ENGINEERING MECHANICS

UNIT-IV

MOMENT OF INERTIA

Introduction:

Consider an elemental area A_i.

Let x_i = distance of C.G. of area A_i from Y-axis.

y_i = distance of C.G. of area A_i from X-axis

Then moment of area about Y-axis = Area × perpendicular distance of C.G. from Y-axis

$$= A_i x_i$$

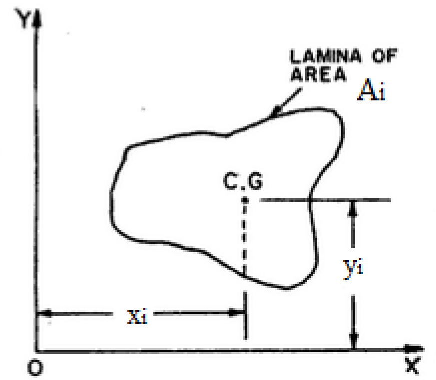


Fig. 4.1

This is known as first moment of area about Y-axis. This first moment of area is used to determine the centre of gravity of the area.

If the first moment of area is again multiplied by the perpendicular distance between the C.G. of the area and Y-axis, then the quantity $A_i x_i \times x_i = A_i x_i^2$ is known as moment of the moment of area or second moment of area or area moment of inertia about Y-axis.

Similarly the moment of area about X-axis is $A_i y_i$ and second moment of area about X-axis is $A_i y_i^2$.

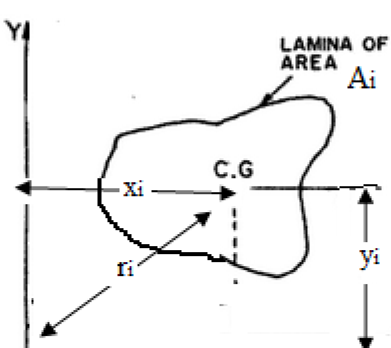
If instead of area, the mass of the body is taken into consideration then second moment is known as second moment of mass or mass moment of inertia.

Area Moment of Inertia:

Definition: The product of area and the square of the distance of the centre of gravity from an axis is known as moment of inertia of the area about that axis.

Moment of inertia is represented by I. Hence moment of inertia about X-axis is represented by I_{xx} and is given by $I_{xx} = \sum A_i y_i^2$. The moment of inertia about Y-axis is represented by I_{yy} and is given by $I_{yy} = \sum A_i x_i^2$.

The moment of inertia is a fourth dimensional term since it is a term obtained by multiplying area by the square of the distance. Hence in S.I. units, if meter is the unit for linear measurements the m^4 is the unit of moment of inertia. If *mm* is



used for linear measurements the mm^4 is the unit for moment of inertia.

Polar Moment of Inertia:

Moment of inertia about an axis perpendicular to the plane of an area is known as polar moment of inertia. It may be denoted as J or I_{zz}.

Thus the moment of inertia about an axis perpendicular to the plane of area at O is called polar moment of inertia at point O and is given by J or $I_{zz} = \sum A_i r_i^2$

Radius of Gyration:

Radius of gyration of a body about an axis is a distance such that its square multiplied by the area gives moment of inertia about the given axis.

Thus mathematically it is defined by the relation $I = AK^2$

$$\therefore K = \sqrt{\frac{I}{A}}$$

Where K = radius of gyration

I = moment of inertia and

A = Area of cross section.

Perpendicular Axis Theorem:

The moment of inertia of an area about an axis perpendicular to its plane (polar moment of inertia) at any point O is equal to the sum of moments of inertia about any two mutually perpendicular axes through same point O and lying in the plane of the area.

If z-z is the axis normal to the plane passing through the point O then as per this theorem $I_{zz} = I_{xx} + I_{yy}$.

Proof:

Let us consider an elemental area dA at a distance of r from O then from the definition

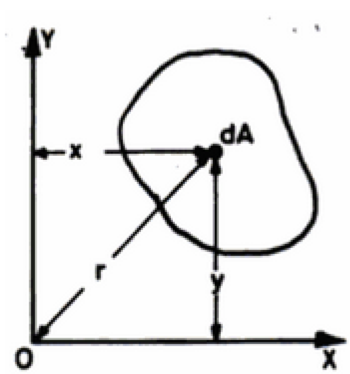


Fig. 4.3

$$I_{zz} = \sum r^2 dA$$

$$= \sum\left(x^2 + y^2\right)dA$$

$$= \sum x^2\, dA + \sum y^2\, dA$$

$$= x^2 A + y^2 A = I_{yy} + I_{xx}$$

$$\therefore I_{zz} = I_{xx} + I_{yy}$$

Parallel Axis Theorem or Transfer Theorem:

Moment of inertia about any axis in the plane of an area is equal to the sum of moment of inertia about a parallel centroidal axis and the product of area and the square of the distance between the two parallel axes.

By this theorem $I_{AB} = I_{GG} + Ay_c^2$

Where I_{AB} = moment of inertia about axis AB.

I_{GG} = moment of inertia about centroidal axis GG parallel to AB

A = the area of the plane figure and

y_c = the distance between axis AB and parallel centroidal axis GG.

Proof:

Consider an elemental parallel strip of area dA at a distance of y from centroidal axis

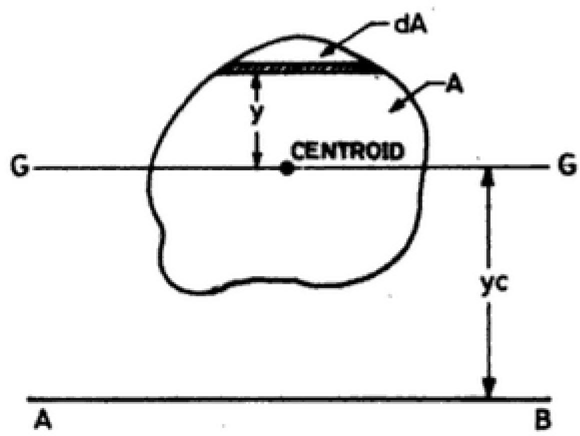


Fig. 4.4

Then, $I_{AB} = \sum \left(y + y_c\right)^2 dA$

$$= \sum\left(y^2 + 2yy_c + y_c^2\right)dA$$

$$= \sum y^2 dA + \sum 2yy_c dA + \sum y_c^2 dA$$

Now $\sum y^2 dA$ = moment of inertia about axis GG = I_{GG}

$$\sum 2yy_c dA = 2y_c \sum y\, dA \times \frac{A}{A}$$

$$= 2y_c A \sum \frac{y\, dA}{A}$$

In the above term $2y_c A$ is constant and $\sum \frac{y\, dA}{A}$ is the distance of the centroid from reference axis GG.

Since GG is passing through the centroid itself $\sum \frac{y\, dA}{A}$ is zero.

Hence the term $\sum 2yy_c dA$ becomes zero.

Now the third term $\sum y_c^2\, dA = y_c^2 \sum dA = A\, y_c^2$

$$\therefore I_{AB} = I_{GG} + Ay_c^2$$

Note: The above equation cannot be applied to any two parallel axes. One of the axes must be centroidal axis only.

Moment of Inertia of standard sections:

i) Moment of Inertia of a Rectangle:

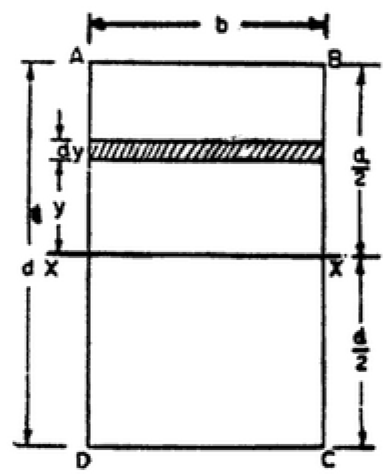
a) About its centroidal axis

Consider a rectangular section ABCD having width = b and depth = d. Let X-X is the horizontal axis passing through the C.G. of the rectangular section. The moment of inertia of the given section about X-X axis is represented by I_{xx}.

Consider a rectangular elementary strip of thickness dy at a distance y from X-X axis.

Area of the strip = $b \cdot dy$

Moment of inertia of the area of the strip about X-X axis = Area of strip × y^2



$$= (b.dy) \times y^2 = by^2 dy$$

Fig. 4.5

Moment of inertia of the whole section will be obtained by integrating the above equation between the limits $-\frac{d}{2}$ to $\frac{d}{2}$.

$$\therefore I_{xx} = \int_{-d/2}^{d/2} by^2 dy = b \int_{-d/2}^{d/2} y^2 dy$$

$$= b\left[\frac{y^3}{3}\right]_{-d/2}^{d/2} = \frac{b}{3}\left[\left(\frac{d}{2}\right)^3 - \left(-\frac{d}{2}\right)^3\right]$$

$$= \frac{b}{3}\left[\left(\frac{d^3}{8}\right) - \left(-\frac{d^3}{8}\right)\right] = \frac{b}{3}\left[\frac{d^3}{8} + \frac{d^3}{8}\right]$$

$$\therefore I_{xx} = \frac{b}{3}.\frac{2d^3}{8} = \frac{bd^3}{12}$$

Similarly, the moment of inertia of the rectangular section about Y-Y axis passing through the C.G. of the section is given by

$$I_{yy} = \frac{db^3}{12}$$

Area of strip, $dA = d \times dx$

Moment of inertia of strip above Y-Y axis $= dA \times x^2$

$$= (d \times dx) \times x^2$$

$$= d \times x^2 \times dx$$

Moment of inertia of the whole section will be obtained by integrating the above equation between the limits $-\frac{b}{2}$ to $\frac{b}{2}$.

$$\therefore I_{yy} = \int_{-b/2}^{b/2} dx^2 dx = d\int_{-b/2}^{b/2} x^2 dx$$

$$= d\left[\frac{x^3}{3}\right]_{-b/2}^{b/2} = \frac{d}{3}\left[\left(\frac{b}{2}\right)^3 - \left(-\frac{b}{2}\right)^3\right]$$

$$= \frac{d}{3}\left[\left(\frac{b^3}{8}\right) - \left(-\frac{b^3}{8}\right)\right] = \frac{d}{3}\left[\frac{b^3}{8} + \frac{b^3}{8}\right]$$

$$\therefore I_{yy} = \frac{d}{3}.\frac{2b^3}{8} = \frac{db^3}{12}$$

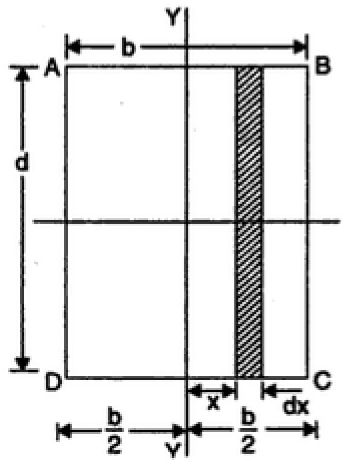


Fig. 4.6

b) About it's Base

Consider a rectangular section ABCD having width = *b* and depth = *d*. We want to find the moment of inertia of the rectangular section about the line CD, which is the base of the rectangular section.

