 \ \exists N_\varepsilon^c$ s.t $\forall i,j \geq N_\varepsilon \ d(p_i, p_j) < \varepsilon$

Proof: (1) Given any $\varepsilon > 0$ Choose $N_\varepsilon^c = \boxed{}$

(2) whenever $i,j \geq N_\varepsilon$

Structure

final $d(p_i, p_j) < \varepsilon$

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Theorem: If $p_i \to p$ then $\underline{\{p_i \mid i \in \mathbb{N}\}}$ is $\underline{\text{Cauchy}}$.

Given: $p_i \to p$ $\forall \varepsilon > 0\ \exists N_\varepsilon$ s.t $\forall i \geq N_\varepsilon\ d(p_i, p) < \varepsilon$

Show: p_i Cauchy: $\forall \varepsilon > 0\ \exists N_\varepsilon^c$ s.t $\forall i,j \geq N_\varepsilon\ d(p_i, p_j) < \varepsilon$

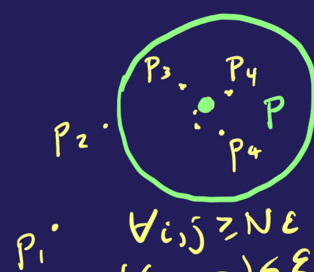
$\underline{\text{Proof}}$: ① Given any $\varepsilon > 0$ Choose $N_\varepsilon^c = \boxed{}$

② whenever $i,j \geq N_\varepsilon$

Structure

Fill in using Given

final $d(p_i, p_j) < \varepsilon$



$\forall i,j \geq N_\varepsilon$
$d(p_i, p) < \varepsilon$
$d(p_j, p) < \varepsilon$

Theorem: If $p_i \to p$ then $\underline{\{p_i \mid i \in \mathbb{N}\} \text{ is Cauchy}}$.

Given: $p_i \to p$ $\forall \varepsilon > 0$ $\exists N_\varepsilon$ s.t $\forall i \geq N_\varepsilon$ $d(p_i, p) < \varepsilon$

Show: p_i Cauchy: $\forall \varepsilon > 0$ $\exists N^c_\varepsilon$ s.t $\forall i,j \geq N_\varepsilon$ $d(p_i, p_j) < \varepsilon$

<u>Proof</u>: (1) Given any $\varepsilon > 0$ Choose $N^c_\varepsilon = \boxed{N_{\varepsilon/2}}$

Structure

(2) whenever $i, j \geq N^c_\varepsilon$ we have

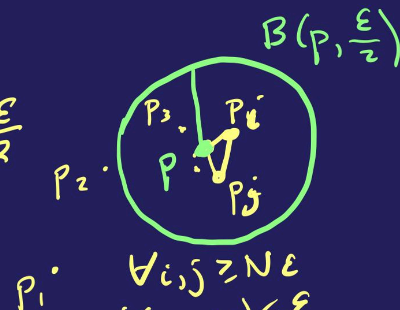
$i \geq N_{\varepsilon/2}$ AND $j \geq N_{\varepsilon/2}$

(3) $d(p_i, p) < \frac{\varepsilon}{2}$ AND $d(p, p_j) < \frac{\varepsilon}{2}$

(4) $d(p_i, p) + d(p, p_j) < \varepsilon$

$\qquad\qquad\quad \lor\!|$

final $d(p_i, p_j) < \varepsilon$



$\forall i,j \geq N_\varepsilon$
$d(p_i, p) < \varepsilon$
$d(p_j, p) < \varepsilon$

Theorem: If $p_i \to p$ then $\underline{\{p_i \mid i \in \mathbb{N}\} \text{ is Cauchy}}$.

