Puzzling Pairs!

By Serenity Allen

Introduction (Objective)

This game is inspired by Langford pairs, which were introduced by chemist and mathematician C. Dudley Langford in 1958. Langford pairings are a way of rearranging a set of N pairs, $\{1, 1, 2, 2, …, N, N\}$, such that every pair, $\{k, k\}$, in the set appears in the sequence with k numbers between the pair. The purpose of this activity is to solve the Langford pairing puzzle for different values of N. We will investigate the puzzle using a few simple cases, leading to a justification for why solving the puzzle with certain values of N is impossible.

Materials

- [Puzzling Pairs! Slides](#)
- Unifix Cubes
- White Board (if white boards are unavailable, paper will be just fine)
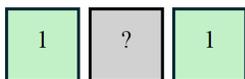- Dry Erase Markers (writing utensil)

Game Instructions (Procedure)

We are given the following pairs of numbers:

1	1	2	2	3	3
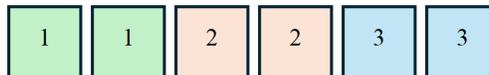
With each pair there is a constraint that must be maintained:

The pair of | 1 | will have **ONE** space between them.

1	?	1

The pair of [2] will have **TWO** spaces between them.

| 2 | ? | ? | 2 |

The pair of [3] will have **THREE** spaces between them.

| 3 | ? | ? | ? | 3 |

In general, each set of {k, k} pairs in the sequence will have k numbers between them.

Main Puzzle to Solve

For which values of N is a solution to this puzzle possible?

Exploration

We suggest using Unifix cubes to model the rules of the game with $N = 3$. Once participants are comfortable with the game rules using the colored blocks, show how the puzzle can be represented with a set of numbers.

Working in groups of 2-3 people, encourage participants to find a solution to the puzzle, using blocks and/or numbers, for $N = 3$ and $N = 4$. Some participants may want to work on the puzzle alone; however, to ensure collaboration, encourage participants to communicate their strategies for problem solving, and briefly discuss any mathematical questions that arise: *How many solutions are there? Will this be the case for every set of N pairs? Will this puzzle always have a solution for any given set of N pairs?*

Once participants have agreed upon a solution for $N = 3$ and $N = 4$, instruct them to attempt the puzzle with $N = 5$. *Although solving the puzzle for $N = 5$ and $N = 6$ is impossible, we suggest not telling the groups until they have had an appropriate amount of productive struggle. The goals for this part of the activity are to encourage communication through the conjecture*

and application of strategies for problem solving, and to develop understanding of the necessary

*conditions for the puzzle to have a solution.*

Continue to explore the puzzle with $N = 7$ encouraging groups to apply previously

conjectured strategies. A solution to this puzzle is possible, and should arise within a few

minutes. Groups should compare solutions to the puzzle. Once several people have found a

solution, have them compare to determine if their solution is unique. *Hopefully, participants will*

discover different solutions to the puzzle. There are 26 solutions when N = 7.

Method of Solution

As groups work to solve the puzzle for different values of N, encourage them to

communicate with each other and explain their problem solving techniques. We will explore one

such problem solving technique below. When solving the puzzle of any given N, we use a

systematic approach starting with the largest number in the set, as there are a limited number of

spots that those numbers can be placed within the sequence.

<u>**Case 1**</u>: $N = 3$: {1, 1, 2, 2, 3, 3}

For this case, the numbers are arranged among six spots, as there are three pairs of two

numbers.
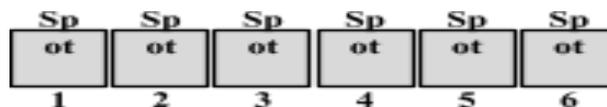
There are a limited number of spots where [3] could be placed. In particular, [3] could

either be placed in Spot 1 and Spot 5: [3] [Spot 2] [Spot 3] [Spot 4] [3] [Spot 6] (Variation 1),

or it could be placed in Spot 2 and Spot 6: [Spot 1] [3] [Spot 3] [Spot 4] [Spot 5] [3] (Variation 2).

