[Linear Algebra MAT313 Spring 2024]
[Professor Sormani]
Lesson 23
Eigenspaces and the Characteristic Polynomial

Please be sure to mark down the date and time that you start this lesson. Carefully take notes on pencil and paper while watching the lesson videos. Pause the lesson to try classwork before watching the video going over that classwork. If you work with any classmates, be sure to write their names on the problems you completed together.

You will cut and paste the photos of your notes and completed classwork and a selfie taken holding up the first page of your work in a googledoc entitled:

MAT313S24-lesson23-lastname-firstname

and share editing of that document with me sormanic@gmail.com and with our graders. If you have a question, type* QUESTION *in your googledoc next to the point in your notes that has a question and email me with the subject* MAT313 QUESTION. *I will answer your question by inserting a photo into your googledoc or making an extra video.

Before starting, check if your Lesson 17 and Lesson 18 are both complete and corrected. Finish fixing these lessons if they are not yet fixed and email me to check them. You will be using determinants in this lesson.

This lesson has two parts: (take a break between parts!)

Part I Finding eigenvalues using the characteristic polynomial
- **Review the definition of eigenvalue and eigenvector**
- **Learn how Determinants can be used to find eigenvalues of matrices.**
- **Review 2x2, 3x3, and nxn determinants without row reduction**

Part II Finding eigenspaces and eigenvectors for each eigenvalue
- **Each eigenvalue has many eigenvectors which we will see can be found by solving a homogeneous system.**
- **The collection of eigenvectors in the solution set is called the eigenspace for that eigenvalue.**

Part I Finding eigenvalues using the characteristic polynomial
Watch [Playlist 313F22-23-1-1to9]

Today's Topics

Eigenvalues, Eigenvectors

Characteristic Polynomials

Eigenspaces

Brief Overview and Review

Defn: Given a square matrix A we say that λ (lambda) is an eigenvalue of A with eigenvector $\vec{v}$ if

$$A\vec{v} = \lambda\vec{v}$$

where $\vec{v} \neq 0$ but $\lambda \in \mathbb{R}$ (even $\mathbb{C}$)
real (or complex)

Notice:

$$A\vec{v} - \lambda\vec{v} = \vec{0}$$

$$A\vec{v} - \lambda I\vec{v} = \vec{0} \qquad \text{because } I\vec{v} = \vec{v}$$

$$(A - \lambda I)\vec{v} = \vec{0}$$

Theorem: λ is an eigenvalue for A with eigenvector $\vec{v} \neq \vec{0}$ if $(A - \lambda I)\vec{v} = \vec{0}$

Given an eigenvalue λ then find the eigenvectors by solving the homogeneous system $\left([A - \lambda I] \middle| \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \right)$

for null space $\{ \vec{v} = \dots\dots \mid \dots\dots \}$

The solution set is called the eigenspace for λ.

All the vectors in the eigenspace except for $\vec{v} = \vec{0}$ are eigenvectors for λ.

What do we do if we do not know the eigenvalues for A?

By the theorem above:

λ is an eigenvalue for A iff $\left([A - \lambda I] \middle| \vec{0} \right)$ has a solution set with nonzero vectors.

This happens when $A - \lambda I$ is singular.

This happens when

$$\det(A - \lambda I) = 0$$

This is actually a polynomial with λ as the variable

Called the characteristic polynomial.

Review determinants

Review eigenvalues + eigenvectors

Prove the theorem

Practice find Characteristic Polynomials and Solving them for λ.

Solving for eigenspaces.

Review Determinants

Review Eigenvalues and Eigenvectors

Prove the Theorem above

Find Characteristic Polynomals

Factor and Solve for eigenvalues

Solve $[A-\lambda I \mid \vec{0}]$ to find eigenspaces

Check our eigenvectors $A\vec{v} = \lambda\vec{v}$

Review Determinants

$\det(A-\lambda I)$ is a determinant with variables λ inside.

For example $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

$$\det(A - \lambda I) = \det\left(\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$= \det\left(\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} (1-\lambda) & (2-0) \\ (2-0) & (4-\lambda) \end{pmatrix}$$

λ is a unknown variable

Recall 2×2 det

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Characteristic Polynomial

$$= (1-\lambda)(4-\lambda) - (2-0)(2-0)$$

$$= 1 \cdot 4 - \lambda 4 - 1\lambda + \lambda^2 - 4$$

$$= \lambda^2 - 5\lambda + 0$$

Characteristic Polynomial.