Given: $p_i \to p$ $\forall \varepsilon > 0$ $\exists N_\varepsilon$ s.t $\forall i \geq N_\varepsilon$ $d(p_i, p) < \varepsilon$

Show: p_i Cauchy: $\forall \varepsilon > 0$ $\exists N^c_\varepsilon$ s.t $\forall i, j \geq N_\varepsilon$ $d(p_i, p_j) < \varepsilon$

Proof: (1) Given any $\varepsilon > 0$ Choose $N^c_\varepsilon = \boxed{N_{\varepsilon/2}}$ — (1) by Given $\exists N_\varepsilon$

Structure

(2) Whenever $i, j \geq N^c_\varepsilon$ we have — (2) by choice of N^c_ε in Step 1.

$i \geq N_{\varepsilon/2}$ AND $j \geq N_{\varepsilon/2}$

(3) $d(p_i, p) < \frac{\varepsilon}{2}$ AND $d(p, p_j) < \frac{\varepsilon}{2}$ — (3) by Given $\forall i \geq N_\varepsilon$ $d(p_i, p) < \varepsilon$

(4) $d(p_i, p) + d(p, p_j) < \varepsilon$ — (4) $\frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\geq$ (final) $d(p_i, p_j) < \varepsilon$ — (final) Triangle Ineq $d(x, y) \leq d(x, z) + d(z, y)$

Justify downwards

QED

Justifications explain how each step follows from steps above it

Defn: A metric space, X, is complete if all Cauchy sequences converge in X

Example: $X = \mathbb{R} \setminus \{0\}$

x y

all real numbers except 0

$d(x,y) = |x-y|$

HW1 • Easy to check this is a metric space HW

X is not complete because $x_j = \frac{1}{j} \in \mathbb{R}$ does not converge in X

0 is not in X

HW2 • Prove: $x_j = \frac{1}{j}$ is a Cauchy sequence

Show $\forall \varepsilon > 0 \ \exists N_\varepsilon$ s.t. $\forall i,j \geq N_\varepsilon$ $\left|\frac{1}{i} - \frac{1}{j}\right| < \varepsilon$.

HW3 • Prove $x_j = \frac{1}{j}$ has no limit $x \in X$

Proof by contradiction

Assume on the contrary $x_j \to x \neq 0$

To complete HW1 you need to check the three rules of a metric space:

Defn

A metric space is a collection of points, X, with a distance $d : X \times X \to \mathbb{R}$

- $d(x,y) \geq 0$ $\forall x,y \in X$ and $d(x,y) = 0 \iff x = y$
- $d(x,y) = d(y,x)$ $\forall x,y \in X$
- $d(x,y) \leq d(x,z) + d(z,y)$ $\forall x,y,z \in X$

Everyone must set up the structure of the proof and start HW2 and HW3. You need to finish it if you are a math major. Ask me to check your proof and type CHECK THIS if you are a math major or would like me to look at it anyway.

Theorem: If $\{x_j \mid j \in \mathbb{N}\}$ is Cauchy and x_j has a converging subsequence $x_{j_k} \to x$ then x_j converges to the same limit, x.

Given: $\forall \varepsilon > 0\ \exists N^c_\varepsilon$ s.t. $\forall i,j \ge N^c_\varepsilon\ |x_i - x_j| < \varepsilon$

$\exists j_k$ s.t. $\forall \varepsilon > 0\ \exists N^s_\varepsilon$ s.t. $\forall k \ge N^s_\varepsilon\ |x_{j_k} - x| < \varepsilon$

Show $\forall \varepsilon > 0\ \exists N_\varepsilon$ s.t. $\forall i \ge N_\varepsilon\ |x_i - x| < \varepsilon$

$\boxed{\text{HW4}}$ Complete Proof.

$$|x_i - x| \le \underbrace{|x_i - x_{j_k}|}_{\text{Cauchy}} + \underbrace{|x_{j_k} - x|}_{\text{subseq conv}}$$

Thm: If $\{x_j \mid j \in \mathbb{N}\}$ is Cauchy then it is bounded: $\exists B$ s.t. $|x_j| \le B\ \forall j \in \mathbb{N}$.