Starting with Variation 1, [3] [Spot 2] [Spot 3] [Spot 4] [3] [Spot 6] , we can now place

[1] and [2] keeping in mind the conditions. The value, [2] can **NOT** be placed in Spot

2, since its pair would be in Spot 5, which has already been taken by [3]. In addition, [2]

can **NOT** be placed in Spot 4, as its pair would be placed in Spot 1, which has been taken by

[3] as well. Hence, the only spots that [2] can be placed in is Spot 3 and Spot 6:

[3] [Spot 2] [2] [Spot 4] [3] [2] . The remaining two places are Spot 2 and Spot 4, and

placing a [1] in these locations, maintains the conditions. The final solution is the following:

[3] [1] [2] [1] [3] [2] .

Similarly, this process of arranging $N = 3$: {1, 1, 2, 2, 3, 3} can be done with Variation 2:

[Spot 1] [3] [Spot 3] [Spot 4] [Spot 5] [3] . The final solution of Variation 2, and that follows all

conditions, is the following: [2] [3] [1] [2] [1] [3] .

Although there are two variations to solve the puzzle for $N = 3$: {1, 1, 2, 2, 3, 3} there is

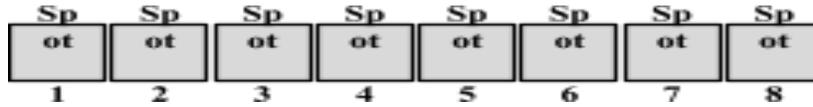only one unique solution since Variation 1, [3] [1] [2] [1] [3] [2] , is the

reflection (reversal) of Variation 2, [2] [3] [1] [2] [1] [3] .

Case 2: $N = 4$: $\{1, 1, 2, 2, 3, 3, 4, 4\}$

Sp ot	Sp ot	Sp ot	Sp ot	Sp ot	Sp ot	Sp ot	Sp ot
1	2	3	4	5	6	7	8

To find a solution, use a similar strategy to the set $N = 3$: $\{1, 1, 2, 2, 3, 3\}$. Starting with the largest number in the set, we have the following variations:

- Variation 1: The value [4] can be placed in Spot 1 and Spot 6. The value [3] can then be placed in Spot 3 and Spot 7 or in Spot 4 and Spot 8. If you try placing a [3] in Spot 4 and Spot 8, the values of [1] and [2] cannot be placed in the sequence following the rules of the puzzle. Thus, a [3] must be placed in Spot 3 and Spot 7. When completed, Variation 1 will have the solution:

[4] [1] [3] [1] [2] [4] [3] [2].

- Variation 2: The value [4] can be placed in Spot 2 and Spot 7. The value [3] can then be placed in Spot 1 and Spot 5 or in Spot 4 and Spot 8. If you try placing a [3] in Spot 1 and Spot 5 or in Spot 4 and Spot 8, the values of [1] and [2] cannot be placed in the sequence following the rules of the puzzle. Thus, Variation 2 does not lead to a solution.

- Variation 3: The value [4] can be placed in Spot 3 and Spot 8. This would yield the same solution as Variation 1, in reverse.

When completed, $N = 4$: {1, 1, 2, 2, 3, 3, 4, 4} has one unique solution:

[4] [1] [3] [1] [2] [4] [3] [2] OR the reversal

[2] [3] [4] [2] [1] [3] [1] [4] .
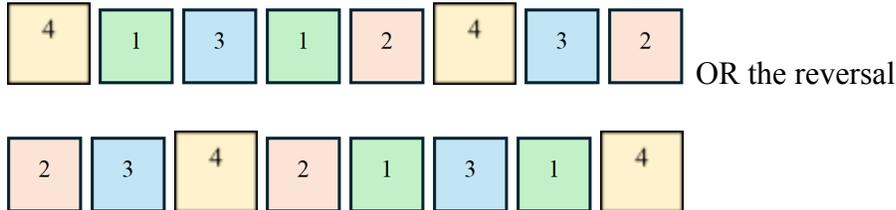
Justification

The solution to the main question, "For which values of N is a solution to this puzzle possible?", also comes down to pairs! To investigate this further, let's start with the solution for $N = 3$: {1, 1, 2, 2, 3, 3}.