Recall that λ is an eigenvalue for $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

if $(I - \lambda A \mid \vec{0})$ has nonzero solutions

which happens when $\det(I - \lambda A) = 0$

Thus the solutions to $\lambda^2 - 5\lambda + 0 = 0$

We factor to solve for eigenvalues λ

$\lambda(\lambda - 5) = 0$ So $\lambda = 0$ or $\lambda = 5$

Classwork

Find the characteristic polynomial of $A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix}$

and solve for the eigenvalues of A.

Hint 3×3 det trick:

Hint: $A - \lambda I$ is

Classwork

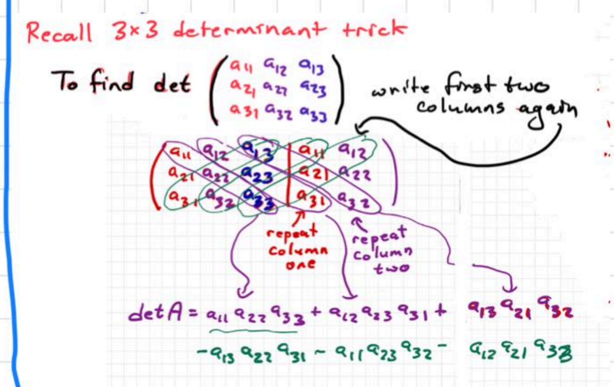
$\det(A-\lambda I)$

Find the characteristic polynomial of $A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix}$

and solve for the eigenvalues of A.

solve $\det(A-\lambda I) = 0$ by factoring

Hint 3×3 det trick:



Hint: $A - \lambda I$ is

$$\begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (4-\lambda) & 1 & 1 \\ 1 & (4-\lambda) & 1 \\ 1 & 1 & (4-\lambda) \end{pmatrix}$$

Be sure to include parentheses

$$\det(A - \lambda I) = \det\begin{pmatrix} (4-\lambda) & 1 & 1 \\ 1 & (4-\lambda) & 1 \\ 1 & 1 & (4-\lambda) \end{pmatrix}$$

scratch work

$$\begin{array}{cccccc} & & & & \text{first} & \text{second} \\ \end{array}$$

$$\left(\begin{array}{ccc|cc} (4-\lambda) & 1 & 1 & (4-\lambda) & 1 \\ 1 & (4-\lambda) & 1 & 1 & (4-\lambda) \\ 1 & 1 & (4-\lambda) & 1 & 1 \end{array}\right)$$

scratch work

$$\left(\begin{array}{ccc|cc} (4-\lambda) & 1 & 1 & (4-\lambda) & 1 \\ 1 & (4-\lambda) & 1 & 1 & (4-\lambda) \\ 1 & 1 & (4-\lambda) & 1 & 1 \end{array}\right)$$

keep parentheses

$$\det(A - \lambda I) = (4-\lambda)(4-\lambda)(4-\lambda) + 1\cdot 1\cdot 1 + 1\cdot 1\cdot 1$$

$$- 1(4-\lambda)\cdot 1 - (4-\lambda)\cdot 1\cdot 1 - 1\cdot 1\cdot(4-\lambda)$$

The characteristic polynomial

$$= (16 - 8\lambda + \lambda^2)(4-\lambda) + 2$$

$$- 4 + \lambda - 4 + \lambda - 4 + \lambda$$

multiply it all out

$$= 16\cdot 4 - 8\lambda 4 + \lambda^2\cdot 4 - 16\lambda + 8\lambda^2 - \lambda^3 + 2$$

$$- 12 + 3\lambda$$

$-32 - 16 + 3$
$= -45$

combine common terms

$$= -\lambda^3 + 12\lambda^2 - 45\lambda + 54$$

$4 + 8 = 12$

$16\cdot 4 + 2 - 12$
$= 64 + 2 - 12 = 54$

How can we solve $\det(A-\lambda I)=0$?!!

Think about why we are solving it.

Want $A-\lambda I$ to be singular.

Look at $A-\lambda I = \begin{pmatrix} (4-\lambda) & 1 & 1 \\ 1 & (4-\lambda) & 1 \\ 1 & 1 & (4-\lambda) \end{pmatrix}$

This is singular if row reduction gives a row of zeroes.

This happens if two rows are the same.

