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Ch 8: Internal Loading within Rigid Bodies

One of the fundamental assumptions we make in Statics is that we are dealing with ‘rigid’ bodies which do not deform (bend or change shape) when forces and moments are applied. While we know that this assumption is not true for real materials, we are building the analytical tools necessary to consider deformation. The tool which you learn in this chapter is to compute the internal loading, which are the forces and moments within a body, as a function of the applied loads. In courses you take after Statics, the internal loading will allow you to determine deformations and ultimately select the appropriate shape, size, and material for structural elements to safely hold the applied loads. Note that the use of the words ‘loads’ or ‘loading’ as opposed to ‘forces’ is intentional when we are referring to both forces and bending moments.

Learning Outcomes: Upon completion of this chapter you should be able to:

1. Develop a free body diagram that includes internal forces.
2. Solve for the internal loading at a single location by solving the equations of static equilibrium.
3. Construct shear and bending moment diagrams using section cut equations.
4. Construct shear and bending moment diagrams using graphical methods.
5. Construct shear and bending moment diagrams using calculus-based equations.

Application Box

The controlling design parameter for most engineering systems is deformation. Thankfully, due to a property called elasticity, most materials will bend, stretch, and compress, long before they ultimately break. For example, when designing the floor in a new building, the floor is often limited to deflecting less than the length of the span in inches, divided by 360. Any more deformation than this would be considered disconcerting to the building residents and also start damaging surface materials (like drywall). For example, for a 20-foot span, the deflection would need to be less than

<m>\frac{20 ft \cdot \frac{12in}{1ft}}{360}=0.667 in<\m>

$$\frac{20ft \cdot \frac{12in}{1ft}}{360} = 0.667in$$

To meet this deformation limit, we need to consider the magnitude and location of applied loads, the size/shape of the floor beams, and the material the floor beams are made from. As deflection is an internal property of the flooring materials, the first step is to determine the internal loadings that arise from the externally applied loads, which you will learn in this chapter.

Overview of Internal Loading

Recall that in Ch 6 Structures you were introduced to an axial form of internal loading which is tension or compression forces. This section will explain the internal shear and internal bending moment, the other two internal loadings in two-dimensional systems.

Learning Outcomes: Upon completion of this section you should be able to:

1. Identify the three basic internal loadings in 2D systems.
2. Determine the relationship between the externally applied loads and internal loading.
3. Articulate the standard sign convention for internal forces.

Definition

Internal loads are present within a rigid body, which allows us to solve for the state of stress (force per unit area) at any location in a rigid body. Let us review something you already are familiar with, straight, two-force truss members (from Ch. 6 Structures). These two-force members are in equilibrium with equal and opposite forces applied to either end. Now let us visualize that we cut a truss member in tension (see Figure AA below, similar to Method of Sections in Ch. 6) at some point along its length. To maintain equilibrium, internal tension forces must be added at the cut location. These new forces applied at the location of the cut are **internal forces** and we can note that they are applied in the **axial** direction as they run along the long axis of the two-force member.

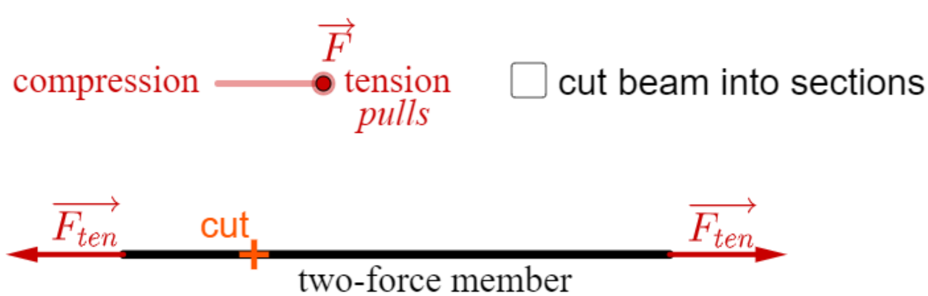


Figure AA - In this interactive start by changing the axial force direction of this two-force member FBD from tension (which pulls) to compression (which pushes). Next, check the box to cut the beam into sections. Each cut section FBD must also have an internal axial force to balance the external force. You can also move the location of the cut to see that the value of the internal axial force is constant the entire length. Source: https://www.geogebra.org/m/enpagkwd

Let us now examine another two-force member in static equilibrium (Figure BB). This time, the two-force member is L shaped instead of straight, but the external forces still share the same line of action to maintain equilibrium. If we cut through this L-shaped rigid body, we can again obtain two rigid bodies that must also be in equilibrium. However, if we apply only axial forces as we did on the straight member, we will quickly notice that these two new rigid bodies are not in static equilibrium as the forces, while equal in magnitude, are not in opposing directions. This means that these rigid bodies are not in equilibrium and that something is missing!

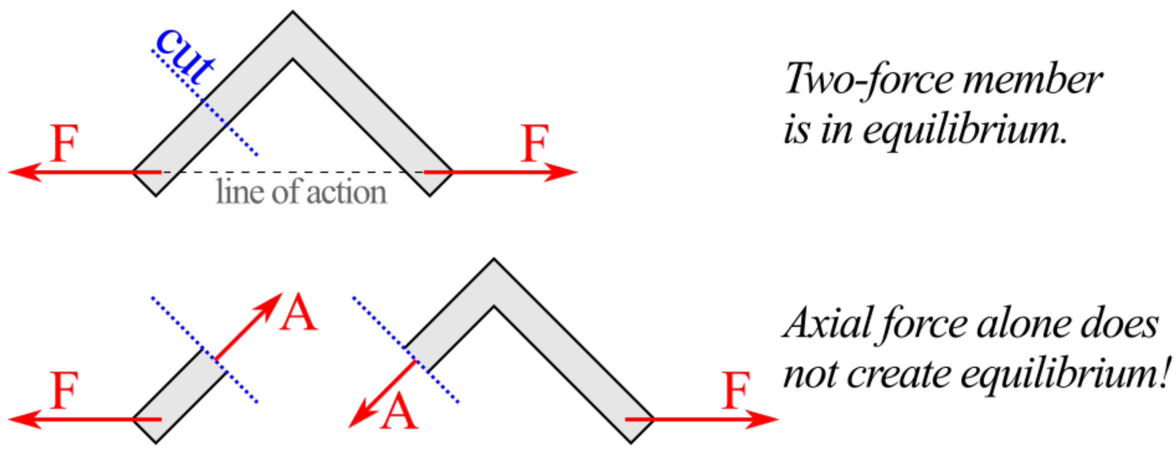


Figure BB - (top) A non-straight, two-force member shown (top) in equilibrium and after being cut into sections and an axial force applied (like in Figure AA) but this axial force alone does not create equilibrium. We will revisit this example in the section below and Figure CC. Source: unlicensed SVG by Dan Baker

If you will recall, for a 2D case such as we are representing here, we used three equations to solve for the three degrees-of-freedom to maintain static equilibrium

- <m>\Sigma F_x = 0<\m> prevented translation in the x-direction,
- <m>\Sigma F_y = 0<\m> prevented translation in the y-direction, and
- <m>\Sigma M_z = 0<\m> prevented rotation.

As the cut bodies are also in equilibrium, we must balance forces in the x and y directions as well as rotation. Assuming the material is rigid, the connection between the two halves of the link resists both translation and rotation, so we can model this connection as a fixed support (reference supports in Ch. 5) and replace the removed half of the link with a force reaction and a couple-moment reaction (as shown in Figure CC below). We resolve the force reaction into components parallel and perpendicular to the cut and give these forces special names in the context of internal loads:

- The internal force component perpendicular to the cut (or along the member) is the axial force <m>\vec{A}<\m>. This is the same internal tension or compression force that we assumed to be the only significant internal load for trusses.
- The internal force component parallel to the cut (or across the member) is the shear force <m>\vec{V}<\m>. The word shear refers to the shearing sense between adjacent planes in the material. You can get a feel for shearing adjacent planes by sliding two pieces of paper together.
- The internal couple moment is called a bending moment <m>\vec{M}<\m> because it has the effect of trying to bend the material by rotating the plane on which it acts.

To be concise, ‘shear force’ is often referred to as simply ‘shear’ and the ‘bending moment’ as ‘moment’ and collectively the three are referred to as the ‘internal loading’.

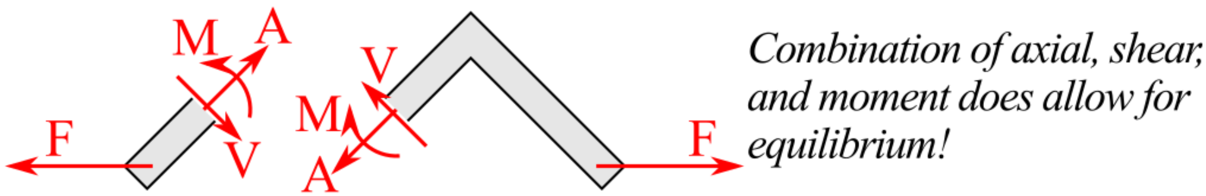


Figure CC - Using the same example from BB, a non-straight, two-force member is cut into sections. The full internal loading can be represented by three internal loads, axial force <m>\vec{A}<\m>, shear force <m>\vec{V}<\m>, and bending moment <m>\vec{M}<\m>. Now each of these two cut pieces of the original rigid body can be in equilibrium.

Thinking Deeper [box]

The internal loadings are a simplification of what actually is a more complex loading distributed across the section plane. We can represent this complex force distribution with a single resultant force <m>\overrightarrow{F}<\m> and couple <m>\overrightarrow{M}<\m> (which were introduced as force system resultants in Ch 4). This resultant force can be resolved into two orthogonal components which are a parallel-to-the-cut shear force <m>\overrightarrow{V}<\m> and perpendicular to the cut axial force <m>\overrightarrow{A}<\m>. The couple <m>\overrightarrow{M}<\m> is referred to as the internal bending moment and represents the net rotational effect of the force system on the surface of the cut (i.e. how the force system is trying to rotate the plane of the cut).