- Label the number of positions; for this example we would have six positions:
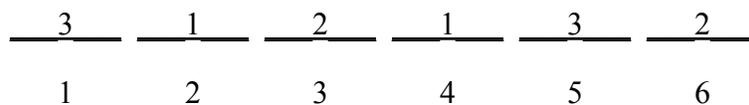
  _____ _____ _____ _____ _____ _____
   1   2   3   4   5   6

- Fill in the solution:

  __3__ __1__ __2__ __1__ __3__ __2__
   1   2   3   4   5   6

- Sort the numbers in the solution by the value of their corresponding position:

  Even Positions: 1, 1, 2
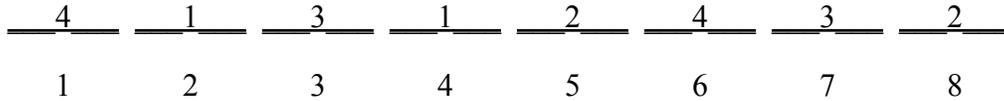
  Odd Positions: 3, 2, 3

Now let's explore the solution for $N = 4$: {1, 1, 2, 2, 3, 3, 4, 4}.

- Label the number of positions; for this example we would have eight positions:

$$\underline{\hspace{1cm}} \ \underline{\hspace{1cm}} \ \underline{\hspace{1cm}} \ \underline{\hspace{1cm}} \ \underline{\hspace{1cm}} \ \underline{\hspace{1cm}} \ \underline{\hspace{1cm}} \ \underline{\hspace{1cm}}$$

  $$1 \qquad 2 \qquad 3 \qquad 4 \qquad 5 \qquad 6 \qquad 7 \qquad 8$$

- Fill in the solution:

$$\underline{\ 4\ } \ \underline{\ 1\ } \ \underline{\ 3\ } \ \underline{\ 1\ } \ \underline{\ 2\ } \ \underline{\ 4\ } \ \underline{\ 3\ } \ \underline{\ 2\ }$$

  $$1 \qquad 2 \qquad 3 \qquad 4 \qquad 5 \qquad 6 \qquad 7 \qquad 8$$

- Sort the numbers in the solution by the value of their corresponding position:

  Even Positions: 1, 1, 4, 2

  Odd Positions: 4, 3, 2, 3

When k is even, one value of k is in an even position and the other value of k is in an odd position. When k is odd, both values of k are either in an even position or both values of k are in an odd position. Since odd values of k come in pairs, the puzzle has a solution if and only if there is an even number of odd pairs in the set: $\{1, 1, 2, 2, \ldots, N, N\}$.

Application

Once participants have seen the justification for Langford's Problem, ask them to solve the puzzle for $N = 8$. Help participants realize that even and odd number pairs can only be placed in certain spots. This observation should (hopefully) lead to a quicker solution, and offers a nice application of the justification to the puzzle. *In our experience, participants used this method to find solutions up to N=11.*

Conclusion (Teacher Reflection)

This guide offers one justification for a solution to this puzzle. When mathematicians from a wide variety of backgrounds tackle this puzzle together, it is possible for multiple interesting approaches to emerge. For example, an algebraist may notice that the values of N for

which a solution is possible are the values that are congruent to -1 (mod 4) or 0 (mod 4).

Someone who likes pictorial representations, may start to draw interesting pictures using colors

or lines to connect the pairs. The exploration offered in this guide follows the outline of the

Puzzling Pairs! Slides; however, we encourage you to explore other fun extensions that Langford

pairs can provide.

Miscellaneous/ (Extensions & Connections)

- Langford Quilts

- Planar Solutions

- Modular Arithmetic

- Known Solutions

  - Open Question: no one knows the number of unique solutions for $N \geq$ 30!