Proof: ① Take $\varepsilon = 5 > 0$ $\exists N_5$

For all $i,j \ge N_5$ $|x_j - x_i| < \varepsilon$

② So Let $i = N_5$ and $y = x_i$ $\forall j \ge N_5\ |x_j - y| < 5$

So $y - 5 < x_j < y + 5$ $\forall j \ge N_5$

③ Choose $B = \max\{|x_1|, |x_2|, \dots |x_{N_5}|, y+5, 5-y\}$

$\boxed{\text{HW5}}$ Complete the proof.

Theorem: The Real Line defined using the Continuoum Hypothesis is Complete

Show: Every Cauchy Sequence Converges

Recall Continuoum Hypothesis:

If a set has an upper bound then it has a least upper upper bound called a sup.

So in particular a bounded seq $\{x_n \mid n \in \mathbb{N}\}$ has $s_n = \sup\{x_n, x_{n+1}, x_{n+2}, \ldots\} < \infty$

$s_n \geq s_{n+1}$ decreasing seq which is bounded below

So $\lim_{n\to\infty} s_n$ exists.

This limit is called $\limsup_{n\to\infty} x_n = L$

Can prove $\exists$ subseq $x_{j_k} \to L$

Proof of Thm:

1. Given a Cauchy Seq in $\mathbb{R}$
2. The sequence is bounded
3. A subseq conv to the limsup
4. Since the seq is Cauchy it then also converges to the same limit. QED

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Part II An Introduction to Series

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Watch the [Series-1to6 Playlist] and do HW1-HW4. HW4 has three proofs in it so it is as hard as three problems.

Series

$$\sum_{j=1}^{\infty} a_j = a_1 + a_2 + a_3 + \ldots \text{ forever}$$

we say the series diverges if there is no limit

Defn: $\displaystyle\sum_{j=1}^{\infty} a_j = \lim_{n\to\infty} \sum_{j=1}^{n} a_j = \lim_{n\to\infty} S_n$

Otherwise the series converges

$$S_n = \text{partial sum} = \sum_{j=1}^{n} a_j = a_1 + a_2 + \ldots + a_n$$

$$T_n = \text{tail} = \sum_{j=n+1}^{\infty} a_j = \lim_{N\to\infty} \sum_{j=n+1}^{N} a_j$$

$$\sum_{j=1}^{\infty} a_j = S_n + T_n$$

Examples of Diverging Series

$\boxed{\text{Ex 1}}$ $\displaystyle\sum_{j=1}^{\infty} 2^j = 2 + 2^2 + 2^3 + 2^4 + \dots = \lim_{n\to\infty} \sum_{j=1}^{n} 2^j$

This diverges because

$$\sum_{j=1}^{n} 2^j = 2 + 2^2 + \dots + 2^n$$

$$\geq \underbrace{1 + 1 + \dots + 1}_{n \text{ terms}} = n$$

$$\lim_{j\to\infty} \sum_{j=1}^{n} 2^j \geq \lim_{j\to\infty} n = \infty$$ diverges to infinity

$\boxed{\text{Ex 2}}$ Harmonic Series diverges

$$\sum_{j=1}^{\infty} \frac{1}{j} = \frac{1}{1} + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}} + \dots$$

$$\geq 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}} + \dots$$

$$\geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$$

can keep grouping terms

$$\frac{1}{3} + \frac{1}{4} \geq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \geq \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

$$\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} \geq \frac{8}{16} = \frac{1}{2}$$

Examples of Diverging Series

Ex 1 $\displaystyle\sum_{j=1}^{\infty} 2^j = 2 + 2^2 + 2^3 + 2^4 + \dots = \lim_{n\to\infty} \sum_{j=1}^{n} 2^j$

This diverges because

$$\sum_{j=1}^{n} 2^j = 2 + 2^2 + \dots + 2^n$$
$$\geq \underbrace{1 + 1 + \dots + 1}_{n \text{ terms}} = n$$

$$\lim_{j\to\infty} \sum_{j=1}^{n} 2^j \geq \lim_{j\to\infty} n = \infty$$ diverges to infinity