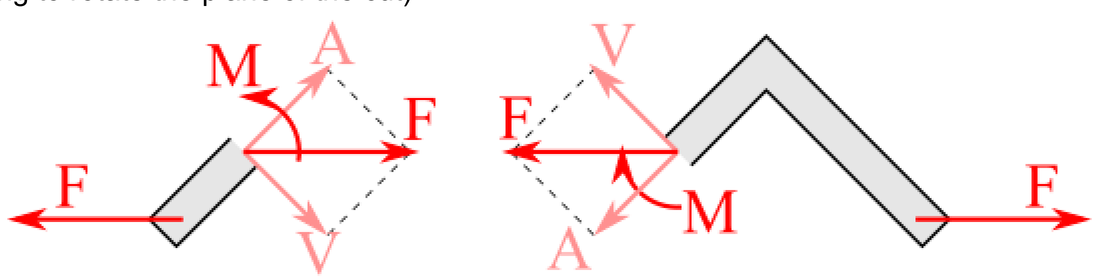
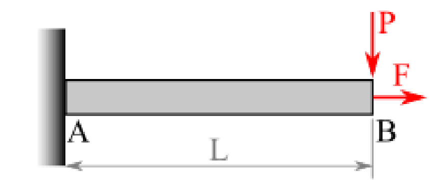
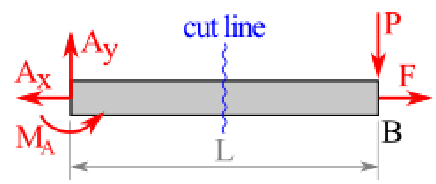
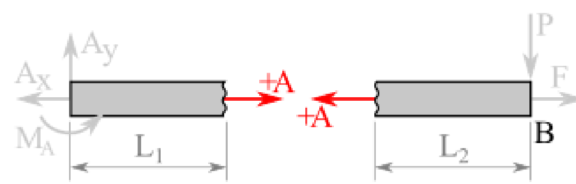
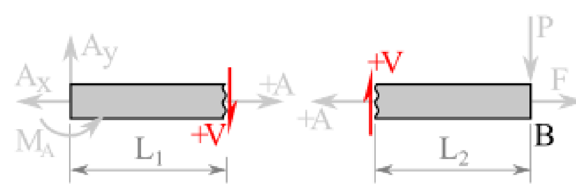
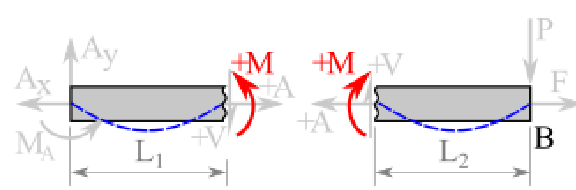
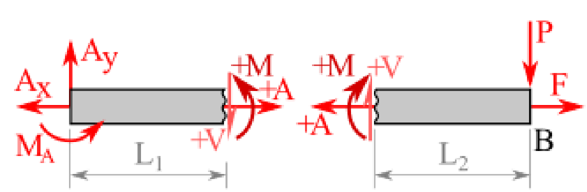


Figure DD - Using the same example shown in Figures BB and CC, we can see that the axial force <m>\vec{A}<\m> and shear force <m>\vec{V}<\m> are components of the force equal and opposite to F on each FBD. The bending moment <m>\vec{M}<\m> resists the force-couple created by the two <m>\vec{F}<\m> forces.

Sign Conventions

Using a standard sign convention allows us to have consistency across our solutions for the calculation of internal loading. These sign conventions are consistent with the signs of the shear and moment diagrams you will learn to draw in later in this chapter. The standard convention for each of the three internal loads is summarized in Table aa below.

[Table caption] Table aa - The table below summarizes the construction of an FBD revealing the axial <m>\vec{A}<\m>, shear <m>\vec{V}<\m>, and moment <m>\vec{M}<\m> vectors to a cantilever beam using a standard sign convention.

Let us examine the assumed positive internal loadings in a cantilever beam which is supported by a fixed connection at A. Applied forces P (vertical) and F (axial) are applied to the beam at the free end B.	
First, draw an FBD of your un-cut beam. Notice that only the applied loads and support reactions are included on this un-cut beam FBD. The internal loads are only exposed (visible in your FBD) after the beam is cut.	
After you cut the beam, you have two rigid-body sections. For clarity, the internal loads will be added one at a time (with the other loads greyed out). The first internal load is the **axial force <m>\vec{A}<\m> which is defined as positive when in tension (pulling)**. Notice that the axial forces are equal and opposite on the two beam sections. This will also be true of the shear force <m>\vec{V}<\m> and bending moment <m>\vec{M}<\m>.	
The next internal load is the **shear force <m>\vec{V}<\m> which is positive when the shear is down on the right face of the cut and up on the left face**. An alternate definition is that the **positive shear forces try to rotate a body in the clockwise (or opposite RHR) direction**. This second definition is quite useful for any orientation of a beam beyond horizontal.	
The final internal loading is the **bending moment <m>\vec{M}<\m> which is positive if the bending direction would tend to bend the beam into a 'smiling' U-shape** (see dotted blue line). Thus, negative moments bend beam into a frowning ∩-shape.	
Therefore, a horizontal beam will always have the assumed positive internal loads shown on these two FBD's. Either the left or right section will allow you to solve for the internal loadings at this cut location.	

The values of axial force <m>\vec{A}<\m>, the shear force <m>\vec{V}<\m>, and the bending moment <m>\vec{M}<\m> may be positive, zero, or negative depending on the scenario. Once your free body diagram including the assumed positive internal loads has been finalized, you then solve for the unknown values and signs of the axial force <m>\vec{A}<\m>, the shear force <m>\vec{V}<\m>, and the bending moment <m>\vec{M}<\m> just like any other FBD. In other words, you draw the internal loads in assumed positive directions on the FBD, but then when you solve for the internal load values, use the same tools you mastered in Ch 5 Equilibrium of Rigid Bodies. This workflow typically includes:

- A horizontal x and vertical y axes system
- <m>\sigma F_x=0\text{, }\sigma F_y=0\text{, and }\sigma M_z=0<\m> to solve for the three unknowns.

Methods to solve for internal loadings

The remainder of this chapter focuses on four techniques to solve for the internal loadings of axial <m>\vec{A}<\m>, shear <m>\vec{V}<\m>, and moment <m>\vec{M}<\m>. All four of the techniques will provide the same answers and we will revisit our recommendations on which ones are often the most efficient at the end of the chapter. The four techniques are:

- Body Internal Loadings at a Single Cut Location
- Beam Shear and Moment Diagrams using Cuts
- Beam Shear and Moment Diagrams using Graphical Method
- Beam Shear and Moment Diagrams using Calculus Equations

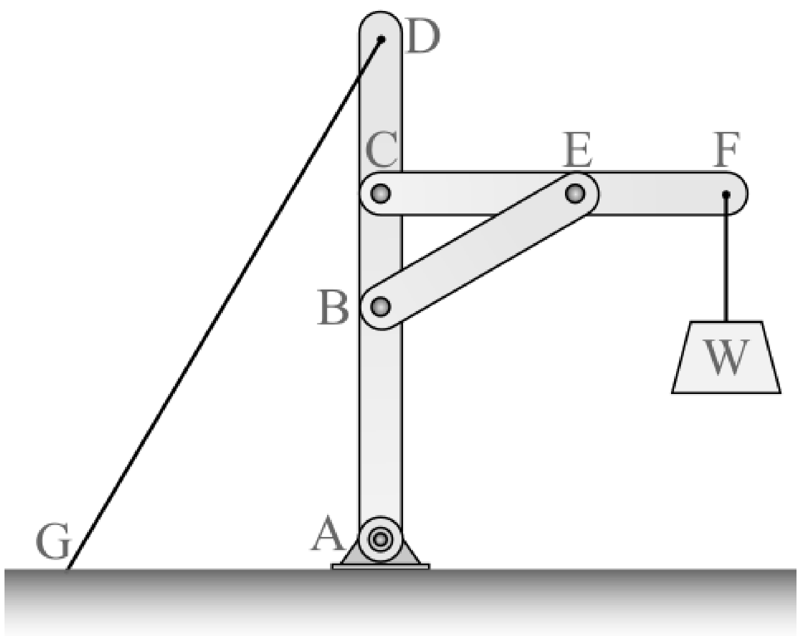
Section Cut to find Internal Loadings at a Single Point

This section will cover the tools needed to compute the internal axial force, shear force, and bending moment at a designated point in a multiforce rigid body.

Learning Outcome: Upon completion of this section you should be able to:

1. Use static equilibrium to calculate the unknown internal loadings at a single point on a rigid body.

If we consider a frame structure of massless bodies as shown in Fig. EE, we can see that the system is made of two two-force members (GD and BE) and two multiforce members (*AD* and *CF*). The internal loading within the two-force members would be purely axial forces (just the like the truss members in Ch 6), but the multiforce members will be subject to the complete set of internal loadings including axial force, shear force, and bending moment.



[caption] Figure EE - A frame structure containing a combination of two-force (BE) and multiforce (AD and CF) members to support the weight at E.

To find the axial force, shear force, and bending moment within the multiforce members, we can once again create an imaginary *cut*. An example of this is shown in Fig. X, where we have indicated the location of a cut through member *AD* between locations *D* and *C*. The free body diagram is shown with pin joint forces at locations *A* and *C* and forces from the two-force members at locations *B* and *D*.

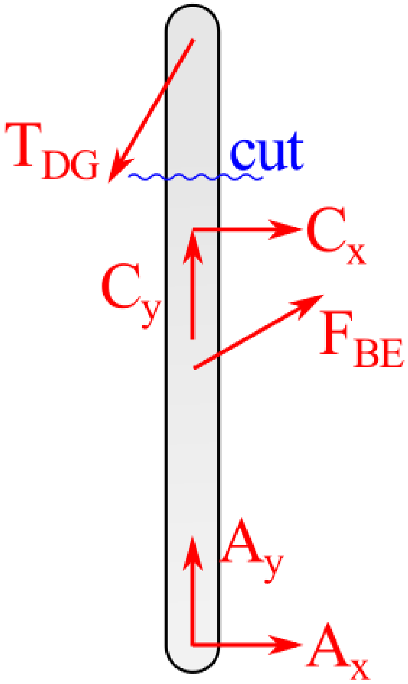


Figure FF - Free body diagram of frame member *AD* with a dotted line indicating the imaginary cut used to analyze the internal forces within the member.