Consider a rectangular elementary strip of thickness *dy* at a distance *y* fr

Area of the strip = *b* . *dy*

Moment of inertia of the area of the strip about X-X axis = Area of strip × y^2

$$= (b.dy) \times y^2 = by^2 dy$$

Moment of inertia of the whole section will be obtained by integrating the above equation between the limits 0 *to* d.

∴ Moment of inertia of the whole section about the line CD

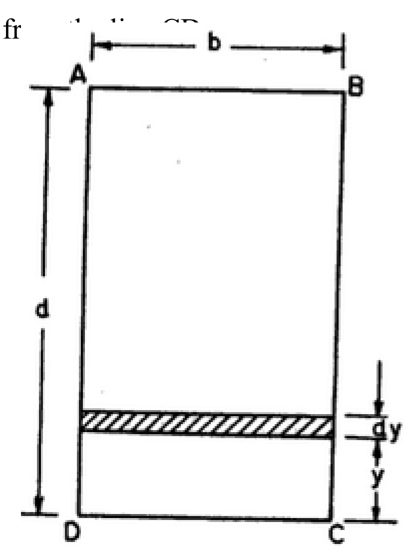


Fig. 4.7

$$= \int_0^d by^2 dy = b\int_0^d y^2 dy$$

$$= b\left[\frac{y^3}{3}\right]_0^d = \frac{bd^3}{3}$$

c) Hollow rectangular section

Consider a hollow rectangular section in which ABCD is the main section and EFGH is the cut-out section.

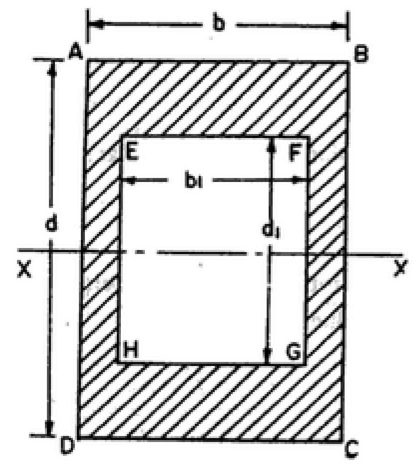


Fig. 4.8

The moment of inertia of the main section ABCD about X-X axis is given by

$$\frac{bd^3}{12}$$

Where b = width of main section and

d = depth

The moment of inertia of the cut-out section EFGH about X-X axis is given by

$$\frac{b_1 d_1^{\ 3}}{12}$$

Where b_1 = width of the cut-out section and

d_1 = depth of the cut-out section.

$\therefore I_{xx}$ = Moment of inertia of rectangle ABCD – Moment of inertia of rectangle EFGH about X-X axis

$$= \frac{bd^3}{12} - \frac{b_1 d_1^{\ 3}}{12}$$

ii) Moment of inertia of right angled triangle:

Consider a triangle AOB of base width = 'b' and height = 'h'. Consider a small strip of thickness dy at a distance y from X-axis.

Area of the strip, dA = Length $DE \times dy$ …. (i)

Considering two similar triangles ADE and AOB,

$$\frac{DE}{OB} = \frac{AD}{AO}$$

Where OB = b, AO = h and AD = $(h - y)$

$$\therefore \frac{DE}{b} = \frac{(h-y)}{h}$$

$$DE = \frac{b\,(h-y)}{h}$$

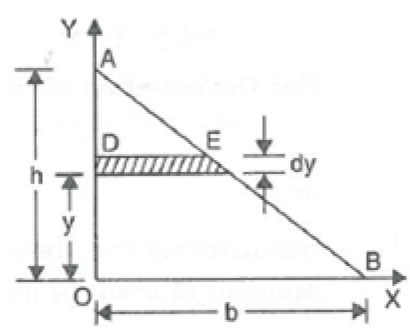


Fig. 4.9

Substituting the value of DE in equation (i), we get

Area of strip, $dA = \frac{b\,(h-y)}{h}\,.\,dy$

Moment of inertia of this strip about X-axis = Area of strip × y^2

$$= \frac{b\,(h-y)}{h}\,.\,dy \times y^2$$

The moment of inertia of the whole rectangular section about X-axis is obtained by integrating the above equation between the limits 0 and h.

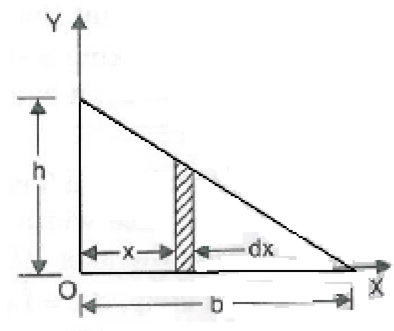


Fig. 4.10

$$\therefore I_{xx} = \int_0^h \frac{b\,(h-y)}{h}\,.\,dy \times y^2$$

$$= \frac{b}{h}\int_0^h (h - y) \times y^2 dy$$

$$= \frac{b}{h}\left[h\left(\frac{y^3}{3}\right)_0^h - \left(\frac{y^4}{4}\right)_0^h\right]$$

$$= \frac{b}{h}\left[\frac{h^4}{3} - \frac{h^4}{4}\right]$$

$$\therefore I_{xx} = \frac{b}{h} \times \left[\frac{h^4}{12}\right] = \frac{bh^3}{12}$$

Similarly, taking a strip of thickness dx parallel to Y-axis at a distance x from the Y-axis, it can be proved that moment of inertia about Y-axis is

$$I_{xx} = \frac{hb^3}{12}$$

About centroidal axis:

Consider a triangular section of base = b and height =h. Let X axis passing through the C.G. of the triangular section and parallel to the base.

The distance between the C.G. of the triangular section and base AB = $\frac{h}{3}$.

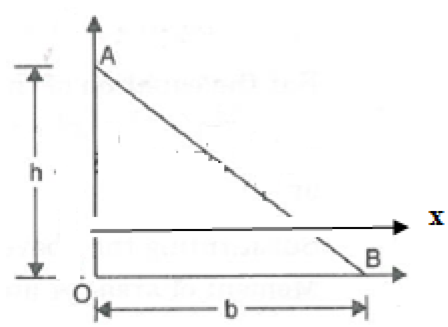


Fig. 4.11

Now, from the theorem of parallel axis, we have

Moment of inertia about base = Moment of inertia about C.G. + Area× (Distance between X-X and BC)^2

$$I_{OB} = I_{xx} + A \times \left(\frac{h}{3}\right)^2$$

$$\therefore I_{XX} = I_{OB} - A \times \left(\frac{h}{3}\right)^2$$

$$= \frac{bh^3}{12} - \left(\frac{b\times h}{2}\right).\left(\frac{h}{3}\right)^2$$

$$= \frac{bh^3}{12} - \frac{bh^3}{18}$$

$$\therefore I_{XX} = \frac{bh^3}{36}$$

Similarly moment of inertia of a right angled triangular section about centroidal Y-Y axis is

$$I_{YY} = \frac{hb^3}{36}$$

iii) Moment of inertia of isosceles Triangle:

a) About Base

Consider a triangle ABC of base width = 'b' and height = 'h'. Consider a small strip of thickness dy at a distance y from X-axis.