Ex 2 Harmonic Series diverges

$$\sum_{j=1}^{\infty} \frac{1}{j} = \frac{1}{1} + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}} + \dots$$
$$\geq 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}} + \dots$$
$$\geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$$

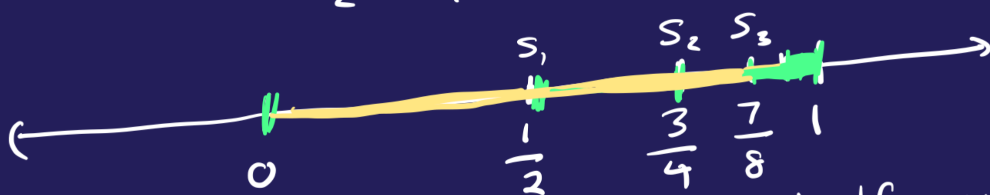
can keep grouping terms

$$\frac{1}{3} + \frac{1}{4} \geq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$
$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \geq \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$
$$\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} \geq \frac{8}{16} = \frac{1}{2}$$

Example: $\sum_{j=1}^{\infty}\left(\frac{1}{2}\right)^j$ converges

$$\sum_{j=1}^{\infty}\left(\frac{1}{2}\right)^j = \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \dots$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$



each step gets halfway closer to 1

S_n are bounded above by 1
and S_n are increasing

Monotone Convergence Thm $\Rightarrow \lim_{n\to\infty} S_n$ exists.

Claim $\sum_{j=1}^{\infty}\left(\frac{1}{2}\right)^j = 1$ $\quad \lim_{n\to\infty} \sum_{j=1}^{n}\left(\frac{1}{2}\right)^j = 1$

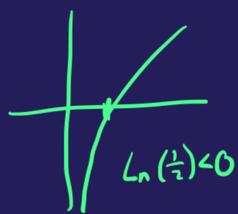
Show: $\forall \varepsilon > 0\ \exists N_\varepsilon$ s.t. $\forall n \geq N_\varepsilon\ |S_n - 1| < \varepsilon$

Proof: ① Given any $\varepsilon > 0$

Choose $N_\varepsilon = \boxed{\frac{\ln(\varepsilon)}{\ln(1/2)} + 1}$

② Whenever $n \ge N_\varepsilon$

Skip Justifications



$$n > \frac{\ln(\varepsilon)}{\ln(1/2)}$$

$$n \ln(\tfrac{1}{2}) < \ln \varepsilon$$

$$\ln(\tfrac{1}{2})^n < \ln(\varepsilon)$$

$$(\tfrac{1}{2})^n < \varepsilon$$

$$\left|\sum_{j=1}^{n} (\tfrac{1}{2})^j - 1\right| < \varepsilon$$

} tricky.

final $|S_n - 1| < \varepsilon$

Need to rigorously prove:

$$\forall n \in \mathbb{N} \quad \sum_{j=1}^{n} \left(\tfrac{1}{2}\right)^j = 1 - \left(\tfrac{1}{2}\right)^n$$

Use <u>proof by induction</u>

Base Case: $n=1$ $\quad \sum_{j=1}^{1} \left(\tfrac{1}{2}\right)^j \overset{?}{=} 1 - \left(\tfrac{1}{2}\right)^1$

$$\sum_{j=1}^{1} \left(\tfrac{1}{2}\right)^j = \left(\tfrac{1}{2}\right)^1 = \tfrac{1}{2}$$

$$1 - \left(\tfrac{1}{2}\right)^1 = 1 - \tfrac{1}{2} = \tfrac{1}{2}$$

Match so base case is proven.