Once the location of the cut has been decided, we can separate the free body diagram of the member into two independent free body diagrams, one above the cut and one below. This is analogous to the Method of Sections technique which you learned in Ch 6. The free-body diagrams for the two sections of the member are shown in Fig. GG. Here the three internal forces are labeled axial <m>\vec{A}<\m>, shear <m>\vec{V}<\m>, and moment <m>\vec{M}<\m>. Either of the free body diagrams can be used to solve for the internal forces, choose the easier one to solve given which interaction forces are known and the complexity of the FBD.

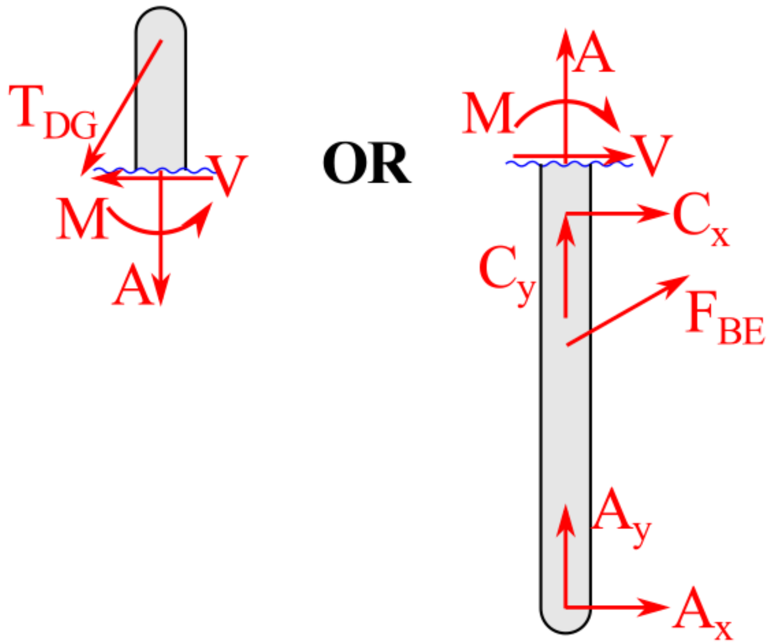
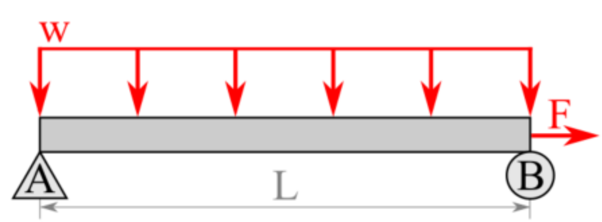
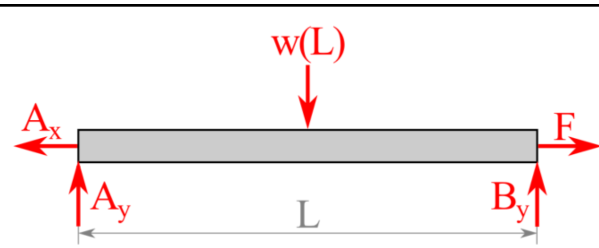
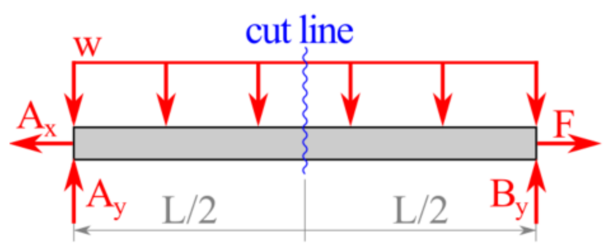
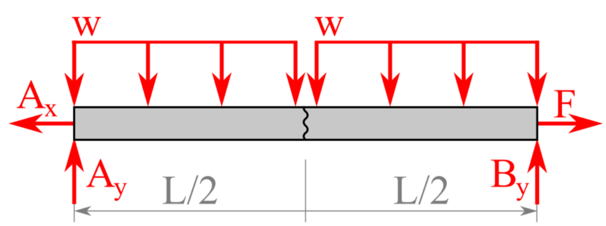
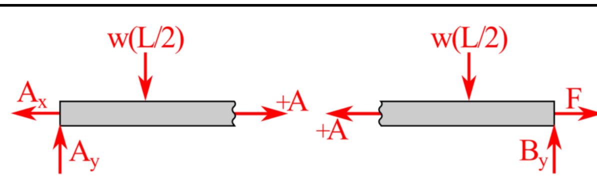


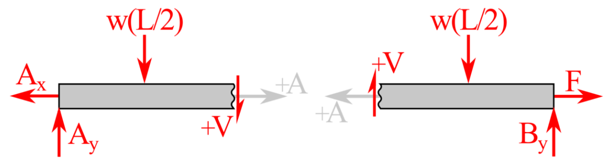
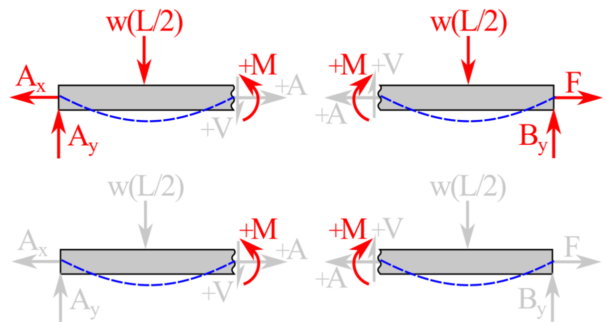
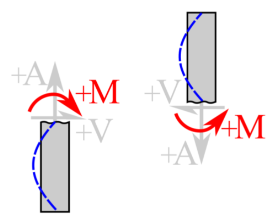
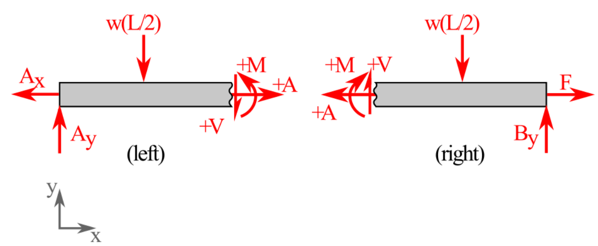
Figure GG - Free body diagrams for the two sections of frame member *AD* after cutting the member. Note that the internal loadings at the cut are all drawing in the assumed positive direction as covered in the Sign Conventions section above. Also note how the loadings would cancel each other out if you imagine putting the two pieces back together.

You can also use this technique to solve for the internal loads on a beam with a distributed load, as shown in Table bb below.

Table bb - The table below summarizes the steps necessary to cut a beam with a distributed load and then to assign assumed positive axial <m>\vec{A}<\m>, shear <m>\vec{V}<\m>, and moment <m>\vec{M}<\m> vectors to a beam.

Let us examine the internal loadings at the midpoint of a beam of length L which is supported by a pin at A and a roller at B. The beam is subjected to a distributed force w (in units of force/length) and a concentrated force F applied to point B.	
The first step to solving the internal loads is to fully solve the external reactions. There are a possible three internal loads at each location, you will need to have all external loads and reactions solved prior to cutting the beam.	 <m>\test{Equilibrium equations to solve for reactions}\\

	\Sigma F_x=0\\ -A_x+F=0\\ \vspace{2 mm} A_x=F \\ \Sigma F_y=0\\ A_y-w(L/2)+B_y=0\\ \vspace{2 mm} A_y=w(L/2)-B_y\\ \Sigma M_A=0\\ -(L/2)(wL/2)+L(B_y)=0\\ \vspace{2 mm} B_y=(1/2)(w/2)<\m> $\Sigma F_x = 0$ $-A_x + F = 0$ $A_x = F$ $\Sigma F_y = 0$ $A_y - w(L/2) + B_y = 0$ $A_y = w(L/2) - B_y$ $\Sigma M_A = 0$ $-(L/2)(wL/2) + L(B_y) = 0$ $B_y = (1/2)(w/2)$
Next, prepare to cut the beam at the specified location, in this case, the midpoint, by re-distributing the load w and locating your cut line.	
Notice that as the beam is cut in two, the distributed load w is cut as well. In the next row each of these distributed load halves will be changed to equivalent point loads of w(L/2) acting through the centroid of each cut half. The next three rows will add each of the three internal loads which are exposed by cutting a beam.	
The first assumed positive internal loadings we will add to the cut faces are the axial forces (A). **Axial forces (A) are positive in tension** (which always pulls). Notice that the axial forces are equal and opposite on the two halves of the cut beam. This will also be true of the shear force (V) and bending moment (M).	

The next assumed positive internal loading is the shear force (V). **Shear forces are positive if the shear is down on the right-side side of the cut and up on the left-hand side**. An alternate definition is that the **positive shear forces try to rotate a body in the clockwise (or opposite RHR) direction**. This second definition is quite useful if you are dealing with a vertical column instead of a horizontal beam.	
The final assumed positive internal loading is the bending moment (M). **Bending moments are positive if the bending direction would tend to bend the beam into a smiling U-shape** (see dotted blue line). Thus, negative moments bend the beam into a frowning ∩-shape. For vertical columns, positive bending moments bend a beam into a C-shape and negative into a backward C-shape (ɔ).	 For a vertical column 
Therefore, a horizontal beam will always have the assumed positive internal loading shown. The **next step is to solve either FBD using the same techniques you have already learned, using your selected coordinate system**. Let's assume you are using a standard x-y coordinate system (as shown) and choose the left-hand FBD to sum your forces in the y-direction. In your ΣFy=0 equation, the unknown shear force (V) will be negative in your equation (as it is pointing down in the negative y-direction). When you solve for your internal loadings, some may be positive and others negative. **Conveniently the values for shear (V) and moment (M) will be exactly the same whether you solve for internal loading at a point (covered in the next section) or use a shear and moment diagram (covered later in section XX).**	 <m>\text{Equations of equilibrium for left segment}\\ \Sigma F_x=0\\ \vspace{2 mm} -A_x+A=0\\ \Sigma F_y=0\\ \vspace{2 mm} A_y-w(L/2)-V=0\\ \Sigma M_{cut}=0\\ \vspace{2 mm} (L/2)(wL/2)-L(A_y)+M=0<\m>