Area of the strip, dA = Length $DE \times dy$ …. (i)

Considering two similar triangles ADE and AOB,

$$\frac{DE}{OB} = \frac{AD}{AO}$$

Where OB = b, AO = h and AD = $(h - y)$

$$\therefore \frac{DE}{b} = \frac{(h-y)}{h}$$

$$DE = \frac{b\,(h-y)}{h}$$

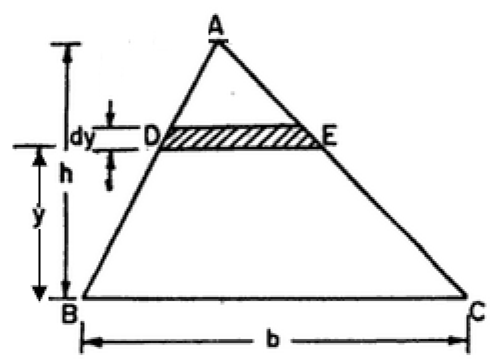


Fig. 4.12

Substituting the value of DE in equation (i), we get

Area of strip, $dA = \frac{b\,(h-y)}{h}\,.\,dy$

Moment of inertia of this strip about the base = Area of strip × y^2

$$= \frac{b\,(h-y)}{h}\,.\,dy\times y^2$$

The moment of inertia of the whole rectangular section about the base is obtained by integrating the above equation between the limits 0 and h.

$$\therefore I_{BC} = \int_0^h \frac{b\,(h-y)}{h}\,.\,dy\times y^2$$

$$= \frac{b}{h}\int_0^h (h - y) \times y^2 dy$$

$$= \frac{b}{h}\left[h\left(\frac{y^3}{3}\right)_0^h - \left(\frac{y^4}{4}\right)_0^h\right]$$

$$= \frac{b}{h}\left[\frac{h^4}{3} - \frac{h^4}{4}\right]$$

$$\therefore I_{BC} = \frac{b}{h} \times \left[\frac{h^4}{12}\right] = \frac{bh^3}{12}$$

Moment of inertia about Y-axis:

Consider a small strip of thickness *dx* at a distance *x* from Y-axis.
Area of the strip, dA = Length $DE \times dx$ …. (*i*)
Considering two similar triangles ADE and AOB,

$$\frac{DE}{BC} = \frac{AE}{AC}$$

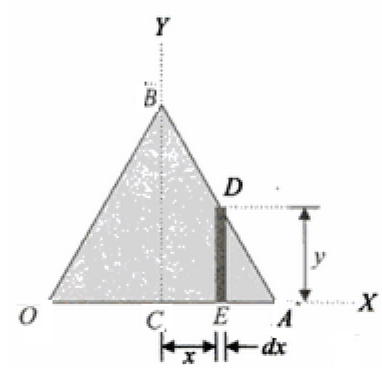


Fig. 4.13

$$\therefore \frac{DE}{h} = \frac{(b/2-x)}{b/2}$$

$$DE = \frac{h\,(b-2x)}{b}$$

Substituting the value of DE in equation (*i*), we get

Area of strip, $dA = \frac{h\,(b-2x)}{b}\,.\,dx$

Moment of inertia of this strip about the base = Area of strip × y^2

$$= \frac{h\,(b-2x)}{b}\,.\,dx \times x^2$$

The moment of inertia of the whole right angled triangular section about the base is obtained by integrating the above equation between the limits 0 and $\frac{b}{2}$.

$$\therefore I_{BC} = \int_0^{b/2} \frac{h\,(b-2x)}{b}\,.\,dx \times x^2$$

$$= \frac{h}{b}\int_0^{b/2}\left(\frac{b}{2} - x\right) \times x^2 dx$$

$$= \frac{h}{b}\left[\frac{b}{2}\left(\frac{x^3}{3}\right)_0^{b/2} - \left(\frac{b^4}{4}\right)_0^{b/2}\right]$$

$$= \frac{h}{b}\left[\frac{b^4}{24} - \frac{b^4}{32}\right]$$

$$\therefore I_{BC} = \frac{h}{b} \times \left[\frac{b^4}{96}\right] = \frac{hb^3}{96}$$

$\therefore$ M.I of entire Isosceles triangle = $2\times\frac{hb^3}{96} = \frac{hb^3}{48}$

b) About Centroidal Axis

Consider a triangular section of base = b and height $=h$. Let X-X is the axis passing through the C.G. of the triangular section and parallel to the base.

The distance between the C.G. of the triangular section and base AB = $\frac{h}{3}$.

Now, from the theorem of parallel axis, we have

Moment of inertia about BC = Moment of inertia about C.G. + Area× (Distance between X-X and BC)^2

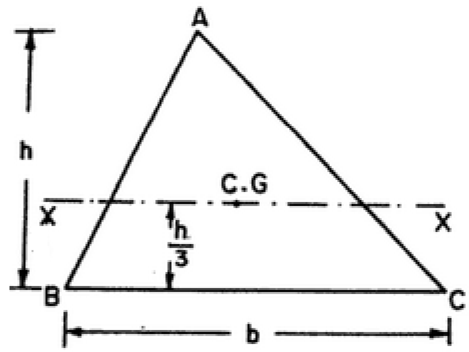


Fig. 4.14

$$I_{BC} = I_G + A\times\left(\frac{h}{3}\right)^2$$

$$\therefore I_G = I_{BC} - A\times\left(\frac{h}{3}\right)^2$$

$$= \frac{bh^3}{12} - \left(\frac{b\times h}{2}\right).\left(\frac{h}{3}\right)^2$$

$$= \frac{bh^3}{12} - \frac{bh^3}{18}$$

$$\therefore I_G = \frac{bh^3}{36}$$

iv) Moment of inertia of circular section:

Consider a circular section of radius R with O as centre. Consider an elementary circular ring of radius r and thickness dr.

Area of circular ring = $2\pi r.\, dr$

In this case first find the moment of inertia of the circular section about an axis passing through O and perpendicular to the plane of the paper. This moment of inertia is also known as polar moment of inertia. Let this axis be Z-Z. Then from the theorem of perpendicular axis, the moment of inertia about X-X axis or Y-Y axis is obtained.

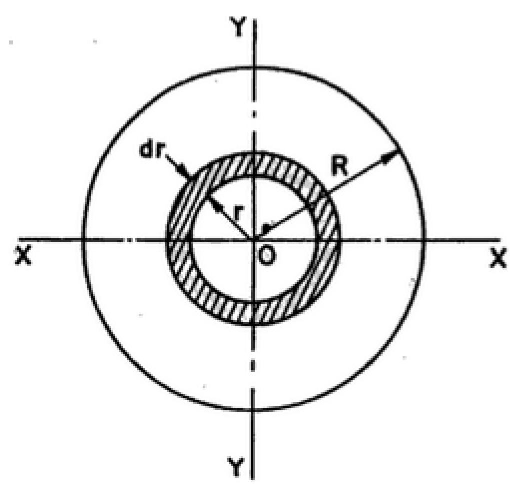


Fig. 4.15

Moment of inertia of the circular ring about an axis passing through O and perpendicular to the plane of the paper = (Area of ring) × (radius of ring from O)^2

$$= (2\pi r.dr).r^2 = 2\pi r^3.dr$$

Moment of inertia of the whole circular section is obtained by integrating the above equation between the limits 0 and R.

∴ Moment of inertia of the whole section about an axis passing through O and perpendicular to the plane of paper is given as

$$I_{zz} = \int_0^R 2\pi r^3.dr = 2\pi\int_0^R r^3.dr$$

$$= 2\pi\left[\frac{r^4}{4}\right]_0^R = \frac{\pi R^4}{2}$$

But $R = \frac{D}{2}$

$$\therefore I_{zz} = \frac{\pi}{2} \times \left(\frac{D}{2}\right)^4 = \frac{\pi D^4}{32}$$

But from the theorem of perpendicular axis, we have

$$I_{zz} = I_{xx} + I_{yy}$$

And due to symmetry $I_{xx} = I_{yy}$

$$\therefore I_{xx} = I_{yy} = \frac{I_{zz}}{2} = \frac{\pi D^4}{64}$$

Moment of inertia of a hollow circular section:

Consider a hollow circular section.

Let D = diameter of outer circle, and

d = diameter of cut-out circle.

Then, the moment of inertia of the outer circle about X-X axis = $\frac{\pi D^4}{64}$

And moment of inertia of the cut-out circle about X-X axis = $\frac{\pi d^4}{64}$

∴ Moment of inertia of the hollow circular section, about X-X axis,

$$\therefore I_{xx} = \frac{\pi D^4}{64} - \frac{\pi d^4}{64} = \frac{\pi}{64}\left[D^4 - d^4\right]$$

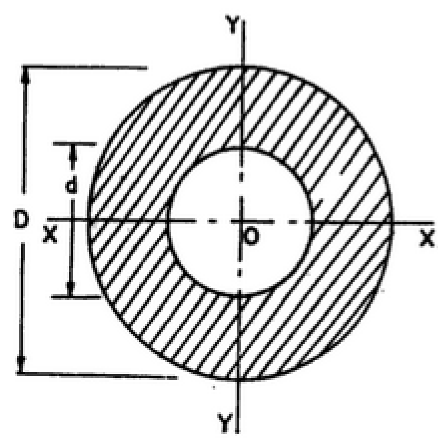


Fig. 4.16

v) Moment of inertia of a semi-circular section:

About diametral axis:

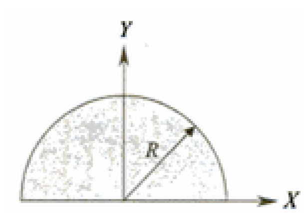


Fig. 4.17

Consider a semicircular area of radius R. We know that moment of inertia of circular section about diametral axis is $I_{xx} = \frac{\pi D^4}{64}$

∴ Moment of inertia of a semi circular section about its diametral axis is given by

$$I_{xx} = \frac{1}{2} \times \frac{\pi D^4}{64} = \frac{\pi D^4}{128}$$

About Centroidal axis:

From parallel axis theorem, we know that

$I_{AB} = I_{xx} + Ay_c^2$

$$\therefore I_{xx} = I_{AB} - Ay_c^2$$

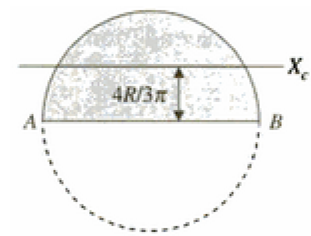


Fig. 4.18

A = Area of the semicircle

$$= \frac{\pi R^2}{2} = \frac{\pi D^2}{8}$$

y_c = distance between the two parallel axes AB and X-X

$$= \frac{4R}{3\pi} = \frac{2D}{3\pi}$$

$$\therefore I_{xx} = \frac{\pi D^4}{128} - \frac{\pi D^2}{8} \times \left(\frac{2D}{3\pi}\right)^2$$

$$= D^4\left(\frac{\pi}{128} - \frac{1}{18\pi}\right)$$

$$= 0.0068598\, D^4 \; or \; 0.11\, R^4$$

vi) Moment of inertia of a quarter circle:

