Tools of counting from Module 4 - Revisited

Counting Unions

$|A \cup B| = |A| + |B| - |A \cap B|$

Counting Cartesian Products

$|A \times B| = |A| * |B|$

Counting Complements (seen briefly in a recitation problem)

$|\overline{A}| = |U| - |A|$

Product, Sum, Subtraction Rules

Product Rule: When a problem asks to make one choice *and then* another, we can multiply the counts of the first choice and the second choice. This is similar to how we count the size of Cartesian Products.

Sum Rule: When a problem asks us to make one choice *or* another *and both cannot happen at the same time*, we can add the counts of each choice together. This is a special case of how we count the size of the union of two sets. In this case, the intersection of both choices is empty.

Subtraction Rule: When a problem asks us to make a lot of choices, it may be easier to count all possible choices and then subtract the invalid choices. This is similar to how we count the size of the complement of a set.

New Tools of counting from Module 5

How many ways are there to choose k things from a set of n elements?

	Repetition not allowed	Repetition allowed
Order matters	$P(n, k)$	n^k
Order does not matter	$\binom{n}{k}$	$\binom{k+n-1}{k}$

Order matters but repetitions are not allowed: **Permutations**

We are choosing from n elements, but the order we choose them matters. That is, choosing A first then B is *not* the same as choosing B first then A.

$$P(n, k) = \frac{n!}{(n-k)!}$$

Order does not matter and repetitions not allowed: **Combinations**

We are choosing from n elements, but the order we choose them does *not* matter. That is, choosing A first then B is the same as choosing B first then A.

So we can start by counting the permutations of these elements, giving us groupings of size k. However, since the order doesn't matter, we have to remove the equivalent permutations. For a group of size k, there are $k!$ possible orderings. That gives us the equation below.

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Order matters and repetitions are allowed: **Product Rule/Cartesian Product**

We are choosing from n elements k times, but the order we choose them matters. We can think of this like a Cartesian product.

$$n^k$$

Order does not matter but repetitions are allowed: **Balls and bins**

We are trying to place k items into n bins. The order we choose to place the balls into the bins does not matter. It does not matter if we place 3 balls in bin 2 first and then 2 balls in bin 1 or vice-versa.

When we choose this technique, we need to identify the balls and the bins in a problem. We have $k$balls and n bins to put them in.

$$\binom{k+n-1}{k}$$

**Caveat on balls and bins: It matters that the balls are *indistinguishable* and the bins are *distinguishable*.

Indistinguishable balls: It does not matter *which* ball goes into a bin. All that matters is how many balls went into a bin.

Distinguishable bins: It does matter *which* bin the number of balls goes into. That is, placing 3 balls in bin A and none in bin B is different from placing 3 balls in bin B and none in bin A.