	$\Sigma F_x = 0$ $-A_x + A = 0$ $\Sigma F_y = 0$ $A_y - w(L/2) - V = 0$ $\Sigma M_{cut} = 0$ $(L/2)(wL/2) - L(A_y) + M = 0$ <m>\text{Equations of equilibrium for right segment}\\ \Sigma F_x=0\\ \vspace{2 mm} -A+F=0\\ \Sigma F_y=0\\ \vspace{2 mm} V-w(L/2)+B_y=0\\ \Sigma M_{cut}=0\\ \vspace{2 mm} -M-(L/2)(wL/2)+L(B_y)=0<\m> $\Sigma F_x = 0$ $-A + F = 0$ $\Sigma F_y = 0$ $V - w(L/2) + B_y = 0$ $\Sigma M_{cut} = 0$ $-M - (L/2)(wL/2) + L(B_y) = 0$

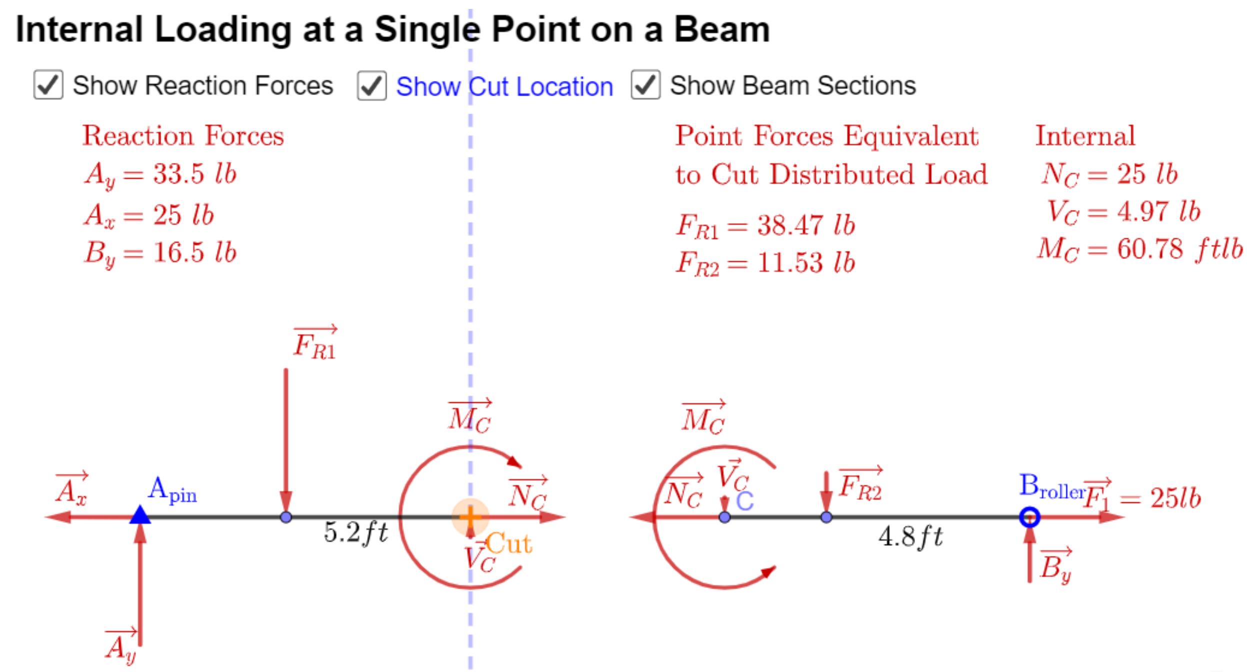


Figure HH: This interactive allows you to change the cut location on a 10 ft long beam with distributed load varying from a value of 10 lb/ft (downward) at point A to a value of 0 lb/ft at point B. As you progressively click the boxes you will be able to see the reaction forces, the cut location, the p, and the beam cut into sections with the internal loadings. Source: https://www.geogebra.org/m/w7m3qdbp

Introduction to Shear and Moment Diagrams

The previous section addressed the analysis of the shear and moment at a single point, which is useful if we know in advance where we expect the rigid body to be the most susceptible to failure. However, often we will not know in advance where this weakest location will be. Thus we need to check the internal loading along the full length of a rigid body to find the locations where the loading is the highest. Shear and moment diagrams are graphs of the internal shear and internal bending moment plotted along the length of the beam. They allow us to see where the maximum loads occur and can also be used to optimize the beam design to reduce the overall weight or material used. Shear and moment are a fundamental tool to compute the value of internal shear force V and bending moment M at every point along a beam. Note that the third internal loading, axial load A, is independent of shear and moment, thus will not be considered in this section.

For simplicity, we will limit our analysis in the remainder of the chapter to beams, which are structural elements capable of withstanding load primarily by resisting bending. Beams can be supported in a variety of ways (see Fig. X).

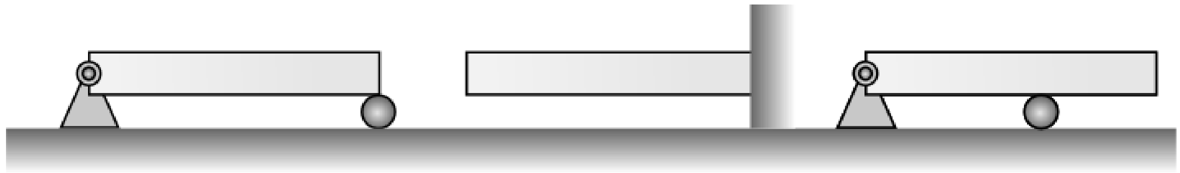


Figure II: Beam support types include: 1) a simply supported beam with a pin on one end and a roller at the other, 2) a cantilevered beam which is fixed on one end, and 3) an overhanging beam which acts as a combination of a simply supported and cantilever beam.

In this first approach to developing shear and moment diagrams, we extend the previous method for computing internal loadings at a specific point.

Learning Outcomes: Upon completion of this section you should be able to:

1. Construct a free body diagram for a section of a beam for any arbitrary cut location.
2. Use beam cuts in each loading segment to construct both a shear diagram and a moment diagram.

Let us will start with a cantilevered beam subjected to a vertical load at its free end. If we assume that the beam is fixed to a wall on its left end and subject to a vertical force P on its right end (as shown in Fig. JJ), we can begin to evaluate the beam's internal loadings.

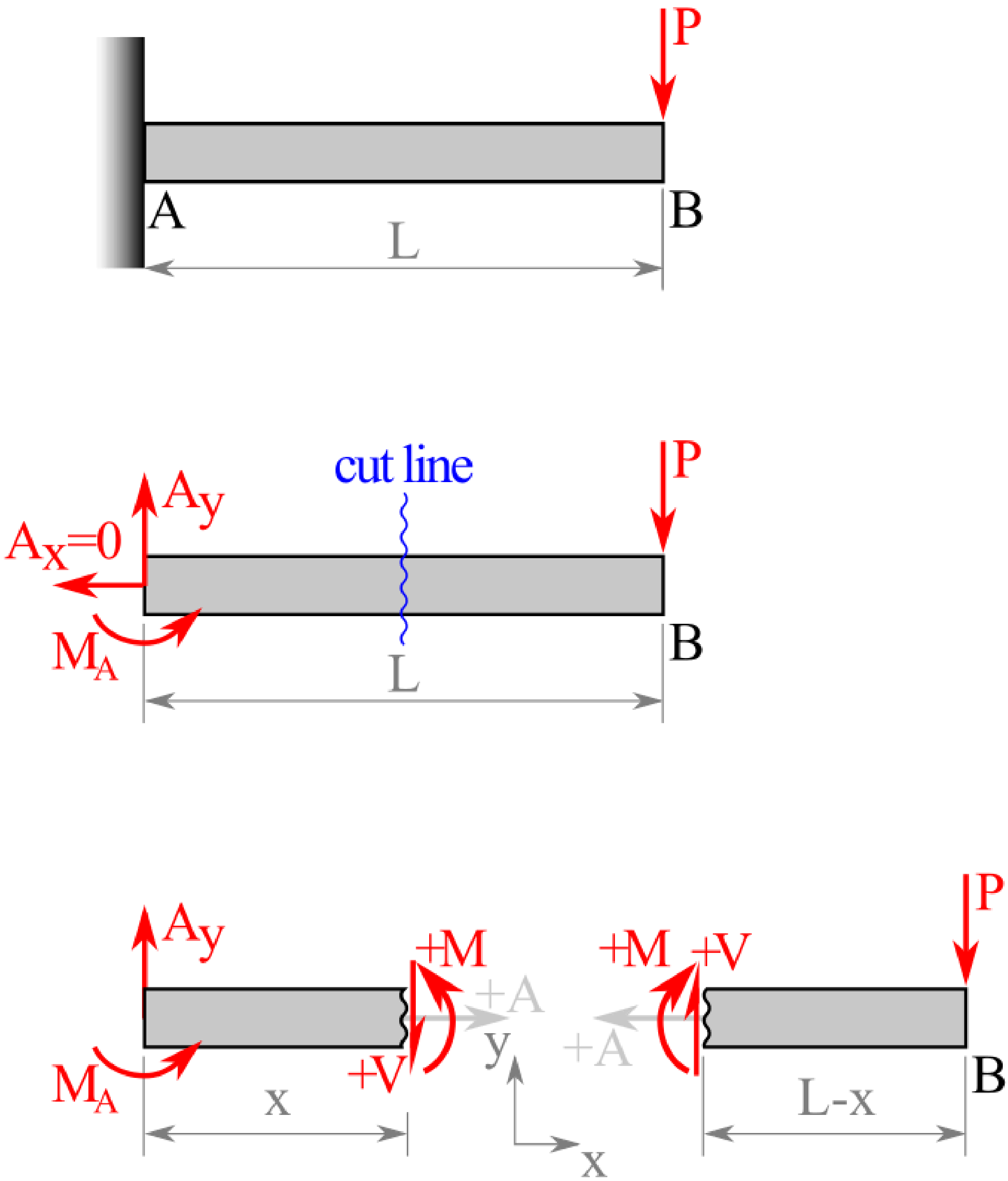


Figure JJ - (top) A cantilever beam *AB* with a vertical load P applied at its free end B can be converted to an FBD (middle). As shown in the FBD, AB has a single loading segment given that there are no changes in the external loadings between A and B. To expose the internal loading the beam is cut, and the internal shear <m>\vec{V}<\m> and moment <m>\vec{M}<\m> are shown on a free body diagram of each section (bottom). The distance x indicates the distance of the cut from the left end of the beam.