About diametral axis:

Consider a quarter circular area of radius R. We know that moment of inertia of circular section about diametral axis is $I_{xx} = \frac{\pi D^4}{64}$

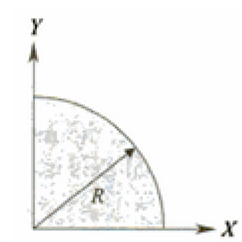


Fig. 4.19

∴ Moment of inertia of a semi circular section about its diametral axis is given by

$$I_{xx} = \frac{1}{4} \times \frac{\pi D^4}{64} = \frac{\pi D^4}{256}$$

About Centroidal axis:

From parallel axis theorem, we know that

$$I_{AB} = I_{xx} + Ay_c^2$$

$$\therefore I_{xx} = I_{AB} - Ay_c^2$$

A = Area of the semicircle

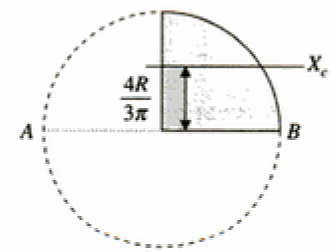


Fig. 4.20

$$= \frac{\pi R^2}{4} = \frac{\pi D^2}{16}$$

y_c = distance between the two parallel axes AB and X-X

$$= \frac{4R}{3\pi} = \frac{2D}{3\pi}$$

$$\therefore I_{xx} = \frac{\pi D^4}{256} - \frac{\pi D^2}{16} \times \left(\frac{2D}{3\pi}\right)^2$$

$$= D^4\left(\frac{\pi}{256} - \frac{1}{36\pi}\right)$$

$$= 0.00343\, D^4 \; or \; 0.0549\, R^4$$

vii) Moment of inertia of area under the curve $x = ky^2$:

Consider an area under a curve whose equation is parabolic and is given by $x = ky^2$ in which $y = b$ when $x = a$.

Suppose it is required to find the moment of inertia of this area about Y-axis. Consider a strip of thickness dx at a distance x from Y-axis.

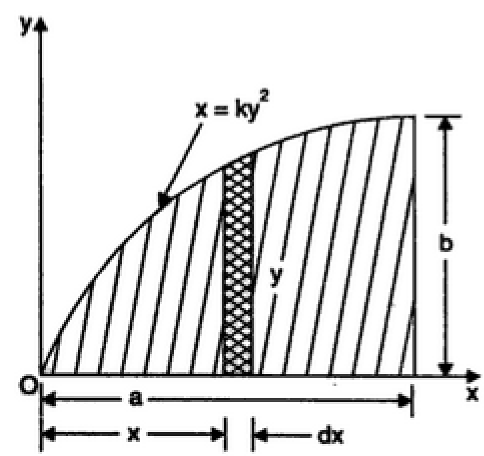


Fig. 4.21

The area of the strip, $dA = y\,dx$

Let us substitute the value y in terms of x.

The equation of the curve is $x = ky^2$.

When $y = b$, $x=a$. Hence above equation becomes $a = kb^2$ or $k = \frac{a}{b^2}$

Now the equation of the curve is $x = \frac{a}{b^2}y^2$.

$$y^2 = \frac{b^2}{a}x$$

$$\therefore y = \frac{b}{\sqrt{a}}\sqrt{x}$$

$\therefore$ Area of the elementary strip, $dA = \frac{b}{\sqrt{a}}\sqrt{x}\,dx$

The moment of inertia of elemental area about Y-axis

$$= dA \times x^2$$

$$= \frac{b}{\sqrt{a}}\, x^{5/2}$$

∴ Moment of inertia of the total area about Y-axis is obtained by integrating the above equation between the limits *0* to *b*.

$$\therefore I_{yy} = \int_0^a \frac{b}{\sqrt{a}}\, x^{5/2}$$

$$= \frac{b}{\sqrt{a}} \int_0^a x^{5/2}$$

$$= \frac{b}{\sqrt{a}} \left[\frac{x^{7/2}}{7/2}\right]_0^a = \frac{2}{7} . \frac{b}{\sqrt{a}} . a^{\frac{7}{2}}$$

$$\therefore I_{yy} = \frac{2}{7} b\, a^3$$

To find the moment of inertia of the given area about X-axis, the same elemental strip of thickness *dx* is considered.

The moment of inertia of this small element about X-axis is equal to the moment of inertia of the rectangle about its base.

∴ Moment of inertia of the element about X-axis

$$= \frac{dx \,.\, y^3}{3}$$

The moment of inertia of the given area about X-axis is obtained by integrating the above equation between the limits *0* and *a*.

$$\therefore I_{xx} = \int_0^a \frac{y^3}{3} dx$$

$$= \int_0^a \frac{\left[\frac{b}{\sqrt{a}} . \sqrt{x}\right]^3}{3}\, dx$$

$$= \frac{b^3}{3a^{3/2}} \int_0^a x^{3/2} dx = \frac{b^3}{3a^{3/2}} \left[\frac{x^{5/2}}{5/2}\right]_0^a$$

$$= \frac{b^3}{3a^{3/2}} . \frac{2}{5} .\, a^{5/2}$$

$$\therefore I_{xx} = \frac{2}{15}\, ab^3$$

viii) Moment of inertia of an Ellipse:

The equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Consider a strip parallel to X-axis at a distance of *y* from X-axis and thickness *dy.*

Area of the strip, $dA = 2x\, dy$

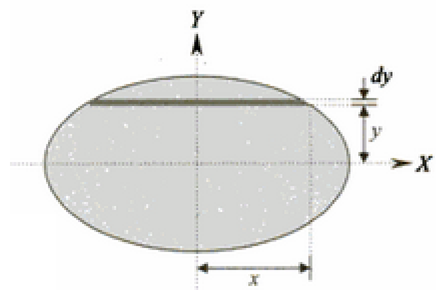


Fig. 4.22

From the equation of the ellipse, $\frac{x^2}{a^2} = 1 - \frac{y^2}{b^2}$

$$x = \frac{a}{b}\sqrt{b^2 - y^2}$$

$$\therefore dA = 2\frac{a}{b}\sqrt{b^2 - y^2}\, dy$$

Moment of inertia of the strip about X-axis = $dA \times y^2$

$$= 2\frac{a}{b}\sqrt{b^2 - y^2} \times y^2 dy$$

Moment of inertia of the entire area about X-axis is obtained by integrating the above equation between the limits $-b$ to b

$$\therefore I_{xx} = \int_{-b}^{b} 2\frac{a}{b}\sqrt{b^2 - y^2}y^2 dy = 2\frac{a}{b}\int_{-b}^{b}\sqrt{b^2 - y^2}y^2 dy$$

Taking $y = b\sin\theta$, we have $dy = b\cos\theta\, d\theta$ and the corresponding limits vary from $-\frac{\pi}{2}\, to\, \frac{\pi}{2}$

Therefore, the integral can be written as

$$\therefore I_{xx} = 2\frac{a}{b}\int_{-\pi/2}^{\pi/2} b^2\sin^2\theta\sqrt{b^2 - b^2\sin^2\theta} \times b\cos\theta\, d\theta$$

$$= 2\frac{a}{b}\int_{-\pi/2}^{\pi/2} b^2\sin^2\theta\,.\, b^2\cos^2\theta\, d\theta$$

$$= 2\frac{a}{b}\,.\,b^4\int_{-\pi/2}^{\pi/2} \sin^2\theta\cos^2\theta\, d\theta$$

$$= 2ab^3\int_{-\pi/2}^{\pi/2}\left(\frac{2\sin\theta\cos\theta}{2}\right)^2 d\theta$$

$$= \frac{ab^3}{2}\int_{-\pi/2}^{\pi/2}\sin^2 2\theta\, d\theta$$

$$= \frac{ab^3}{2} \int_{-\pi/2}^{\pi/2} \left(\frac{1-\cos 4\theta}{2}\right) d\theta$$

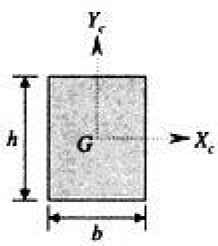
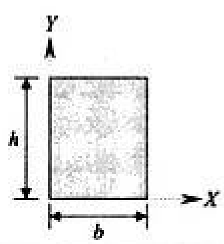
$$= \frac{ab^3}{4} \left[\theta - \frac{\sin\sin 4\theta}{4}\right]_{-\pi/2}^{\pi/2}$$

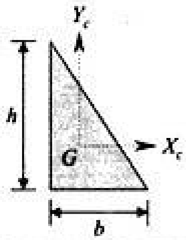
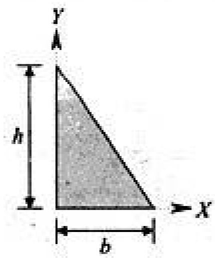
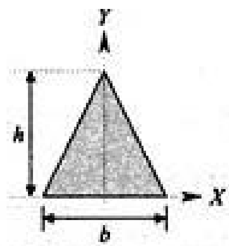
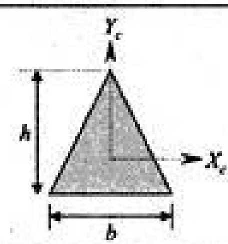
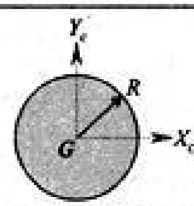
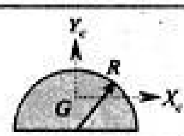
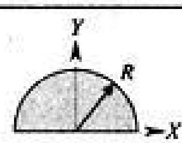
$$\therefore I_{xx} = \frac{\pi ab^3}{4}$$