Notice two rows are the same if $(4-\lambda)=1$

$3=\lambda$

If $\lambda=3$ $\quad A-3I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ which is singular!

Its det is $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 1\cdot1\cdot1 + 1\cdot1\cdot1 + 1\cdot1\cdot1 - 1\cdot1\cdot1 - 1\cdot1\cdot1 - 1\cdot1\cdot1 = 0$

This means $(\lambda-3)$ is a factor of $\det(A-\lambda I)$.

LONG DIVISION OF POLYNOMIALS

$$
\begin{array}{r|l}
 & -\lambda^2 + 9\lambda - 18 \\
\hline
\lambda - 3 & -\lambda^3 + 12\lambda^2 - 45\lambda + 54 \\
-\lambda^2(\lambda-3) & \underline{-\lambda^3 + 3\lambda^2} \\
 & 9\lambda^2 - 45\lambda \\
9\lambda(\lambda-3) & \underline{9\lambda^2 - 27\lambda} \\
 & -18\lambda + 54 \\
-18(\lambda-3) & \underline{-18\lambda + 54} \\
 & 0
\end{array}
$$

Thus $-\lambda^3 + 12\lambda^2 - 45\lambda + 54$

$$= (\lambda-3)(-\lambda^2 + 9\lambda - 18)$$

$$= -(\lambda-3)(\lambda^2-9\lambda+18)$$

$$= -(\lambda-3)(\lambda+a)(\lambda+b)$$

$ab = 18$

$\lambda a + \lambda b = -9\lambda$

$\lambda^2 = \lambda^2$

6×3

$a = -6 \quad b = -3$

$-6\lambda - 3\lambda = -9\lambda$ ✓

$(-6)(-3) = 18$ ✓

$$= -\underbrace{(\lambda-3)}_{3}\underbrace{(\lambda-6)}_{2}\underbrace{(\lambda-3)}_{2}$$

The eigenvalues are $\lambda = 3$

$\lambda = 6$

Classwork:

Use $\det(A-\lambda I)=0$ to find the eigenvalues of A where:

$$A=\begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix}$$

4x4 matrix

To find det of an nxn matrix

Method of Minors

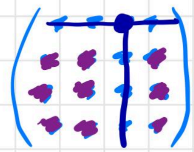
$$\det\begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix}$$

choose a row with some zeroes

$$\begin{matrix}+&-&+&-\\-&+&-&+\\+&-&+&-\\-&+&-&+\end{matrix}$$

← we chose first row

$$= +a_{11}\det A_{11} - a_{12}\det A_{12} + a_{13}\det A_{13} - a_{14}\det A_{14}$$



where A_{ij} is the minor for a_{ij}: found by crossing out row i column j

$$\det\left(A-\lambda I\right)=\det\left(\begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix}-\lambda\begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & \end{pmatrix}\right)$$

↑ Id matrix

$$\det\left(A-\lambda I\right)=\det\left(\begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix}-\lambda\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right)$$

↑ Id matrix

Identity Matrix 4x4

$$\det\left(A-\lambda I\right)=\det\left(\begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix}-\lambda\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right)$$

↑ Id matrix

Identity Matrix 4x4

$$=\det\begin{pmatrix}(1-\lambda)&0&1&1\\0&(1-\lambda)&1&1\\1&1&(1-\lambda)&0\\1&1&0&(1-\lambda)\end{pmatrix}$$

← notice this is just the matrix A with $-\lambda$ on diagonals

$$\det(A-\lambda I) = \det\left(\begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix} - \lambda\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right)$$

Identity Matrix 4x4

↑ Id matrix

$$= \det\begin{pmatrix}(1-\lambda)&0&1&1\\0&(1-\lambda)&1&1\\1&1&(1-\lambda)&0\\1&1&0&(1-\lambda)\end{pmatrix}$$