While we could use either the right or left segment, we'll focus our calculations on the right-hand portion as it doesn't not require the us to solve for the reactions at A first. Using a standard horizontal x and vertical y axes system, the equations of equilibrium are:

\begin{align}

\Sigma F_y&=0=V-P\\

\Sigma M_{cut}&=0=-M-P(L-x)

\end{align}

where L is the length of the beam. Thus we solve for shear <m>V=P<\m> and moment as and <m>M=-P(x-L)<\m>. Hence, the shear force V has constant value over the length of the beam (as it is not a function of the distance x) and the moment M is a linear function of the position along the beam, x.

The previous example may have been relatively simple with only a single loading segment, but a beam can be both supported and loaded in a variety of different ways. Beams with changing loads can be broken into loading segments which are defined by point forces, couple-moments, or the start/end of a distributed load. The beam in Figure KK has three loading segments. The equations for the shear force V and bending moment M can be found by making a cut through each loading segment.

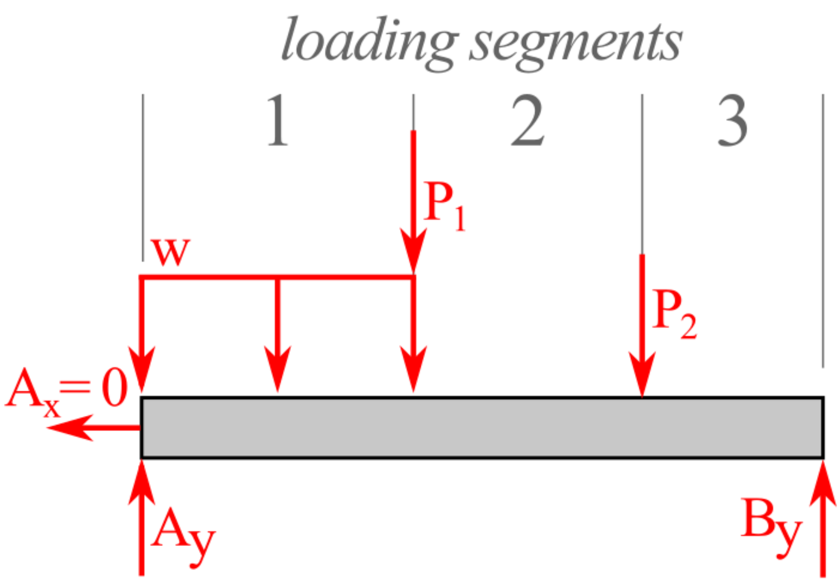


Figure KK - The FBD of a simply supported beam AB with a constant value distributed load <m>w<\m> and two vertical loads <m>P_1\text{ and } P_2<\m>. Cut through the middle of any segment to solve for the shear and moment values in that segment given segment.

The major difference between cutting a beam to solve for the internal loads at a single location (which we covered in the previous section) and cutting the beam to find the equations for the internal loads across each loading segment, is that the loading segment equations will be based upon a variable length distance (often x). Hence, if we perform an imaginary cut through loading segment 2, we can draw a new free body diagram (of the beam left of the cut) now with the internal shear and moment shown in Figure LL.

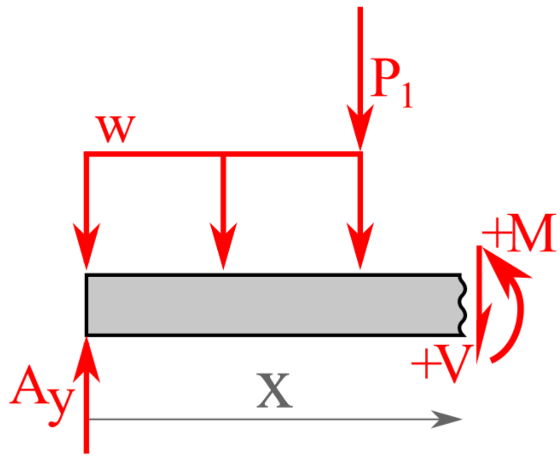


Figure LL - Free body diagram of a section of the beam from Fig.KK to the left of the cut location including the assumed positive shear and moment internal loads. Notice that the cut location is now defined by the variable x instead of a known distance.

We bring the equations of V and M from each loading segment together to plot what are known as the shear and moment diagrams. These diagrams help us visualize the change in the values of V and M for all of the possible locations across the beam. The values (including the signs) of the shear and moment diagrams will match the values you would find when cutting beams at a single point.

Application Box - Local vs. Global Equations

When writing equations for loading segments, it will be up to you to choose either

- **global equations** ≡ with all segments using the same origin and axis system often located at the left end of the beam
- **local equations** ≡ with each segment using a local origin and axis system often located at the left end of the segment

Often using local equations is easier as you can simply use the variable 'x' in your equations (as opposed to 'x + constant' and you do not have to project your y-intercept values back to an axis system which is not adjacent to the segment.

https://www.geogebra.org/m/chssp2ya

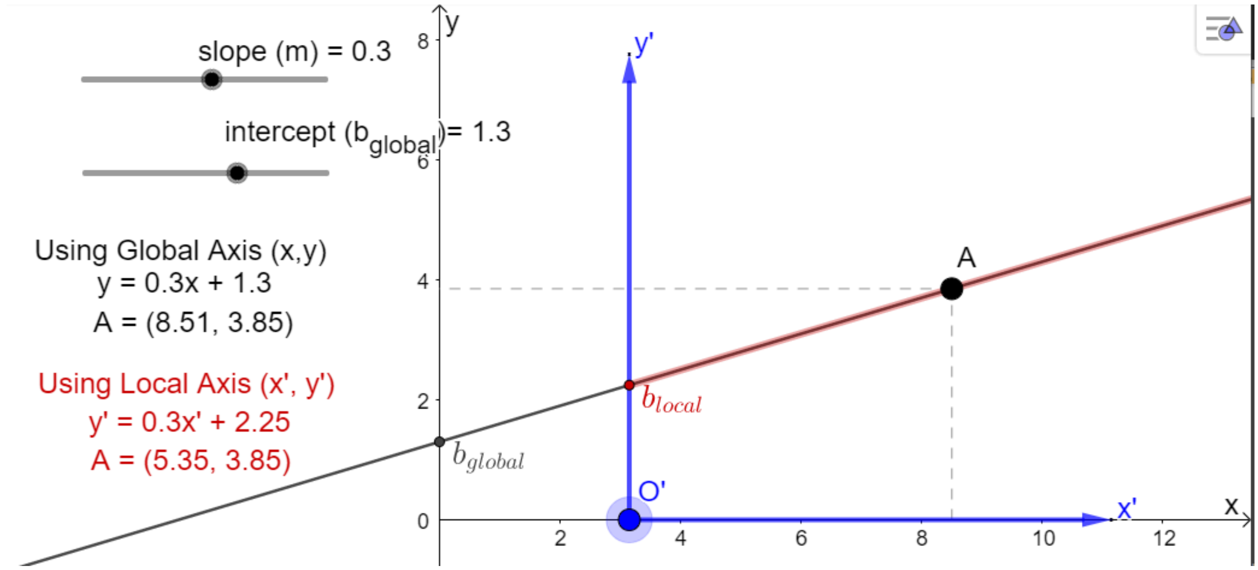


Figure MM - Interactive demonstrating the difference between global and local axes for (1) writing an equation for a line and (2) finding the coordinates of point A (which you can slide along the line). Notice as you change the slope, global intercept, and location of point A that the only values that shift between local and global equations are the y intercept values (b_local vs b_global) and the x coordinate of A.

Graphical Method for Shear and Moment Diagrams

This section focuses on using the graphical method to create shear and moment diagrams for beams.

Learning Outcomes: Upon completion of this section you should be able to:

1. Articulate the mathematical relationship between load, shear, and moment.
2. Construct a shear diagram from the graphical representation of the beam load.
3. Construct a moment diagram from the graphical representation of the shear.

Derivation of Relationship Between Loading, Shear and Moment

Suppose that we have a simply supported beam upon which there is an applied load w(x) which is distributed on the beam by some function of position, x, as shown in Fig. X.

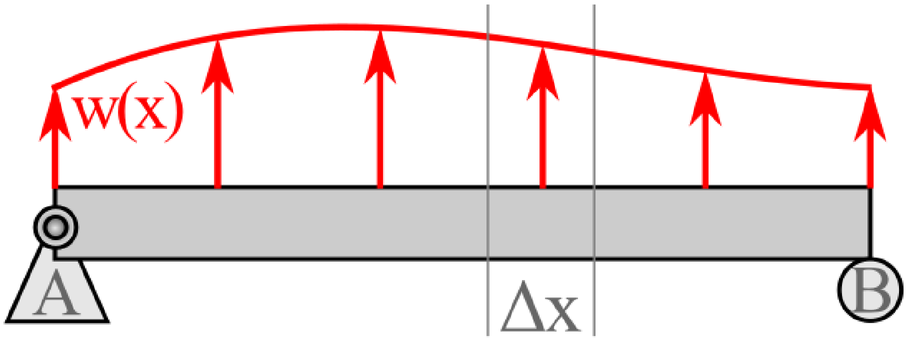
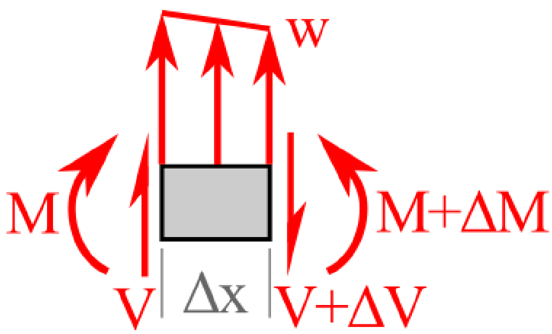


Figure NN - A simply supported beam with a distributed load that is a function of beam position <m>w(x)<\m>.

If we select a small section of this beam to look at closely, we can assume a free body diagram such as shown in Fig. OO.



[caption] Figure OO - A free body diagram of a small section of the beam with a width of <m>\Delta x<\m> Given that we assume <m>\Delta x<\m> is infinitely short, we can also assume that the distributed load value is constant over this small distance.