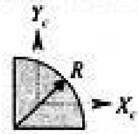
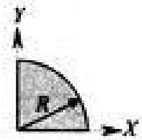
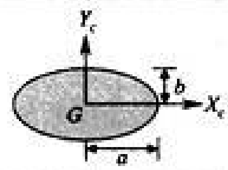
Similarly by taking a strip parallel to Y-axis, it can be shown that the moment of inertia about Y-axis is

$$\therefore I_{yy} = \frac{\pi ba^3}{4}$$

Moment of inertia of regular shapes:

Area	*Shape*	$\bar{I}_{xx}$	$\bar{I}_{yy}$	$\bar{I}_{xy}$
Rectangle (about centroidal axes)		$\frac{bh^3}{12}$	$\frac{hb^3}{12}$	0
Rectangle (about axes along the sides)		$\frac{bh^3}{3}$	$\frac{hb^3}{3}$	$\frac{b^2h^2}{4}$

Area	Shape	$\bar{I}_{xx}$	$\bar{I}_{yy}$	$\bar{I}_{xy}$
Right triangle (about centroidal axes)		$\frac{bh^3}{36}$	$\frac{hb^3}{36}$	$-\frac{b^2h^2}{72}$
Right triangle (about X–Y axes)		$\frac{bh^3}{12}$	$\frac{hb^3}{12}$	$\frac{b^2h^2}{24}$
Isosceles triangle (about X–Y axes)		$\frac{bh^3}{12}$	$\frac{hb^3}{48}$	0
Isosceles triangle (about centroidal axes)		$\frac{bh^3}{36}$	$\frac{hb^3}{48}$	0
Circle (about centroidal axes)		$\frac{\pi R^4}{4}$	$\frac{\pi R^4}{4}$	0
Semicircle (about centroidal axes)		$0.11R^4$	$\frac{\pi R^4}{8}$	0
Semicircle (about diametric axes)		$\frac{\pi R^4}{8}$	$\frac{\pi R^4}{8}$	0

Area	*Shape*	$\bar{I}_{xx}$	$\bar{I}_{yy}$	$\bar{I}_{xy}$
Quarter-circle (about centroidal axes)		$0.055R^4$	$0.055R^4$	$-0.016R^4$
Quarter-circle (about X–Y axes)		$\frac{\pi R^4}{16}$	$\frac{\pi R^4}{16}$	$\frac{R^4}{8}$
Ellipse (about centroidal axes)		$\frac{\pi ab^3}{4}$	$\frac{\pi ba^3}{4}$	0

Moment of inertia of Composite sections:

Moment of inertia of composite sections about an axis can be found by the following steps:

i) Divide the given figure into a number of simple figures.

ii) Locate the centroid of each simple figure by inspection or using standard expressions.

iii) Find the moment of inertia of each simple figure about its centroidal axis. Add the term Ay^2 where A is the area of the simple figure and y is the distance of the centroid of the simple figure about the reference axis.

iv) Sum up moments of inertia of all simple figures to get the moment of inertia of the composite section.

Example. 4.1. *Find the moment of inertia of the angle section about the centroidal axes. Also find the radii of gyration about the same axes.*

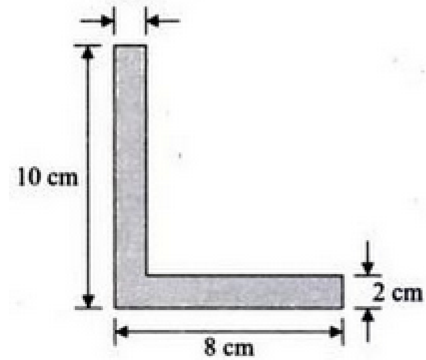


Fig. 4.23

Solution.

S.No.	Element	Area (A_i) (cm^2)	x_i (cm)	y_i (cm)	$A_i x_i$	$A_i y_i$
1.	Rectangle-1	$8\times2 = 1$	$\frac{8}{2} = 4$	$\frac{2}{2} = 1$	$16\times4 =$	$16\times1 =$
2.	Rectangle-2	$2\times8 = 1$	$\frac{2}{2} = 1$	$2 + \frac{8}{2}$	$16\times1 =$	$16\times6 =$

$\sum=$		32			80	112

$$\therefore \bar{x} = \frac{\sum a_i x_i}{\sum a_i} = \frac{80}{32} = 2.5\ cm$$

$$\therefore \bar{y} = \frac{\sum a_i y_i}{\sum a_i} = \frac{112}{32} = 3.5\ cm$$

Moment of inertia calculations:

S.No.	$\left(I_{xx}\right)_i$	$\left(I_{yy}\right)_i$	$A_i(y_i - \bar{y})^2$	$A_i(x_i - \bar{x})^2$
1. 2.	$\frac{8\times2^3}{12} = 5.33$ $\frac{2\times8^3}{12} = 85.33$	$\frac{2\times8^3}{12} = 85.33$ $\frac{8\times2^3}{12} = 5.33$	$16(1 - 3.5)^2 = 100$ $16(6 - 3.5)^2 = 100$	$16(4 - 2.5)^2 = 36$ $16(1 - 2.5)^2 = 36$
$\sum=$	90.66	90.66	200	72

$$\therefore I_{xx} = \sum \left(I_{xx}\right)_i + \sum A_i(y_i - \bar{y})^2$$

$$I_{xx} = 90.66 + 200 = 290.66\ cm^4$$

$$\therefore I_{yy} = \sum \left(I_{yy}\right)_i + \sum A_i(x_i - \bar{x})^2$$

$$I_{yy} = 90.66 + 72 = 162.66\ cm^4$$

Radii of gyration:

The radii of gyration about X and Y axes are determined as

$$k_x = \sqrt{\frac{I_{xx}}{A}} = \sqrt{\frac{290.66}{32}} = 3.01\ cm$$

$$k_y = \sqrt{\frac{I_{yy}}{A}} = \sqrt{\frac{162.66}{32}} = 2.25\ cm$$

Example. 4.2. *In the above problem determine the moment of inertia of the angle section about the base.*

Solution.

As we know the moment of inertia of the composite section about its centroid, using parallel axis theorem, we can determine the moment of inertia of the composite section about the base.

$$I_{Base} = I_{xx} + A\bar{y}^2$$

$$= 290.66 + 32\times(3.5)^2$$

$$I_{Base} = 682.66\ cm^4$$

However, if we don't know the moment of inertia of the composite section about its centroidal axis, we can determine the moment of inertia of the composite section about the base as follows

S.No.	$(I_{xx})_i$	$A_i(y_i)^2$
1.	$\frac{8\times2^3}{12} = 5.33$	$16(1)^2 = 16$
2.	$\frac{2\times8^3}{12} = 85.33$	$16(6)^2 = 576$
$\Sigma=$	90.66	592

$$\therefore I_{Base} = \Sigma\left(I_{xx}\right)_i + \Sigma A_i(y_i)^2$$

$$I_{Base} = 90.66 + 592 = 682.66\ cm^4$$

Example. 4.3. *Find the moment of inertia of I-section about centroidal axes. Also find the radii of gyration about the same axes.*

Solution.

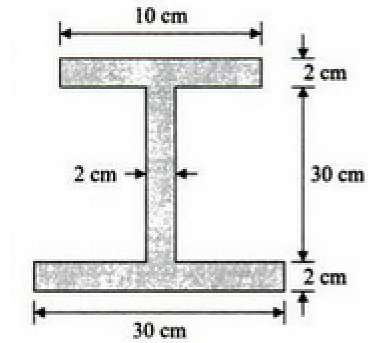


Fig. 4.24

S.No.	Element	Area (A_i) (cm^2)	y_i (cm)	$A_i\, y_i$
1.	Rectangle-1	$30\times2 = 60$	$\frac{2}{2} = 1$	60
2.	Rectangle-2	$30\times2 = 60$	$2 + \frac{30}{2} = 17$	1020
3.	Rectangle-3	$10\times2 = 20$	$32 + \frac{2}{2} = 33$	660
$\Sigma=$		140		1740

$$\therefore \bar{y} = \frac{\Sigma a_i y_i}{\Sigma a_i} = \frac{1740}{140} = 12.43\ cm$$

Moment of inertia calculations:

S.No.	$(I_{xx})_i$	$(I_{yy})_i$	$A_i(y_i - \bar{y})^2$	$A_i(x_i - \bar{x})^2$
1.	$\frac{30\times2^3}{12} = 20$	$\frac{2\times30^3}{12} = 4500$	$60(1 - 12.43)^2 = 7838.69$	0
2.	$\frac{2\times30^3}{12} = 4500$	$\frac{30\times2^3}{12} = 20$	$60(17 - 12.43)^2 = 1253.09$	0
3.	$\frac{10\times2^3}{12} = 6.67$	$\frac{2\times10^3}{12} = 166.67$	$60(33 - 12.43)^2 = 8462.5$	0
$\Sigma=$	4526.67	4686.67	17554.28	0

$$\therefore I_{xx} = \sum \left(I_{xx}\right)_i + \sum A_i(y_i - \bar{y})^2$$

$$I_{xx} = 4526.67 + 17554.28 = 22080.95\ cm^4$$

$$\therefore I_{yy} = \sum \left(I_{yy}\right)_i + \sum A_i(x_i - \bar{x})^2$$

$$I_{yy} = 4686.67\ cm^4$$

Radii of gyration:

The radii of gyration about X and Y axes are determined as

$$k_x = \sqrt{\frac{I_{xx}}{A}} = \sqrt{\frac{22080.95}{140}} = 12.56\ cm$$

$$k_y = \sqrt{\frac{I_{yy}}{A}} = \sqrt{\frac{17554.28}{140}} = 5.79\ cm$$

Example. 4.4 *Find the moment of inertia of the shaded area about the centroidal axes.*

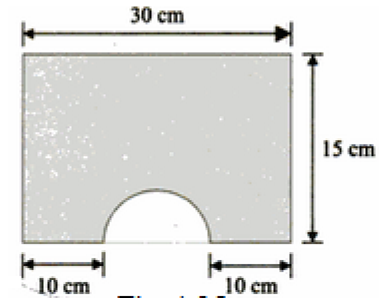


Fig. 4.25

Solution. The given area can be considered to be made up of a rectangle, from which a semicircular area has been removed. Due to symmetry, the x-coordinate of the centroid is $\bar{x} = 15\ cm$. Y-coordinate is determined as follows.