Induction Step:

Assume true for n: $\sum_{j=1}^{n} \left(\tfrac{1}{2}\right)^j = 1 - \left(\tfrac{1}{2}\right)^n$ ← Induction Hypothesis

Show true for $n+1$: $\sum_{j=1}^{n+1} \left(\tfrac{1}{2}\right)^j = 1 - \left(\tfrac{1}{2}\right)^{n+1}$

$$\underset{\text{base}}{1} \xrightarrow{\text{step}} 2 \xrightarrow{\text{step}} 3 \xrightarrow{\text{step}} 4 \frown \quad \frown \frown \frown \; n$$

True $\forall n \in \mathbb{N}$

① $\sum_{j=1}^{n+1} \left(\tfrac{1}{2}\right)^j =$

$=$ } somewhere we must use the induction hypothesis

final $= 1 - \left(\tfrac{1}{2}\right)^{n+1}$

(3) $$= 1 - \left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^{n+1}$$ (3) Induction Hypothesis

(4) $$= 1 - \frac{2}{2^{n+1}} + \frac{1}{2^{n+1}}$$ (4) by laws of fractions

(5) $$= 1 + \frac{-2+1}{2^{n+1}}$$ (5) adding fractions

(6) $$= 1 + \frac{-1}{2^{n+1}}$$ (6) $-2+1=-1$

(final) $$= 1 - \left(\frac{1}{2}\right)^{n+1}$$ (7) $\left(\frac{1}{a}\right)^k = \frac{1}{a^k}$

QED

$\boxed{HW1}$ Write a proof by Induction that $\sum_{j=0}^{n} \left(\frac{1}{4}\right)^j = \frac{1}{3}\left(4 - \left(\frac{1}{4}\right)^n\right)$ Base Case here is $n=0$

$\boxed{HW2}$ Prove that $\sum_{j=0}^{\infty} \left(\frac{1}{4}\right)^j = \frac{4}{3}$ using $\varepsilon - N_\varepsilon$ and also use HW1

Theorem: if $\sum_{j=1}^{\infty} a_j$ converges then $\lim_{j\to\infty} a_j = 0$.

Proof:

① $\sum_{j=1}^{\infty} a_j$ converges — ① Given

② $\{S_n = \sum_{j=1}^{n} a_j \mid n \in \mathbb{N}\}$ is Cauchy — ② Converges $\Rightarrow$ Cauchy

③ $\forall \varepsilon > 0 \ \exists N_\varepsilon^c$ s.t $\forall n, m \geq N_\varepsilon^c \ |S_n - S_m| < \varepsilon$ — ③ defn of Cauchy