← notice this is just the matrix A with $-\lambda$ on diagonals

Take the det using the method of minors

$$\begin{matrix}+&-&+&-\\-&+&-&+\\+&-&+&-\\-&+&-&+\end{matrix}$$

$$= +(1-\lambda)\det A_{11} - 0 \det A_{12} + 1 \det A_{13} - 1 \det A_{14}$$

How to find the minors:

$$\begin{pmatrix}(1-\lambda)&0&1&1\\0&(1-\lambda)&1&1\\1&1&(1-\lambda)&0\\1&1&0&(1-\lambda)\end{pmatrix}$$

$$= (1-\lambda)\det\begin{pmatrix}(1-\lambda)&1&1\\1&(1-\lambda)&0\\1&0&(1-\lambda)\end{pmatrix}$$

$$-0\det\begin{pmatrix}0&1&1\\1&(1-\lambda)&0\\1&0&(1-\lambda)\end{pmatrix}$$

$$+1\det\begin{pmatrix}0&(1-\lambda)&1\\1&1&0\\1&1&(1-\lambda)\end{pmatrix}$$

$$-1\det\begin{pmatrix}0&(1-\lambda)&1\\1&1&(1-\lambda)\\1&1&0\end{pmatrix}$$

To find each of these 3x3 det use 3x3 trick

To find each of these 3x3 det use 3x3 trick

$$= (1-\lambda)\begin{pmatrix} (1-\lambda)(1-\lambda)(1-\lambda) + 1\cdot 0\cdot 1 + 1\cdot 1\cdot 0 \\ -1(1-\lambda)1 - (1-\lambda)\cdot 0\cdot 0 - 1\cdot 1\cdot(1-\lambda) \end{pmatrix} - 0$$

parentheses around the whole det

a real number

$$\left(\begin{array}{ccc|cc} (1-\lambda) & 1 & 1 & (1-\lambda) & 1 \\ 1 & (1-\lambda) & 0 & 1 & (1-\lambda) \\ 1 & 0 & (1-\lambda) & 1 & 0 \end{array}\right)$$

$$+1\begin{pmatrix} 0 + 0 + 1 \\ -1 - 0 - (1-\lambda)1(1-\lambda) \end{pmatrix} - 1\begin{pmatrix} 0 + (1-\lambda)^2 + 1 \\ -1 - 0 - 0 \end{pmatrix}$$

Simplify the Polynomial:

Simplify the Polynomial:

$$= (1-\lambda)\left((1-\lambda)^3 - 2(1-\lambda)\right) - 0$$
$$+ 1\left(-(1-\lambda)^2\right) - 1\left((1-\lambda)^2\right)$$
$$= (1-\lambda)^4 - 2(1-\lambda)^2 - (1-\lambda)^2 - (1-\lambda)^2$$
$$= (1-\lambda)^4 - 4(1-\lambda)^2$$
$$= \left((1-\lambda)^2 - 4\right)(1-\lambda)^2$$
$$= (1 - 2\lambda + \lambda^2 - 4)(1-\lambda)^2$$
$$= (\lambda^2 - 2\lambda - 3)(1-\lambda)^2$$
$$= (\lambda-3)(\lambda+1)(1-\lambda)^2 = 0$$

be very careful with parentheses!

So $\lambda = 3 \quad \lambda = -1 \quad \lambda = 1$

are the eigenvalues of $A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$

Next Part of Lesson

find eigenvectors and eigenspaces.

Hint: If you cannot factor ax^2+bx+c use the following theorem:

THEOREM: "Quadratic Formula" The roots of $ax^2+bx+c=0$ are

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$\boxed{HW1}$ Find the eigenvalues

$$B = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

$\boxed{HW2}$ Find the evalues

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}.$$

$\boxed{HW3}$ Find the evalues

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Do Part I HW1-HW3 above before Part II.
Be sure to check your eigenvalues.

Part II Finding eigenspaces and eigenvectors for each eigenvalue

You may wish to do this on a different day.
Watch [Playlist 313F22-23-2-1to3]

Defn: Given an $n \times n$ matrix A

A real or complex number λ is an **eigenvalue** of A if there is a nonzero vector $\vec{v}$ such that

$$A\vec{v} = \lambda\vec{v}$$

Any such vector $\vec{v}$ is an **eigenvector**.

The set of all such vectors is the **eigenspace** for eigenvalue λ.

Theorem: Given an $n \times n$ matrix A

A real or complex number λ is an **eigenvalue** of A if there is a nonzero solution to the homogeneous system:

$$(A - \lambda I)\vec{v} = \vec{0}$$

Any such vector $\vec{v}$ is an **eigenvector**.

The set of all such vectors is the **eigenspace** for eigenvalue λ.

Corollary: λ is an eigenvalue of $A \Leftrightarrow$

$\Leftrightarrow$ $A - \lambda I$ singular $\Longleftrightarrow \det(A - \lambda I) = 0$

Corollary: The eigenspace of λ is the null space of $A - \lambda I$: $\mathrm{Null}(A - \lambda I)$.