S.No.	Element	Area (A_i) (cm^2)	y_i (cm)	$A_i\,y_i$
1.	Rectangle	$30\times15 = 450$	$\frac{15}{2} = 7.5$	$450\times7.5 = 3375$
2.	Semi circle	$-\frac{\pi}{2}(5)^2 = -39.27$	$\frac{4(5)}{3\pi} = 2.12$	$-39.27\times2.12 = -83.25$
$\Sigma=$		410.73		3291.75

$$\therefore \bar{y} = \frac{\Sigma a_i y_i}{\Sigma a_i} = \frac{3291.75}{410.73} = 8.01\ cm$$

Moment of inertia calculations:

S.No.	$(I_{xx})_i$	$(I_{yy})_i$	$A_i(y_i - \bar{y})^2$	$A_i(x_i - \bar{x})^2$
1.	$\frac{30\times15^3}{12} = 8437.5$	$\frac{15\times30^3}{12} = 33750$	$450(7.5 - 8.01)^2 = 117.05$	0
2.	$-0.11(5)^4 =- 6$	$-\frac{\pi(5)^4}{8} =- 245.4$	$-39.27(2.12 - 8.01)^2 =- 1362.$	0
Σ=	8368.75	33504.56	-1245.31	0

$$\therefore I_{xx} = \Sigma (I_{xx})_i + \Sigma A_i(y_i - \bar{y})^2$$

$$I_{xx} = 8368.75 - 1245.31 = 7123.44\ cm^4$$

$$\therefore I_{yy} = \Sigma (I_{yy})_i + \Sigma A_i(x_i - \bar{x})^2$$

$$I_{yy} = 33504.56\ cm^4$$

Example. 4.5 *Find the moment of inertia of the shaded area about the centroidal axes.*

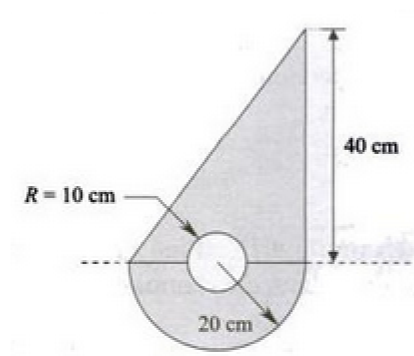


Fig. 4.26

Solution.

S.No.	Element	Area (A_i) (cm^2)	x_i (cm)	y_i (cm)	$A_i x_i$	$A_i y_i$
		$\frac{1}{2}\times40\times40 =$	$\frac{2(40)}{3} = 26.$	$\frac{40}{3} = 13.33$	800×26.67	$800\times13.33 =$

1. 2. 3.	Triangle Semi-circle Circle	$\frac{\pi}{2}(20)^2 = 62$ $-\pi(10)^2 =-$	20 20	$-\frac{4\times20}{3\pi} =-$ 0	628.32×20 -314.26×2	628.32×-8 0
$\Sigma=$		1114.16			27619.2	5329.56

$$\therefore \bar{x} = \frac{\Sigma a_i x_i}{\Sigma a_i} = \frac{27619.2}{1114.16} = 24.79\ cm$$

$$\therefore \bar{y} = \frac{\Sigma a_i y_i}{\Sigma a_i} = \frac{5329.56}{1114.16} = 4.78\ cm$$

Moment of inertia calculations:

S.No.	$\left(I_{xx}\right)_i$	$\left(I_{yy}\right)_i$	$A_i(y_i - \bar{y})^2$	$A_i(x_i - \bar{x})^2$
1. 2. 3.	$\frac{40\times40^3}{36} = 7111$ $0.11(20)^4 = 17$ $-\frac{\pi(10)^4}{4} =- 78$	$\frac{40\times40^3}{36} = 7111$ $\frac{\pi(20)^4}{8} = 62831$ $-\frac{\pi(10)^4}{4} =- 78$	$800(13.33 - 4.78)^2$ $628.32(-8.49 - 4.7$ $-314.16(0 - 4.78)^2$	$800(26.67 - 24.79)^2 =$ $628.32(20 - 24.79)^2 =$ $-314.16(20 - 24.79)^2$
$\Sigma=$	80857.13	126088.98	161946.64	10035.64

$$\therefore I_{xx} = \sum \left(I_{xx}\right)_i + \sum A_i(y_i - \bar{y})^2$$

$$I_{xx} = 80857.13 + 161946.64 = 242803.77\ cm^4$$

$$\therefore I_{yy} = \sum \left(I_{yy}\right)_i + \sum A_i(x_i - \bar{x})^2$$

$$I_{yy} = 126088.98 + 10035.64 = 136124.62\ cm^4$$

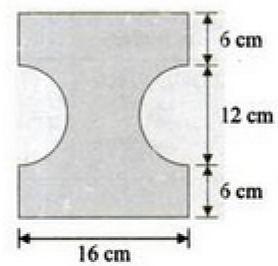


Fig. 4.27

Example. 4.6 *Find the moment of inertia of the shaded area about the centroidal axes.*

Solution.

Since the section has two axes of symmetry, we can readily know that its centroid lies at the centre. i.e., $\bar{x} = 8\ cm\ and\ \bar{y} = 12\ cm$. The area of each semi circular portion cut is $\frac{\pi(6)^2}{2} = 56.55\ cm^2$. The centroid of each cut section lies at $\frac{4(6)}{3\pi} = 2.55\ cm$.

Moment of inertia calculations:

S.No.	$\left(I_{xx}\right)_i$	$\left(I_{yy}\right)_i$	$A_i(y_i - \bar{y})^2$	$A_i(x_i - \bar{x})^2$
1.	$\frac{16\times24^3}{12} = 1843$	$\frac{24\times16^3}{12} = 8192$	0	0
2.	$-\frac{\pi(6)^4}{8} =- 50$	$-0.11(6)^4 =-$	0	$-56.55(2.55-8)^2 =-$
3.	$-\frac{\pi(6)^4}{8} =- 50$	$-0.11(6)^4 =-$	0	$-56.55(2.55-8)^2 =-$
$\Sigma=$	17414.12	7906.88	0	-3359.36

$$\therefore I_{xx} = \sum \left(I_{xx}\right)_i + \sum A_i (y_i - \bar{y})^2$$

$$I_{xx} = 17414.12\ cm^4$$

$$\therefore I_{yy} = \sum \left(I_{yy}\right)_i + \sum A_i (x_i - \bar{x})^2$$

$$I_{yy} = 7906.88 - 3359.36 = 4547.52\ cm^4$$

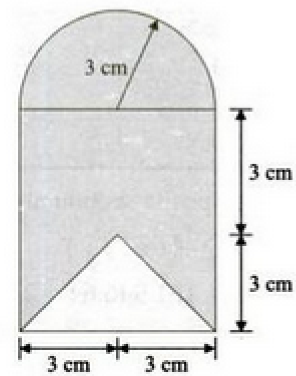


Fig. 4.28

Example. 4.6 *Find the moment of inertia of the shaded area about the horizontal centroidal axis.*

Solution.

The given area can be considered to be made up of a rectangle and a semicircular area, from which a triangular area has been removed. We can see that composite section has axis of symmetry about Y-axis and hence we need to determine y-coordinate of the centroid only.

S.No.	Element	Area (A_i) (cm^2)	y_i (cm)	$A_i y_i$
1.	Square	$6\times6 = 36$	$\frac{6}{2} = 3$	$36\times3 = 108$
2.	Semi circle	$\frac{\pi}{2}(3)^2 = 14.14$	$6 + \frac{4(3)}{3\pi} = 7$	$14.14\times7.27 = 102.8$
3.	Triangle	$-\frac{1}{2}\times6\times3 = -9$	$\frac{3}{3} = 1$	-9
Σ=		41.14		201.8

$$\therefore \bar{y} = \frac{\Sigma a_i y_i}{\Sigma a_i} = \frac{201.8}{41.14} = 4.91\ cm$$

Moment of inertia calculations:

S.No.	$\left(I_{xx}\right)_i$	$A_i(y_i - \bar{y})^2$
1. 2. 3.	$\frac{6\times6^3}{12} = 108$ $0.11(3)^4 = 8.91$ $-\frac{6\times3^3}{36} =- 4.5$	$36(3 - 4.91)^2 = 131.33$ $14.14(7.27 - 4.91)^2 = 78.75$ $- 9(1 - 4.91)^2 - 137.59$
$\Sigma=$	112.41	72.49

$$\therefore I_{yy} = \Sigma\left(I_{yy}\right)_i + \Sigma A_i(x_i - \bar{x})^2$$

$$I_{yy} = 112.41 + 72.49 = 184.9\ cm^4$$

Product of Inertia:

The Fig. 4.29 shows a body of area *A*. Consider a small area *dA*. The moment of this area about X-axis is $y\,.\,dA$. Now the moment of $y\,.\,dA$ about Y-axis is $xy\,dA$. Then $xy\,dA$ is known as the product of inertia of are *dA* with respect to X-axis and Y-axis. The integral $\int xy\,dA$ is known as the product of inertia of area *A* with respect to X and Y axes. This product of inertia is represented by I_{xy}.