④ $|S_{m+1} - S_m| < \varepsilon$ — ④ Taking $n = m+1$

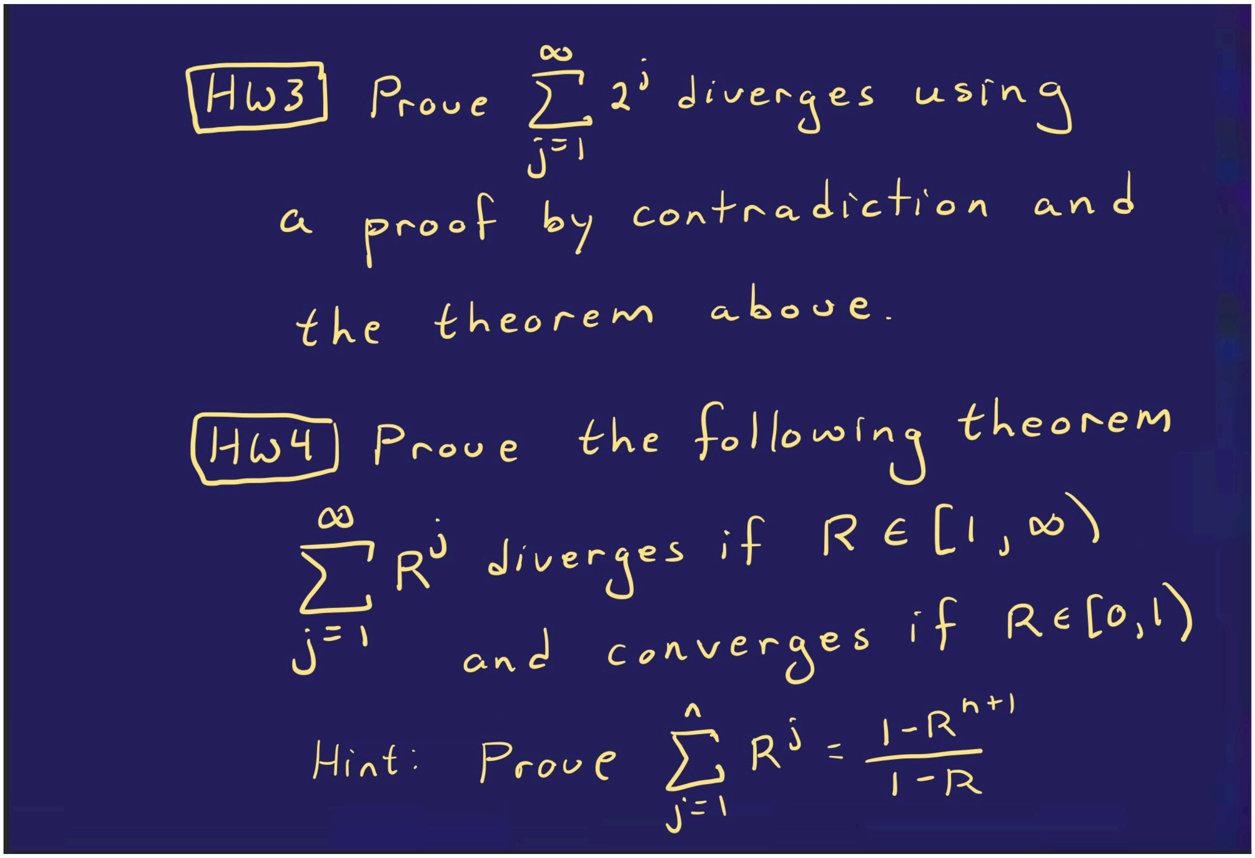
⑤ $|a_{m+1}| < \varepsilon$ — ⑤ $\cancel{a_1} + \cancel{a_2} + \ldots + \cancel{a_m} + a_{m+1} - (\cancel{a_1} + \cancel{a_2} + \ldots + \cancel{a_m}) = a_{m+1}$

⑥ $\forall \varepsilon > 0 \ \exists N_\varepsilon^c$ s.t $\forall m \geq N_\varepsilon^c \ |a_{m+1}| < \varepsilon$ — ⑥ by steps 3-5

⑦ $\forall \varepsilon > 0$ Choose $N_\varepsilon = N_\varepsilon^c + 1$ so $\forall j \geq N_\varepsilon \ |a_j| < \varepsilon$ — ⑦ $j = m+1$, $j \geq N_\varepsilon^c + 1$