Apply force equilibrium in the vertical direction gives allows us to make the following derivation:

\begin{align}
\sum F_y&=0\\
V+w(\Delta x)-(V+\Delta V)&=0\\
\frac {\Delta V}{\Delta x}&=w\\
\lim_{\Delta x \to\ 0} \frac {\Delta V}{\Delta x}\Rightarrow \frac {dV}{dx}=w
\end{align}

$$\sum F_y = 0$$

$$V + w(\Delta x) - (V + \Delta V) = 0$$

$$\frac{\Delta V}{\Delta x} = w$$

$$\lim_{\Delta x \to\ 0} \frac{\Delta V}{\Delta x} \Rightarrow \frac{dV}{dx} = w$$

This final equation tells us that, at a given location, **the slope of the shear V is the value of the loading w.** Furthermore, if we multiply both sides by dx, we can integrate to find that

<me>\Delta V=\int w\ dx<\me>

$$\Delta V = \int w\ dx$$

In words, this equation says that over a given distance, **the change in the shear V is the area under the loading curve, w**.

Now looking at the internal bending moments on the FBD in Figure OO, when we apply moment equilibrium about the centroid G of the element:

\begin{align}
\sum M_G&=0\\\\
-\frac {\Delta x}{2}V-\frac {\Delta x}{2}(V+\Delta V)-M+(M+\Delta M)&=0\\\\
\frac {\Delta M}{\Delta x}&=\frac{1}{2}(2V+\Delta V)\\
\lim_{\Delta x \to\ 0} \frac {\Delta M}{\Delta x}\Rightarrow \frac {dM}{dx}=V
\end{align}

$$\sum M_G = 0$$

$$-\frac{\Delta x}{2}V - \frac{\Delta x}{2}(V + \Delta V) - M + (M + \Delta M) = 0$$

$$\frac{\Delta M}{\Delta x} = \frac{1}{2}(2V + \Delta V)$$

$$\lim_{\Delta x \to\ 0} \frac{\Delta M}{\Delta x} \Rightarrow \frac{dM}{dx} = V$$

This final equation tells us that, **the slope of the moment diagram is the value of the shear.** Furthermore, if we multiply both sides by dx, we can integrate to find that

<me>\Delta M=\int V\ dx<\me>

$$\Delta M = \int V\ dx$$

In words, this equation says that over a given segment, **the change in the moment value is the area under the shear curve**.

Hence, your loading, shear, and moment are related directly by the basic derivatives and integrals that you learned in Calculus I. These relationships are summarized in Table cc below.

Table cc - Summarizing the derivations above we find the following relationships between loading <m>w<\m>, internal shear force <m>V<\m>, and internal bending moment <m>M<\m>.

Relating	Derivatives (slopes)	Integrals (areas)
<m>\text{loading }w\Leftrightarrow \text{shear }V<\m>	<men>\frac {dV}{dx}=w<\m> $\frac{dV}{dx} = w$ the slope of the shear diagram is the value of the loading	<men>\Delta V=\int w\ dx<\men> $\Delta V = \int w \ dx$ the change in the shear diagram is the area under the loading w
<m>\text{shear }V\Leftrightarrow \text{moment }M<\m>	<men>\frac {dM}{dx}=V<\men> $\frac{dM}{dx} = V$ the slope of the moment diagram is the value of the shear	<men>\Delta M=\int V\ dx<\men> $\Delta M = \int V \ dx$ the change in the moment value is the area under the shear curve

Application of Graphical Technique for Shear and Moment Diagrams

This graphical technique is often the fastest and most visually intuitive way to draw shear and moment diagrams. This technique is the visual application of the basic calculus relationships as you integrate from load (w) to shear (V) to moment (m) (and would take derivatives the other direction).

To summarize, as we look at the shear and moment graphs we categorize three different graphical elements which relate the loading, shear, and moment of each diagram: jumps, slopes, and areas:

- **Jumps** ≡ vertical changes in shear and moment diagrams
- **Slopes** ≡ gradual changes in shear and moment diagrams

- **Areas** ≡ change of shear is the area under the load and the change of moment is the area under the shear

Table dd below focuses on why jumps and steps exist. For these relationships to work ALL diagrams must be drawn Left to Right

Table dd - Graphical relationships between loading, shear, and moment of a beam as they relate to jumps, slopes, and areas for each diagram.

Internal Load	Jumps	Slopes	Areas
Shear	Concentrated forces jump shear the same direction and amount ↑ F = ↑ V jump ↓ F = ↓ V jump	Value of load is the slope of the shear + W (↑) = +V slope (↗) − W (↓) = − V slope (↘)	The change in the shear is the area under a distributed loading curve \Delta V=\int w\ dx $\Delta V = \int w\ dx$
Moment	Concentrated couple moments jump moment, either: 1. Opposite the right-hand rule or 2. When you draw couple arrow on the left (like ⤹ or ↻), the arrowhead is in jump direction +⤹ couple = ↓ M jump − ↻ couple = ↑ M jump	Value of shear is the slope of the moment + V = + M slope (↗) − V = − M slope (↘)	The change in the moment is the area under a shear curve \Delta M=\int V\ dx $\Delta M = \int V\ dx$

Use the following interactives Figures PP-??) to practice your drawing of shear and moment diagrams of Diagram interactives.

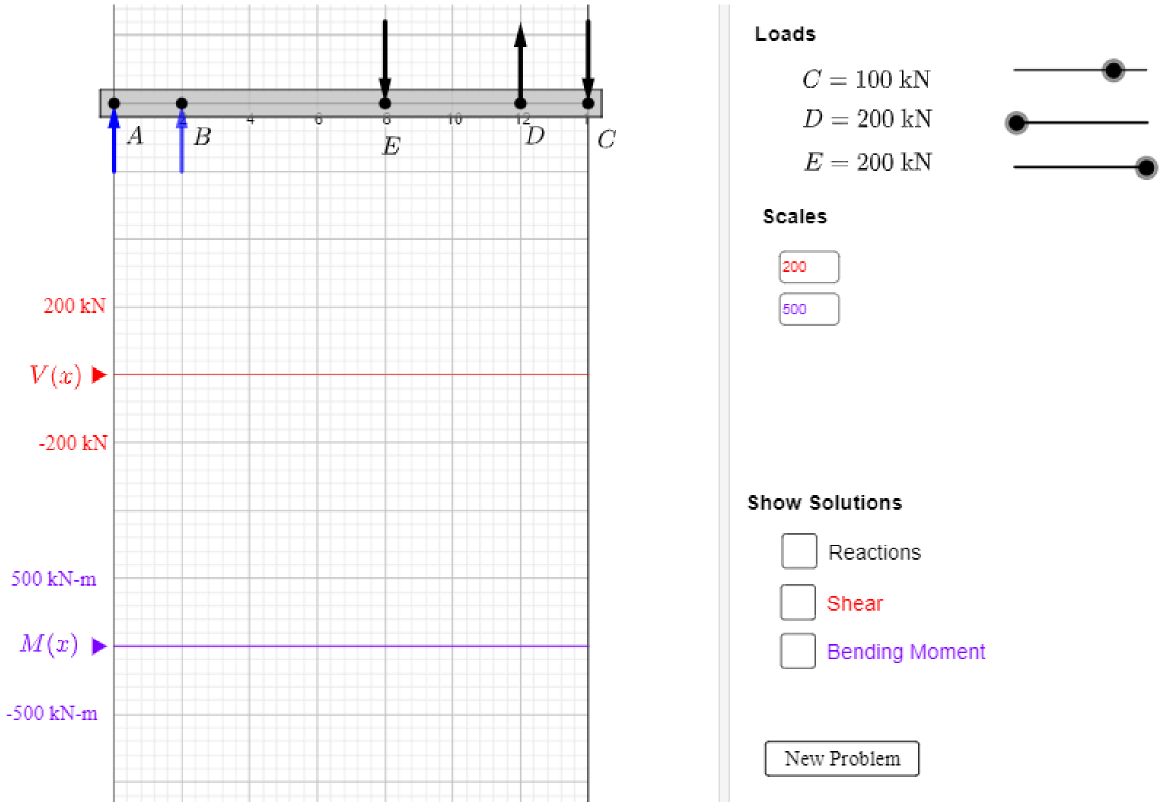


Figure PP - This interactive focuses on the shear and moment diagrams of a beam subject to concentrated forces. Points A and B are the supports and C, D, and E are external forces in kN. The position of each load and length of the beam is shown along the bottom of the beam in units of meters. To create a new problem, you can either move the locations of A-E and adjust the magnitude sliders or if you click 'New Problem', the interactive will generate a new random problem. After viewing a few diagrams to think about how they work, turn off the solutions, generate a new problem, and try to solve it before checking your work with the solutions. Source: https://www.geogebra.org/m/qnhnawam