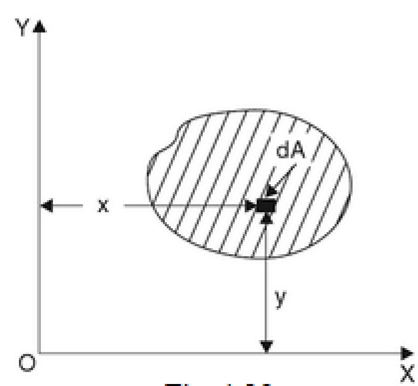


Fig. 4.29

$$\therefore I_{xy} = \int xy\,dA$$

Hence the product of inertia of the plane area is obtained if an elemental area is multiplied by the product of its coordinates and is integrated for entire area.

Note:

i) *The product of inertia may be positive, negative or zero depending upon distance x and y which could be positive, negative or zero.*

ii) *If area is symmetrical with respect to one or both of the axes, the product of inertia will be zero as shown in Fig.4.30. The total area A is symmetrical about Y-axis. The small area dA which is symmetrical about Y-axis has coordinates* (x, y) *and* $(-x, y)$. *The corresponding products of inertia for small area are* $xy\,dA$ and $-xydA$ *respectively. Hence the product of inertia for total area becomes zero.*

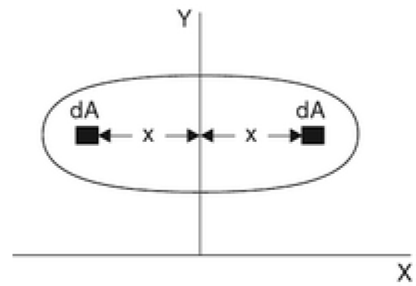


Fig. 4.30

iii) *The product of inertia with respect to centroidal axis will also be zero.*

Principal Axes:

The principal axes are the axes about which the product of inertia is zero.

The product of inertia (I_{xy}) of plane area *A* with respect to X and Y axes is given by

$$I_{xy} = \int xy\,dA$$

But the moment of inertia of plane area *A* about X-axis (I_{xx}) or about Y-axis (I_{yy}) is given by

$$I_{xx} = \int y^2 dA \text{ and } I_{yy} = \int x^2 dA$$

The moment of inertia is always positive but product of inertia may be positive (if both *x* and *y* are positive), may be negative (if one co-ordinate is positive and other is negative) or may be zero (if any co-ordinate is zero).

The fig. 4.31 (a) shows a body of area *A*. Consider a small area *dA*. The product of inertia of the total area *A* with respect to X and Y axes is given as

$$I_{xy} = \int xy\,dA$$

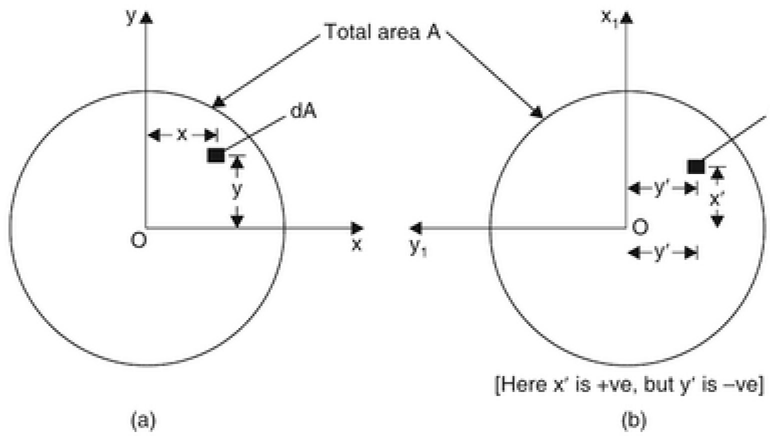


Fig. 4.31

Let now the axes are rotated anticlockwise by 90^0 as shown in Fig. 4.31 (b) keeping the total area *A* in the same position. Let x_1 and y_1 are the new axes. The co-ordinates of the same small area *dA* with respect to new axes are *x'* and *y'*.

Hence the product of inertia of the total area *A* with respect to new axes x_1 and y_1 becomes as

$$I_{x_1 y_1} = \int x' y' \, dA$$

Now let us find the relation between old and new co-ordinates, we get

$$x = -y' \text{ and } y = x'$$

Or
$$y' = -x \text{ and } x' = y$$

$$I_{x_1 y_1} = \int (y)(-x)\, dA = -\int xy\, dA = -I_{xy}$$

The above result shows that by rotating the axes through 90^0, the product of inertia has become negative. This means that the product of inertia which was positive previously has now become negative by rotating the axes through 90^0. Hence product of inertia has changed its sign. It is also possible that by rotating the axes through certain angle, the product of inertia will become zero. The new axes about which product of inertia is zero, are known as principal axes.

Note:

i) *The product of inertia is zero about principal axes.*

ii) *As the product of inertia is zero about symmetrical axis, hence symmetrical axis is the principal axis of inertia for the area.*

iii) *The product of inertia depends upon the orientation of the axes.*

Principal moments of inertia:

Fig. 4.32 (a) shows a body of area *A* with respect to old axes (*x, y*) and new axes (x_1, y_1). *The new axes* x_1 and y_1 have been rotated through an angle θ in anticlockwise direction. Consider a small area *dA*. The co-ordinates of the small area with respect to old axes is (*x, y*) whereas with respect to new axes, the co-ordinates are *x'* and *y'*. The new co-ordinates (*x', y'*) are expressed in terms of old co-ordinates (*x, y*) and angle θ as

$$x' = y \sin sin\,\theta + x \cos cos\,\theta$$

$$y' = y \cos cos\,\theta - x \sin sin\,\theta$$

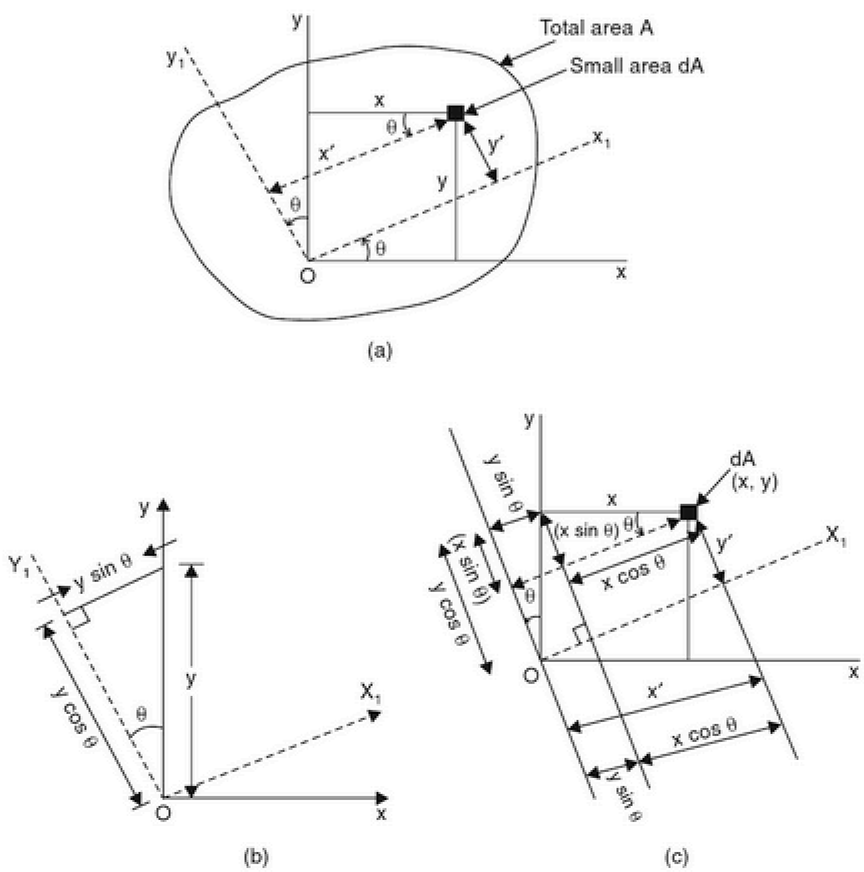


Fig. 4.32

The moment of inertia and product of inertia of area *A* with respect to old axes are

$$I_{xx} = \int y^2 dA,\ I_{yy} = \int x^2 dA \ and \ I_{xy} = \int xy\, dA$$

Also the moment of inertia and product of inertia of area *A* with respect to new axes will be

$$I_{x_1x_1} = \int \left(y^{'}\right)^2 dA,\ I_{y_1y_1} = \int \left(x^{'}\right)^2 dA \ and \ I_{x_1y_1} = \int x^{'}y^{'}\, dA$$

Let us substitute the values of $x^{'}, y^{'}$ in the above equations, we get

$$I_{x_1x_1} = \int \left(y^{'}\right)^2 dA$$

$$= \int (y \cos cos\, \theta\ -\ x \sin sin\, \theta\ )^2 dA$$

$$= \int\left(y^2 cos^2\theta\ +\ x^2 sin^2\theta\ -\ 2xycos\theta\ sin\theta\right)dA$$

$$= \int y^2 cos^2\theta\, dA\ +\int x^2 sin^2\theta\, dA\ -\int 2xycos\theta\ sin\theta\, dA$$

$$= cos^2\theta \int y^2\, dA\ +\ sin^2\theta \int x^2\, dA\ -\ 2cos\theta sin\theta \int xy\, dA$$

$$\therefore I_{x_1x_1} = \left(cos^2\theta\right)I_{xx} + \left(sin^2\theta\right)I_{yy} - (2cos\theta sin\theta)I_{xy} ...(i)$$

$$I_{y_1y_1} = \int \left(x^{'}\right)^2 dA$$

$$= \int (y \sin sin\, \theta\ +\ xcos\ )^2 dA$$

$$= \int\left(y^2 sin^2\theta\ +\ x^2 cos^2\theta\ +\ 2xysin\theta cos\theta\right)dA$$

$$= \int y^2 sin^2\theta\, dA\ +\int x^2 cos^2\theta\, dA\ +\int 2xysin\theta cos\theta\, dA$$

$$= sin^2\theta \int y^2\, dA + cos^2\theta \int x^2\, dA + 2sin\theta cos\theta \int xy\, dA$$

$$\therefore I_{y_1y_1} = \left(sin^2\theta\right)I_{xx} + \left(cos^2\theta\right)I_{yy} + (2sin\theta\ cos\theta)I_{xy} ...(ii)$$