⑧ $\lim_{j \to \infty} a_j = 0$ — ⑧ Defn of limit

QED

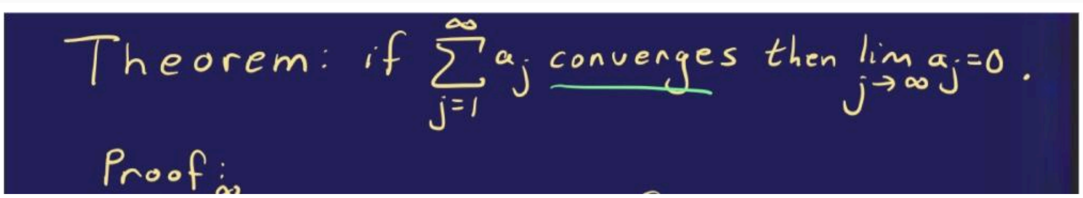


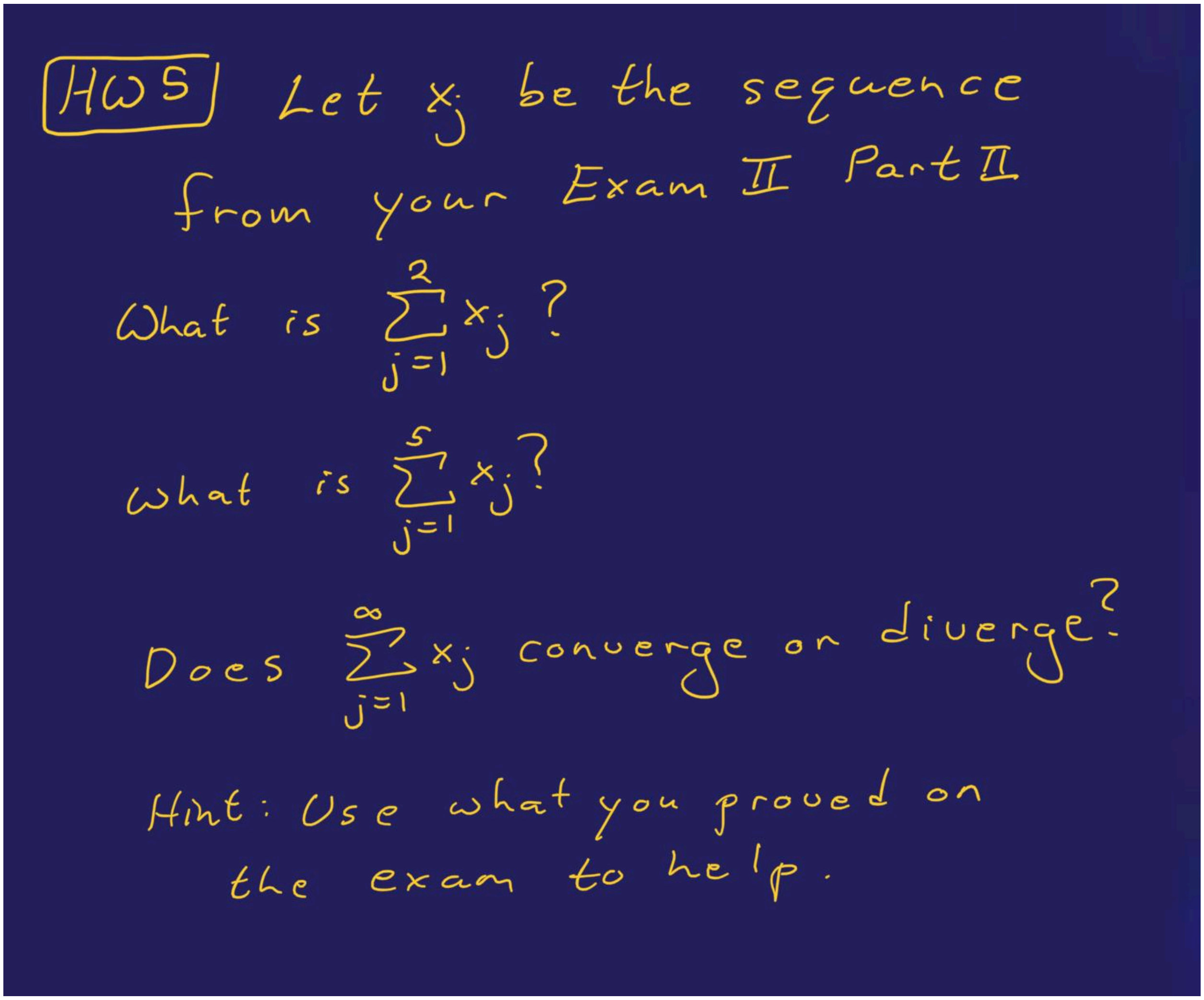
HW4 has three parts:

Part I an induction proof of the formula for the sum in the hint.

Part II a proof of convergence using that formula when R in [0,1). Try showing it converges to L=1/(1-R). Be sure to solve up for j to find the right choice of N. The formula for N will depend on R. You will use R in [0,1) to justify the choice of N is good.

Part III is a proof that the sum diverges. You can use proof by contradiction and this theorem to help:





Before you submit your lesson, see the [solution hints]() and fix your work following those hints.

Also remember a selfie holding one page of homework is required in every lesson.