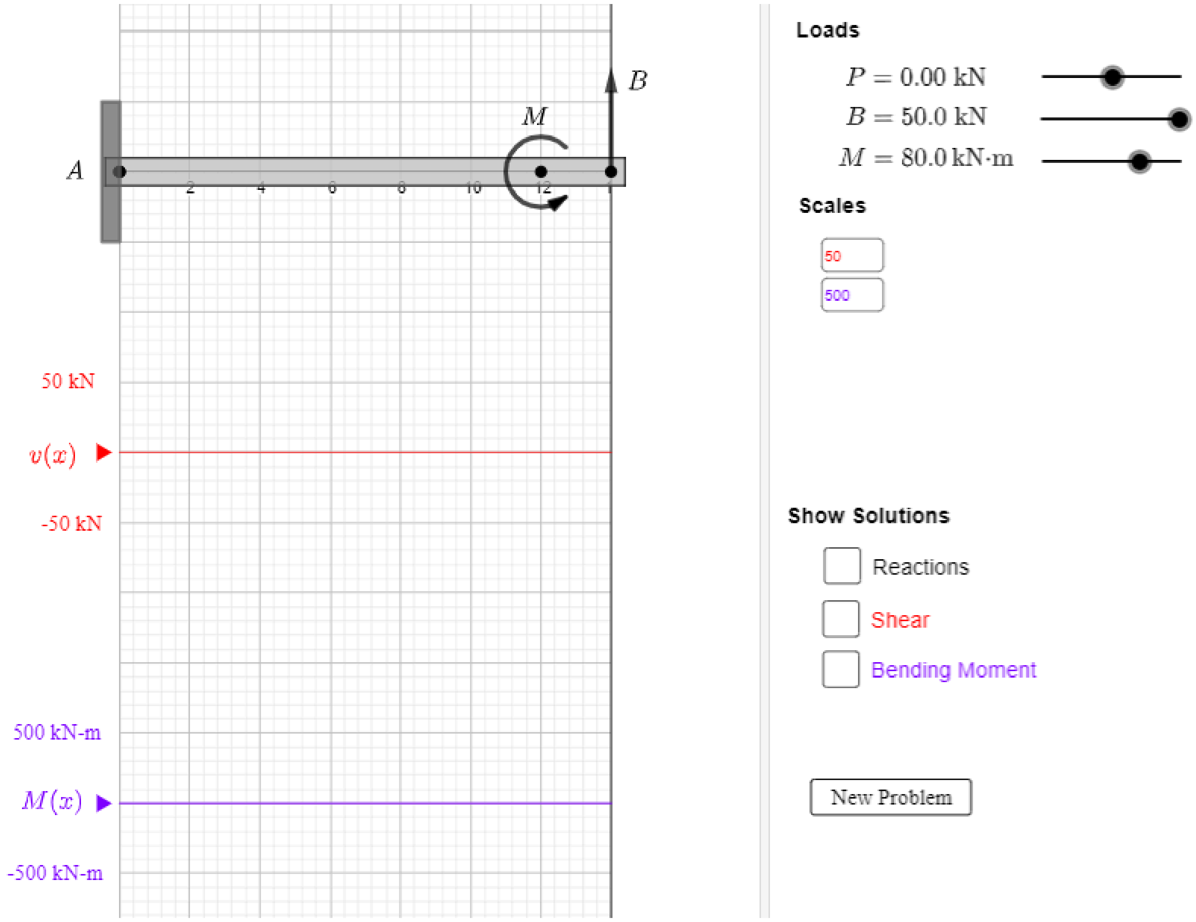


Figure QQ - This interactive focuses on the shear and moment diagrams of a cantilever beam subject to a couple-moment M and two concentrated forces, P and B. Point A is a fixed (rigid) support. The position of each load and length of the beam is shown along the bottom of the beam in units of meters. To create a new problem, you can either move the locations of P, B, and M and adjust the magnitude sliders or if you click ‘New Problem’ the interactive will generate a new random problem. After viewing a few diagrams to think about how they work, turn off the solutions, generate a new problem, and try to solve it before checking your work with the solutions. Source: https://www.geogebra.org/m/wzynat8x

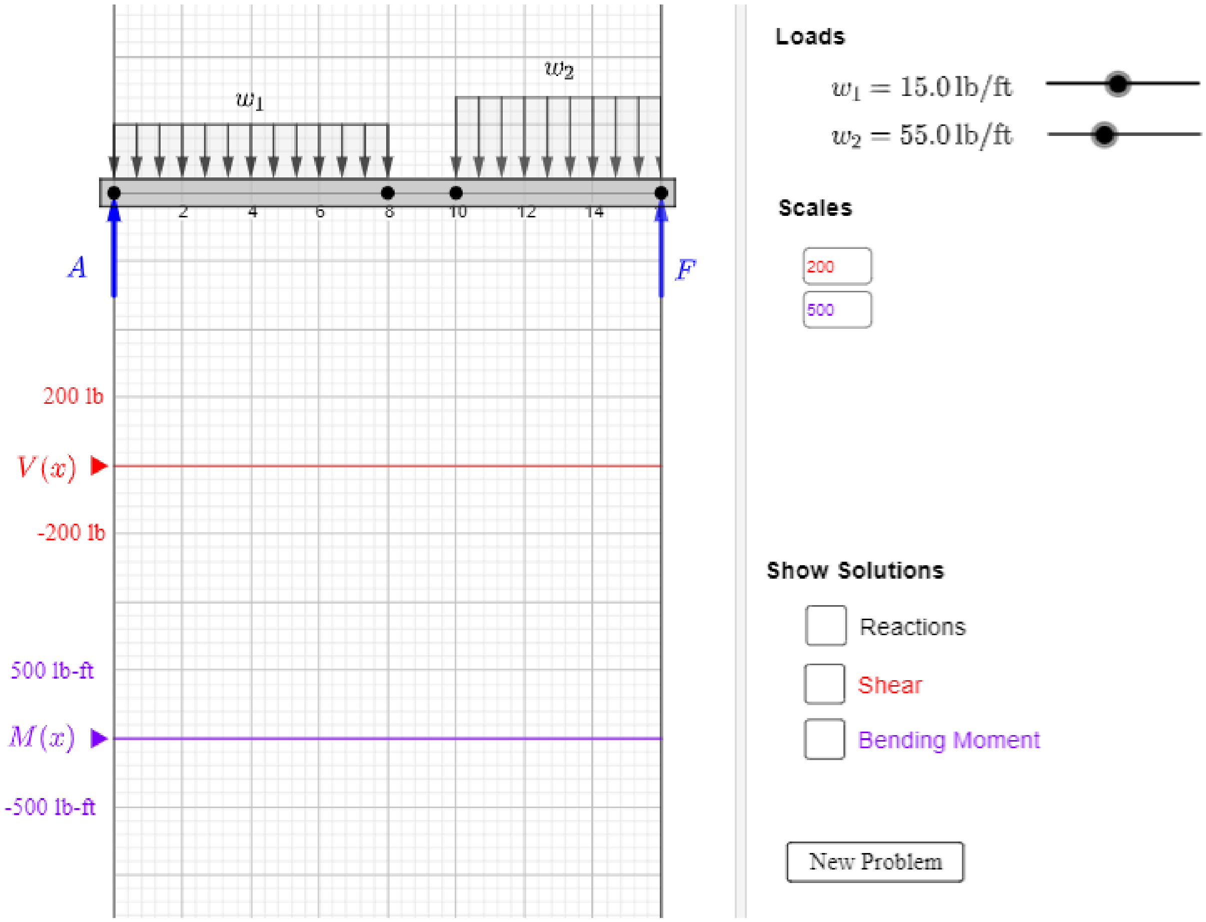


Figure RR - This interactive focuses on the shear and moment diagrams of a simply-supported beam subject to two uniform value distributed loads, <m>w_1\text{ and }w_2<\m> in pounds-force per foot (lb/ft). Forces A and F are the reaction forces. The position of each load and length of the beam is shown along the bottom of the beam in units of feet. To create a new problem, you can either move the locations of the distributed loads endpoints and adjust the magnitude sliders, or if you click ‘New Problem’ the interactive will generate a new random problem. After viewing a few diagrams to think about how they work, turn off the solutions, generate a new problem, and try to solve it before checking your work with the solutions. Source:https://www.geogebra.org/m/q7zdsmu3

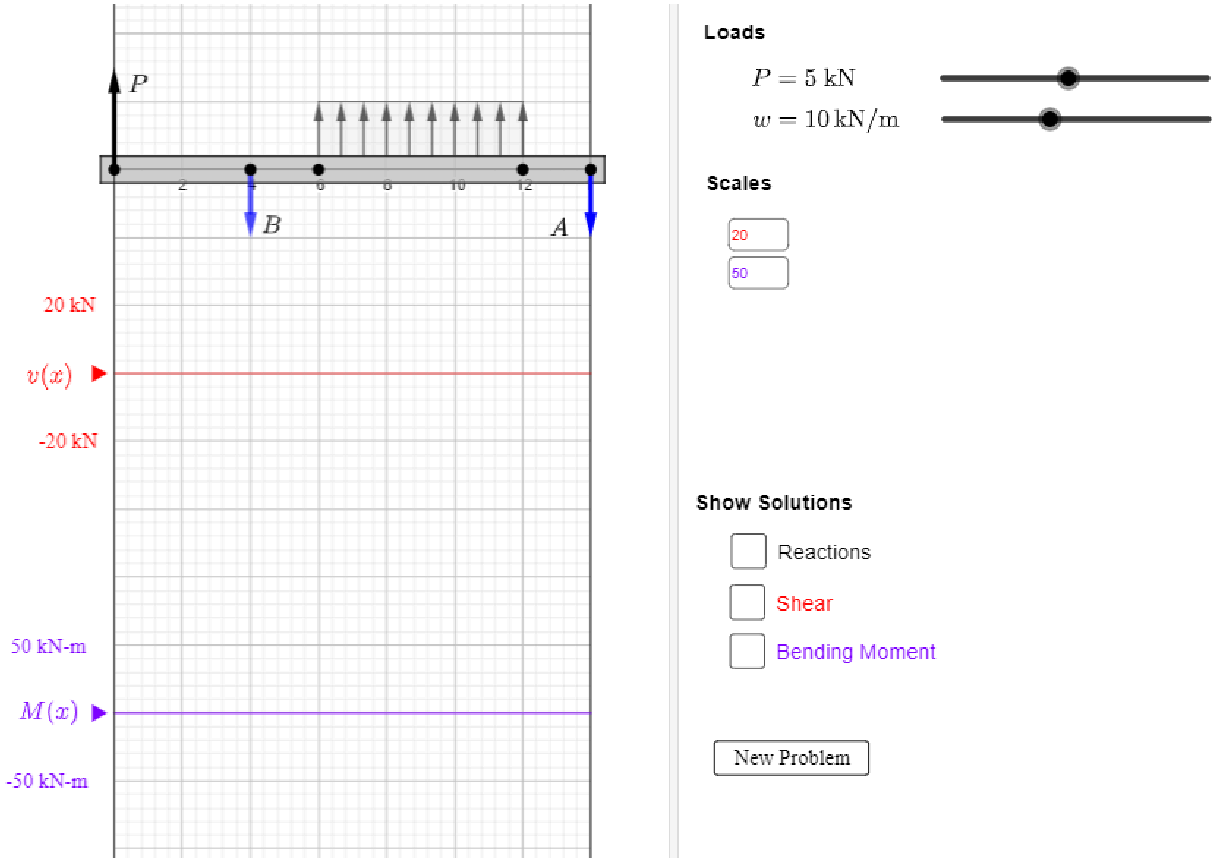


Figure SS - This interactive focuses on the shear and moment diagrams of a beam subject to one concentrated force <m>P<\m> and a uniform value distributed load, <m>w<\m>. Forces A and B are vertical reactions. The position of each load and length of the beam is shown along the bottom of the beam in units of meters. To create a new problem, you can either move the locations of the loads and adjust the magnitude sliders, or if you click 'New Problem' the interactive will generate a new random problem. After viewing a few diagrams to think about how they work, turn off the solutions, generate a new problem, and try to solve it before checking your work with the solutions. Source: https://www.geogebra.org/m/qqv7vew7

Calculus-Based Equations for Shear and Moment Diagrams

As you have learned, the graphical-based technique is based upon the application of derivatives as graphical slopes and integrals as graphical areas. However, there are times that the graphical technique falls short due to the areas that you need to find being more complicated than rectangles or triangles. One example is a triangular load (1st-degree polynomial) which integrates into a quadratic (2nd-degree polynomial) shear force distribution, and then you need to find the area under the shear to compute the cubic (3rd-degree polynomial) moment. For these types of problems, it is advisable to either use the Equations from Section Cuts or add in some calculus-based computations.