$\boxed{\text{Proof}}$ of the Theorem:

$$A v = \lambda \vec{v} \iff (A - \lambda I)\vec{v} = 0$$

Must prove both directions

$\boxed{\text{Part I}}$ $\underset{\text{given}}{A\vec{v} = \lambda\vec{v}} \implies \underset{\text{show}}{(A - \lambda I)\vec{v} = 0}$

1. $A\vec{v} = \lambda\vec{v}$ — 1. Given
2. $A\vec{v} = \lambda I\vec{v}$ — 2. $I\vec{v} = \vec{v}$
3. $A\vec{v} - \lambda I\vec{v} = \vec{0}$ — 3. subtracting $\lambda I\vec{v}$ on both sides
4. $(A - \lambda I)\vec{v} = \vec{0}$ — 4. $(A - B)\vec{w} = A\vec{w} - B\vec{w}$ where $B = \lambda I$ $\vec{w} = \vec{v}$

Part I done

$\boxed{\text{Part II}}$ $(A - \lambda I)\vec{v} = \vec{0} \implies A\vec{v} = \lambda\vec{v}$

1. $(A - \lambda I)\vec{v} = \vec{0}$ — 1. Given
2. $A\vec{v} - \lambda I\vec{v} = \vec{0}$ — 2. $(A - B)\vec{w} = A\vec{w} - B\vec{w}$
3. $A\vec{v} = \lambda I\vec{v}$ — 3. add to both sides
4. $A\vec{v} = \lambda\vec{v}$ — 4. $I\vec{v} = \vec{v}$

Part II done

QED

Classwork: Find the eigenspace for $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ for $\lambda = 5$

The eigenspace = Null $(A - \lambda I)$

$$= \text{Null}\left(\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \text{Null}\begin{pmatrix} (1-5) & 2 \\ 2 & (4-5) \end{pmatrix} = \text{Null}\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -4 & 2 & 0 \\ 2 & -1 & 0 \end{array}\right) \xrightarrow{\rho_1 \to \frac{-1}{4}\rho_1} \left(\begin{array}{cc|c} \boxed{1} & -1/2 & 0 \\ 2 & -1 & 0 \end{array}\right) \xrightarrow{\rho_2 \to \rho_2 - 2\rho_1} \left(\begin{array}{cc|c} \boxed{1} & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Echelon Form
Reduced Echelon Form!

$$\boxed{\begin{aligned} 1x_1 + (-1/2)x_2 &= 0 \\ 0 &= 0 \end{aligned}}$$

Solve for leaders

$x_1 = \frac{1}{2}x_2$

$x_2 = x_2$ free

$$\text{Null}(A - \lambda I) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (1/2)\,x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \;\middle|\; x_2 \in \mathbb{R} \right\}$$

eigenspace

$\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$ is an eigenvector

We have many eigenvectors
choosing different values of $x_2 \in \mathbb{R}$ (not $x_2 = 0$)

For example take $x_2 = 4$ $\quad 4\begin{pmatrix} 1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ is an eigenvector

Lets check our eigenvector $\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1\left(\frac{1}{2}\right) + 2(1) \\ 2\left(\frac{1}{2}\right) + 4(1) \end{pmatrix} = \begin{pmatrix} 2.5 \\ 5 \end{pmatrix} = \begin{pmatrix} 5/2 \\ 5 \end{pmatrix}$$

$$5\begin{pmatrix} 1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5/2 \\ 5 \end{pmatrix}$$ match! Checks!

HW4

HW Find the eigenspace for $\lambda = 0$

and $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

In this case $\text{Null}(A - \lambda I) = \text{Null}(A - 0I) = \text{Null}(A)$

Be sure to check your answer!