We use the same fundamental equations in this section as you used in the graphical method
<me>\Delta V=\int w\ dx and \Delta M=\int w\ dx<\me>

The difference is that instead of areas and slopes you will start by writing an equation for a load in a specific loading segment (see Table ee below) and then take an integral of that equation to find your shear force equation. Note that both integral equations above compute the change in the shear and moment, thus you will need starting values for each segment. The starting values come from either 1) the ending value of the previous segment or 2) point loads (point forces for shear diagrams and point couple moments for moment diagrams). Because of the requirement for these segment starting values, no segment can be computed in isolation from the other segments. Physically this means that the shear and moment along a beam are not just due to the loading in one segment, but instead are related to the loading on the rest of the beam as well.

Table ee - Description of the equations for various loading types in a specific loading segment. Note that the top two loading cases (point and uniform distributed loads) can easily be handled with the graphical method and do not require the use of the calculus-based method.

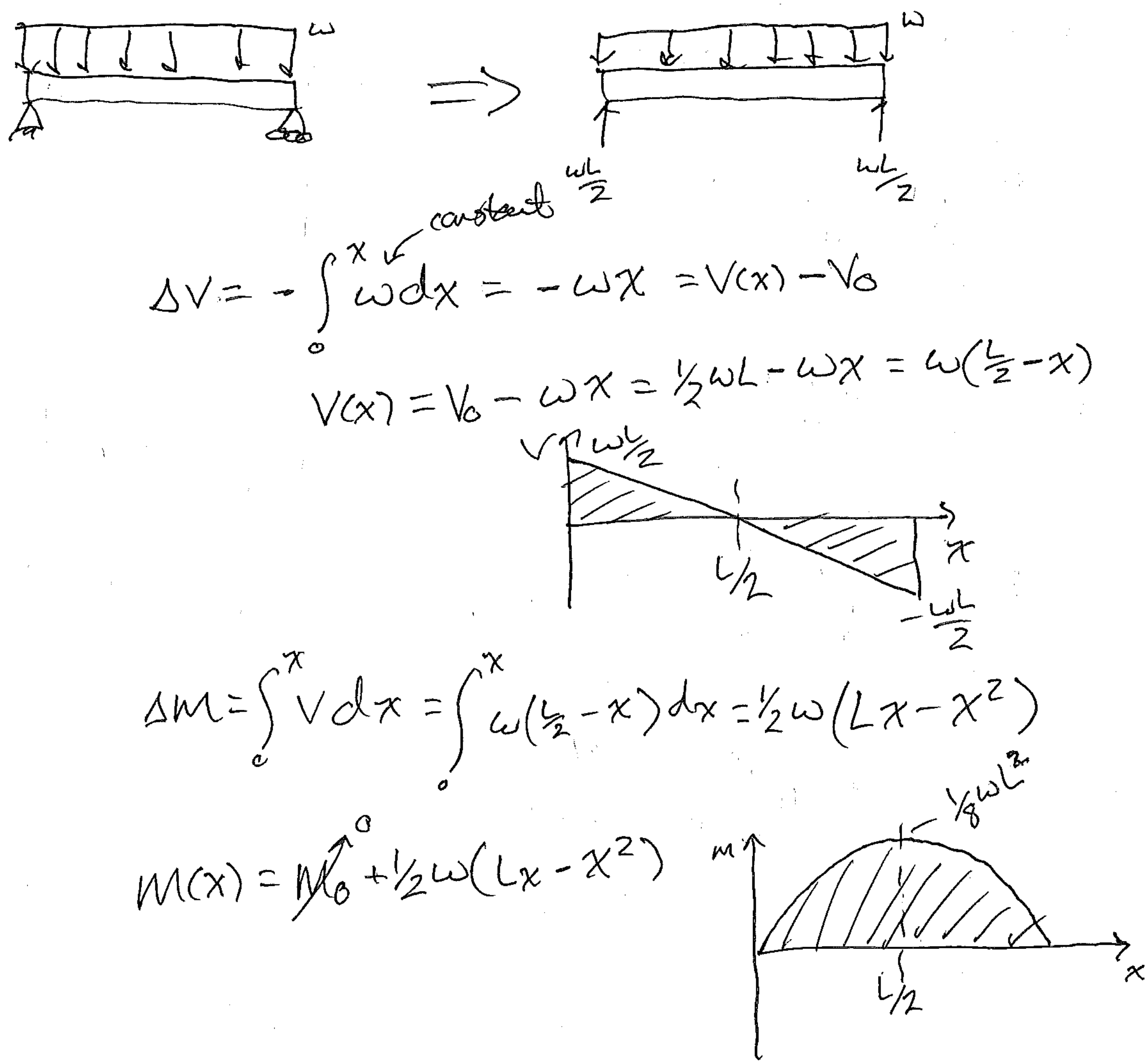
Loading type in a segment	**Equation**	**Description**
Spaces between point loads	<me>w=0<\m>	No load between point loads will result in no change in the shear (V) between point loads (only happens when self-weight of the beam is ignored)
Uniform distributed loads (rectangular area)	<me>w =\text{ constant value)<\me>	Constant value is negative (-) if the load is downward (↓) and positive (+) if the load is upward (↑)
Linearly changing loads (triangular or trapezoidal area)	<me>w = mx + b<\me>	Equation defined by the arrow tips of the distributed load[1] m = slope (rise/run) x = distance from datum (global or local to segment, see Global vs Local functions for segment equations above) b = y-intercept @ x = 0
Any other loading distribution	<me>w = f(x)<\me>	Equation defined by the arrow tips of the distributed load[1]

[1]As most gravitational distributed loads are drawn with the arrows pointing down and resting on the beam, slide distributed loads along their line of action so that their tails are on the beam and their tips define your equation.

Application of the Calculus-based Method for Shear and Moment Diagrams

Now

1. You can either start with this method from the start of your problem or, to be more time-efficient, use the graphical method until the point you need to find areas of more complicated shapes than rectangles and triangles.
2. You will need to have solved the loading segment to the left of your desired segment.
3. Write an equation for the loading w in the segment (see table XX above)
4. Integrate the loading equation <m>w(x)<\m> to find the change in the shear V and include the shear value at the beginning of your loading segment including the influence of any point loads at that location (equivalent to the integration constant)
5. Integrate the shear equation <m>w(x)<\m> to find the change in the bending moment M and include the moment value at the beginning of your loading segment including the influence of any point couple moments at that location (equivalent to the integration constant)
6. To find maximum values
    a. For shear, use <m>V(x)<\m> equation to compute
    b. For moment, use <m>M(x)<\m> to compute. If the value of the maximum moment is in the middle of a loading segment (often where the shear value is zero), set the <m>V(x)=0<\m>, solve for x and then put that distance value into the moment <m>M(x)<m> equation. See the example below)



Chapter summary

You have likely already realized that engineering (and life) that there can be multiple ways to solve a problem. The four different techniques to compute internal loadings are a demonstration of this design flexibility. Feel free to use Table ff below to help you think through the advantages and disadvantages of each method. In the end, the decision of which method to use is often up to you, and the better you know each method, the easier it will be to choose the one which is both applicable and computationally efficient.

[table caption] Table ff - Summary of the methods to solve for internal loadings along with the advantages and disadvantages of each.

Method	Description	Advantages	Disadvantages
Section Cut to find Internal Loadings at a Single Point	Cut a rigid body at a specific location to expose and then solve for the internal loadings at that location	is computationally efficient works for any shape rigid body (not just beams) takes advantage of tools you have already learned in previous chapters	requires knowledge of specialized sign conventions for internal loadings only reveals internal loading values at a single point does not reveal all values of shear and moment for a body
Section Cut Equations for Shear and Moment Diagrams	Break a beam into loading segments and develop equations for shear and moment using a cut through each se	allows computation without calculus computations or knowledge	requires knowledge of specialized sign conventions for internal loadings is computationally time-intensive computational steps lack conceptual cross-checks
Graphical Method for Shear and Moment Diagrams	Use mathematical relationship among loading, shear, and moment to graphically create shear and moment diagrams	requires only simple computations to support conceptual relationships conceptual nature allows many cross-checks for accuracy handles point loads, uniform distributed loads, and couple moments	requires solid calculus knowledge of slopes and areas related to derivatives and integrals requires you learn a few simple graphical rules and that you work from left to right
Calculus-Based Equations for Shear and Moment Diagrams	Use mathematical relationship among loading, shear, and moment to develop equations for shear	allows shear and moment diagram computation of complicated loading distributions (e.g.	requires accurate computation of derivatives and integrals

	and moment diagrams	non-uniform distributed loads)	requires you learn a few simple graphical rules for concentrated forces and couple moments

Application Box: What should be included a complete shear and moment diagram problem?

Now as you set off to solve your own shear and moment diagrams, keep in mind all the elements that need to be included.

1. (If reactions are unsolved) A FBD with distributed loads turned into equivalent concentrated loads and the computations shown to find the reactions.
2. Large (~4 inches wide) and legible diagrams, including:
    a. A separate (from element 1) 'true loading' FBD of the beam with the actual reaction forces/couples and given loads.
    b. Shear and moment diagrams drawn directly below the 'true loading' FBD indicating the correct values (zero, uniform, linear, or polynomial) and the polynomial sections with correct concavity.
3. Values of V&M for EVERY point of inflection and max/min on the diagram.
4. Relevant distances (if they are not directly interpretable from the diagrams above).
5. Enough work to justify your results related to your chosen method(s) of Equations from Section cuts, Graphical, Calculus-based methods.