Classwork

Find the eigenspace for $\lambda=3$ of $A=\begin{pmatrix}4&1&1\\1&4&1\\1&1&4\end{pmatrix}$

$$\text{Null}(A-\lambda I)=\text{Null}\left(\begin{pmatrix}4&1&1\\1&4&1\\1&1&4\end{pmatrix}-3\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right)$$

$$=\text{Null}\begin{pmatrix}(4-3)&1&1\\1&(4-3)&1\\1&1&(4-3)\end{pmatrix}=\text{Null}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$$

$$\left(\begin{array}{ccc|c}\boxed{1}&1&1&0\\1&1&1&0\\1&1&1&0\end{array}\right)\xrightarrow[\rho_3\to\rho_3-\rho_1]{\rho_2\to\rho_2-\rho_1}\left(\begin{array}{ccc|c}\boxed{1}&1&1&0\\0&0&0&0\\0&0&0&0\end{array}\right)$$ Reduced Echelon Form

$$\boxed{\begin{aligned}\boxed{1x_1}+1x_2+1x_3&=0\\0&=0\\0&=0\end{aligned}}$$

Solve for leaders $x_1=-x_2-x_3$

free $x_2=x_2$, $x_3=x_3$

$$\text{Null}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}=\left\{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}-x_2-x_3\\x_2\\x_3\end{pmatrix}\;\middle|\;x_2,x_3\in\mathbb{R}\right\}$$

$$\text{eigenspace}=\left\{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=x_2\begin{pmatrix}-1\\+1\\0\end{pmatrix}+x_3\begin{pmatrix}-1\\0\\1\end{pmatrix}\;\middle|\;x_2,x_3\in\mathbb{R}\right\}$$

two eigenvectors

many more:

for example $x_2=4$, $x_3=5$ $\quad 4\begin{pmatrix}-1\\+1\\0\end{pmatrix}+5\begin{pmatrix}-1\\0\\1\end{pmatrix}=\begin{pmatrix}-4\\4\\0\end{pmatrix}+\begin{pmatrix}-5\\0\\5\end{pmatrix}=\begin{pmatrix}-9\\4\\5\end{pmatrix}$ is also an eigenvector

$$\begin{pmatrix}4&1&1\\1&4&1\\1&1&4\end{pmatrix}\begin{pmatrix}-1\\+1\\0\end{pmatrix}=\begin{pmatrix}4(-1)+1(1)+1\cdot 0\\1(-1)+4(1)+1(0)\\1(-1)+1(1)+4(0)\end{pmatrix}=\begin{pmatrix}-3\\3\\0\end{pmatrix}=3\begin{pmatrix}-1\\+1\\0\end{pmatrix}\checkmark$$

$$\begin{pmatrix}4&1&1\\1&4&1\\1&1&4\end{pmatrix}\begin{pmatrix}-1\\0\\1\end{pmatrix}=$$ You check this!

$\boxed{HW}$ Find eigenspace for $\lambda=6$ and matrix $\begin{pmatrix}4&1&1\\1&4&1\\1&1&4\end{pmatrix}$

$\boxed{HW}$ What if you try $\lambda=2$? What happens? ← warning not an eigenvalue!

HW5 and HW6 above

$\boxed{\text{HW}}$ Find the eigenspaces of $A = \begin{pmatrix} 1&0&1&1 \\ 0&1&1&1 \\ 1&1&1&0 \\ 1&1&0&1 \end{pmatrix}$

Recall: $\lambda = 3 \quad \lambda = -1 \quad \lambda = 1$

are the eigenvalues of $A = \begin{pmatrix} 1&0&1&1 \\ 0&1&1&1 \\ 1&1&1&0 \\ 1&1&0&1 \end{pmatrix}$

So find the eigenspace for each.

Be sure to check your answers!

Check $A\vec{v} = \lambda \vec{v}$

HW7 above is very long. So it counts as HW7 and HW8. Here's the solution for eigenvalue=1

Solution for $\lambda = 1$:

$$\left(\begin{array}{cccc|c} 1-1 & 0 & 1 & 1 & 0 \\ 0 & 1-1 & 1 & 1 & 0 \\ 1 & 1 & 1-1 & 0 & 0 \\ 1 & 1 & 0 & 1-1 & 0 \end{array}\right) = \left(\begin{array}{cccc|c} 0&0&1&1&0 \\ 0&0&1&1&0 \\ 1&1&0&0&0 \\ 1&1&0&0&0 \end{array}\right) \xrightarrow{\rho_1 \leftrightarrow \rho_3} \left(\begin{array}{cccc|c} 1&1&0&0&0 \\ 0&0&1&1&0 \\ 0&0&1&1&0 \\ 1&1&0&0&0 \end{array}\right) \xrightarrow{\rho_4 \to \rho_4 - \rho_1} \left(\begin{array}{cccc|c} \boxed{1}&1&0&0&0 \\ 0&0&\boxed{1}&1&0 \\ 0&0&1&1&0 \\ 0&0&0&0&0 \end{array}\right)$$

subtract λ on diagonal

$$\xrightarrow{\rho_3 \to \rho_3 - \rho_2} \left(\begin{array}{cccc|c} 1&1&0&0&0 \\ 0&0&1&1&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{array}\right) \qquad \begin{aligned} x_1 + x_2 &= 0 \\ x_3 + x_4 &= 0 \end{aligned} \qquad \begin{aligned} x_1 &= -x_2 \\ x_2 &= x_2 \text{ (free)} \\ x_3 &= -x_4 \\ x_4 &= x_4 \text{ (free)} \end{aligned}$$

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \;\middle|\; x_2, x_4 \in \mathbb{R} \right\}$$

both eigenvectors. Can check them.

$$\begin{pmatrix} 1&0&1&1 \\ 0&1&1&1 \\ 1&1&1&0 \\ 1&1&0&1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1+0+0+0 \\ 0+1+0+0 \\ -1+1+0+0 \\ -1+1+0+0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = 1 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \checkmark$$

$$\begin{pmatrix} 1&0&1&1 \\ 0&1&1&1 \\ 1&1&1&0 \\ 1&1&0&1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} = \text{you show this} = 1 \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

Do the other eigenvalues too

Note that if an eigenspace has two directions, you can use Gram-Schmidt to find an orthonormal basis to get perpendicular eigenvectors.

HW9:

Check the four eigenvectors you found in HW7 are perpendicular by taking dot products.

Hint for HW9: Quickly review Lesson 4 for the definition of dot product.

Hint for HW9: Check all combinations so that is 12 dot products to check.

HW10:

HW Fill in the justifications of this proof.

Theorem: If $A^T = A$ and

$A\vec{v}_1 = \lambda_1 \vec{v}_1$ and $A\vec{v}_2 = \lambda_2 \vec{v}_2$

and $\lambda_1 \neq \lambda_2 \neq 0$

then $\vec{v}_1 \cdot \vec{v}_2 = 0$

You may use the following theorems as justifications:

Thm: $A^T\vec{v} \cdot \vec{w} = \vec{v} \cdot A\vec{w}$ Thm: $A(\vec{v}+\vec{w}) = A\vec{v} + A\vec{w}$

Thm: $k\vec{v} \cdot \vec{w} = (k\vec{v}) \cdot \vec{w} = \vec{v} \cdot (k\vec{w})$ Thm: $A(k\vec{v}) = kA\vec{v}$

Also remember for real numbers

$\frac{1}{x}x = x\frac{1}{x} = \frac{x}{x} = 1$ when $x \neq 0$

$x = kx \Rightarrow x = 0$ or $k = 1$

Proof:

1. $\vec{v}_1 \cdot \vec{v}_2 = \frac{1}{\lambda_2}\lambda_2 \vec{v}_1 \cdot \vec{v}_2$
2. $= \frac{1}{\lambda_2} \vec{v}_1 \cdot (\lambda_2 \vec{v}_2)$
3. $= \frac{1}{\lambda_2} \vec{v}_1 \cdot (A\vec{v}_2)$
4. $= \frac{1}{\lambda_2} (A^T\vec{v}_1) \cdot \vec{v}_2$
5. $= \frac{1}{\lambda_2} (A\vec{v}_1) \cdot \vec{v}_2$
6. $= \frac{1}{\lambda_2} (\lambda_1 \vec{v}_1) \cdot \vec{v}_2$
7. $= \frac{\lambda_1}{\lambda_2} \vec{v}_1 \cdot \vec{v}_2$
8. So $\vec{v}_1 \cdot \vec{v}_2 = \frac{\lambda_1}{\lambda_2} \vec{v}_1 \cdot \vec{v}_2$ but $\frac{\lambda_1}{\lambda_2} \neq 1$
9. $\vec{v}_1 \cdot \vec{v}_2 = 0$

The next few homework problems have you practice finding eigenvectors and eigenvalues from beginning to end: find the characteristic polynomial, factor it to find the eigenvalues, then find the eigenspaces by solving the homogeneous systems:

HW11:

$\boxed{\text{HW}}$ Find eigenvalues and eigenspaces of $\begin{pmatrix} 2 & -3 \\ 1 & 6 \end{pmatrix}$ using Characteristic Polynomial Method:

Solution: $\begin{pmatrix} 2 & -3 \\ 1 & 6 \end{pmatrix}$

$$\det\begin{pmatrix} 2-\lambda & -3 \\ 1 & 6-\lambda \end{pmatrix} = (2-\lambda)(6-\lambda)-(-3)(1)=0$$
$$= 12-6\lambda-2\lambda+\lambda^2+3$$
$$= \lambda^2-8\lambda+15=0$$
$$(\lambda-3)(\lambda-5)=0$$

These are our evalues

$$\lambda=3 \quad \lambda=5$$

$\lambda=3$ Solve the homog system

$$\left(\begin{array}{cc|c} 2-3 & -3 & 0 \\ 1 & 6-3 & 0 \end{array}\right) = \left(\begin{array}{cc|c} -1 & -3 & 0 \\ 1 & 3 & 0 \end{array}\right) \xrightarrow{\rho_1 \to -\rho_1} \left(\begin{array}{cc|c} 1 & 3 & 0 \\ 1 & 3 & 0 \end{array}\right) \xrightarrow{\rho_2 \to \rho_2-\rho_1} \left(\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$1x_1+3x_2=0$

$x_1=-3x_2$

$x_2=x_2$ (free)

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3x_2 \\ x_2 \end{pmatrix} = x_2\begin{pmatrix} -3 \\ 1 \end{pmatrix} \;\middle|\; x_2 \in \mathbb{R} \right\}$$

$\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ is evector

Check $\begin{pmatrix} 2 & -3 \\ 1 & 6 \end{pmatrix}\begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -9 \\ 3 \end{pmatrix}$ ✓ $3\begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -9 \\ 3 \end{pmatrix}$ ✓

$\lambda=5$ Solve homog system

$$\left(\begin{array}{cc|c} 2-5 & -3 & 0 \\ 1 & 6-5 & 0 \end{array}\right) = \left(\begin{array}{cc|c} -3 & -3 & 0 \\ 1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{\rho_1 \to -\frac{1}{3}\rho_1} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right) \xrightarrow{\rho_2 \to \rho_2-\rho_1} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$1x_1+1x_2=0 \qquad x_1=-x_2$

$x_2=x_2$ (free)

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_2 \end{pmatrix} = x_2\begin{pmatrix} -1 \\ 1 \end{pmatrix} \;\middle|\; x_2 \in \mathbb{R} \right\}$$

evector is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Check

$$\begin{pmatrix} 2 & -3 \\ 1 & 6 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2-3 \\ -1+6 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \end{pmatrix}$$

$5\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \end{pmatrix}$ ← Match!

HW12: Needs complex numbers!

HW Find eigenvalues and eigenspaces for $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ using the characteristic polynomial and then solving for eigenspaces

Solution Hints

$$\det\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = 0$$

$$(-\lambda)(-\lambda) - (-1)(1) = 0$$

$$\lambda^2 + 1 = 0$$

$$\lambda^2 = -1$$

has no real solutions!

$i = \sqrt{-1}$ $\quad \lambda = \pm i$

$-i = -\sqrt{-1}$

$\lambda = +i$ Find its evector

Solve $\quad -i(i) = -i^2 = -(-1) = +1$

$$\left(\begin{array}{cc|c} \boxed{-i} & -1 & 0 \\ 1 & -i & 0 \end{array}\right) \xrightarrow{\rho_1 \to i\rho_1} \left(\begin{array}{cc|c} 1 & -i & 0 \\ 1 & -i & 0 \end{array}\right)$$

$$\xrightarrow{\rho_2 \to \rho_2 - \rho_1} \left(\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$x_1 - i x_2 = 0 \qquad x_1 = i x_2$

$x_2 = x_2$ (free)

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} i x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} i \\ 1 \end{pmatrix} \;\middle|\; x_2 \in \mathbb{C} \right\}$$

So the evector is $\begin{pmatrix} i \\ 1 \end{pmatrix}$ free in $\mathbb{C}$

Check

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} 0i - 1\cdot 1 \\ 1\cdot i + 0\cdot 1 \end{pmatrix} = \begin{pmatrix} -1 \\ i \end{pmatrix} \checkmark$$

$$i\begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} i^2 \\ i \end{pmatrix} = \begin{pmatrix} -1 \\ i \end{pmatrix}$$

Now Find evector for $\lambda = -i$ yourself!