Adding equations (*i*) and (*ii*), we get

$$I_{x_1x_1} + I_{y_1y_1} = I_{xx}\left[sin^2\theta + cos^2\theta\right] + I_{yy}\left[sin^2\theta + cos^2\theta\right] + (2sin\theta\ cos\theta)I_{xy} - (2cos\theta sin\theta)I_{xy}$$

$$\therefore I_{x_1x_1} + I_{y_1y_1} = I_{xx} + I_{yy} ...(iii)$$

The equation (*iii*) shows that sum of moments of inertia about old axes (*x, y*) and new axes (x_1, y_1) are same. Hence the sum of moments of inertia of area *A* is independent of orientation of axes. Now let us find the value of $I_{x_1x_1} - I_{y_1y_1}$

$$I_{x_1x_1} - I_{y_1y_1} = I_{xx}\left[cos^2\theta - sin^2\theta\right] + I_{yy}\left[sin^2\theta - cos^2\theta\right] - (4cos\theta sin\theta)I_{xy}$$

$$= I_{xx}\left[cos^2\theta - sin^2\theta\right] - I_{yy}\left[cos^2\theta - sin^2\theta\right] - (4cos\theta sin\theta)I_{xy}$$

$$= I_{xx} - I_{yy}\left[cos^2\theta - sin^2\theta\right] - 2\times 2\ cos\theta sin\theta \times I_{xy}$$

$$\therefore I_{x_1x_1} - I_{y_1y_1} = I_{xx} - I_{yy}(\cos cos\ 2\theta\ ) - 2I_{xy} \sin sin\ 2\theta\ ...(iv)$$

Adding equation (*iii*) and equation (*iv*), we get

$$2I_{x_1x_1} = \left[I_{xx} + I_{yy}\right] + \left[\left(I_{xx} - I_{yy}\right) \cos cos\ 2\theta\ - 2I_{xy} \sin sin\ 2\theta\ \right]$$

$$\therefore I_{x_1x_1} = \frac{\left(I_{xx}+I_{yy}\right)}{2} + \frac{\left(I_{xx}-I_{yy}\right)}{2}\cos cos\ 2\theta\ - I_{xy} \sin sin\ 2\theta ...(v)$$

To find the value of $I_{y_1y_1}$ subtract equation (*iv*) from equation (*iii*), we get

$$2I_{y_1y_1} = \left[I_{xx} + I_{yy}\right] - \left[\left(I_{xx} - I_{yy}\right) \cos cos\ 2\theta\ - 2I_{xy} \sin sin\ 2\theta\ \right]$$

$$\therefore I_{y_1y_1} = \frac{\left(I_{xx}+I_{yy}\right)}{2} - \frac{\left(I_{xx}-I_{yy}\right)}{2}\cos cos\ 2\theta\ + I_{xy} \sin sin\ 2\theta\ ...(vi)$$

Product of inertia about new axes

Let us find the value of $I_{x_1y_1}$ in terms of I_{xy} and angle θ.

$$I_{x_1y_1} = \int x^{'} y^{'} \, dA$$

Let us substitute the values of $x^{'}, y^{'}$ in the above equation, we get

$$I_{x_1y_1} = \int (y \sin sin\,\theta + x \cos cos\,\theta)(y \cos cos\,\theta - x \sin sin\,\theta)\, dA$$

$$= \int (y^2 sin\theta \cos cos\,\theta - xysin^2\theta + xycos^2\theta - x^2 cos\theta sin\theta)\, dA$$

$$= \int y^2 sin\theta \cos cos\,\theta\, dA - \int xysin^2\theta\, dA + \int xycos^2\theta\, dA - \int x^2 cos\theta sin\theta\, dA$$

$$= \frac{2sin\theta cos\theta}{2}\int y^2\, dA - sin^2\theta \int xy\, dA + cos^2\theta \int xy\, dA - \frac{2cos\theta sin\theta}{2}\int x^2 dA$$

$$= \frac{sin2\theta}{2}.\, I_{xx} + I_{xy}\left(cos^2\theta - sin^2\theta\right) - \frac{sin2\theta}{2}.\, I_{yy}$$

$$\therefore I_{x_1y_1} = \frac{\left(I_{xx} - I_{yy}\right)}{2} \sin sin\, 2\theta + I_{xy} \cos cos\, 2\theta$$

Direction of principal axes:

We have already defined the principal axes. Principal axes are the axes about which the product of inertia is zero.

∴ For principal axes, $I_{x_1y_1} = 0$

$$\frac{\left(I_{xx} - I_{yy}\right)}{2} \sin sin\, 2\theta + I_{xy} \cos cos\, 2\theta = 0$$

$$\frac{\left(I_{xx} - I_{yy}\right)}{2} \sin sin\, 2\theta = - I_{xy} \cos cos\, 2\theta$$

$$\frac{\sin sin\, 2\theta}{\cos cos\, 2\theta} = \frac{-2I_{xy}}{I_{xx} - I_{yy}} = \frac{2I_{xy}}{I_{yy} - I_{xx}}$$

$$\tan tan\, 2\theta = \frac{2I_{xy}}{I_{yy} - I_{xx}}$$

The above equation will give the two values of 2θ or θ. These two values of θ will differ by 90^0. By substituting the values of θ in equations (*v*) and (*vi*), the values of principle of moments of inertia can be obtained.

$$I_{x_1x_1} = \frac{(I_{xx}+I_{yy})}{2} \pm \sqrt{\frac{(I_{xx}-I_{yy})^2}{2} + I_{xy}^2}$$

Product of inertia of a right angled triangle:

Consider a right-angled triangle of base *b* and height *h*. If we take a thin strip parallel to the base at a distance *y* from the base and of infinite small thickness *dy,* then the area of the strip is

$$dA = b' dy$$

From similar triangles, we know that

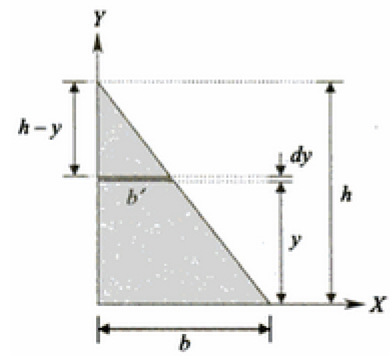


Fig. 4.33

$$b' = \frac{b}{h}(h - y)$$

$$\therefore dA = \frac{b}{h}(h - y)\, dy$$

The product of inertia of the strip about X-Y axes is

$$dI_{xy} = \left[\frac{b'}{2}\right] y\, dA$$

$$= \left[\frac{b'}{2}\right] y\, \frac{b}{h}(h - y)\, dy$$

$$= \frac{1}{2}\frac{b^2}{h^2}(h - y)^2 y\, dy$$

Therefore, the product of inertia of the triangle about X-Y axes is obtained as

$$I_{xy} = \frac{1}{2}\frac{b^2}{h^2}\int_0^h (h - y)^2 y\, dy$$

$$= \frac{1}{2}\frac{b^2}{h^2}\left[\frac{h^2y^2}{2} + \frac{y^4}{4} - \frac{2hy^3}{3}\right]_0^h$$

$$\therefore I_{xy} = \frac{b^2h^2}{24}$$

By applying the parallel axis theorem for product of inertia, we can obtain the product of inertia of the triangle about the centroidal axes:

$$I_{xy} = I_{base} - A\overline{xy}$$

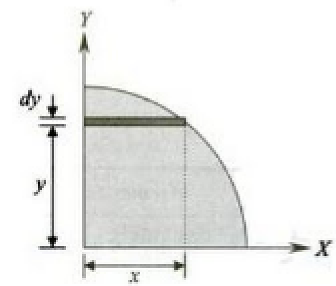
$$= \frac{b^2h^2}{24} - \frac{bh}{2} \times \frac{b}{3} \times \frac{h}{3}$$

$$\therefore I_{xy} = -\frac{b^2h^2}{72}$$

Product of inertia of quarter circle:

Consider a thin strip parallel to X-axis at a distance *y* from X-axis and of infinite small thickness *dy*. Then the area of the strip is

$$dA = x\, dy$$



The product of inertia of this strip is

$$dI_{xy} = \left(\frac{x}{2}\right) y\, dA = \left(\frac{x}{2}\right) y\, x\, dy$$

Therefore, the product of inertia of the entire area about X-Y axes is

$$I_{xy} = \int_0^R \frac{1}{2} x^2 y\, dy$$

We know that the equation of a circle is $x^2 + y^2 = R^2 \Rightarrow x^2 = R^2 - y^2$

$$I_{xy} = \int_0^R \frac{1}{2}\left(R^2 - y^2\right) y\, dy$$

$$= \frac{1}{2}\int_0^R \left(R^2 y - y^3\right) dy$$

$$\therefore I_{xy} = \frac{1}{2}\left(\frac{R^4}{2} - \frac{R^4}{4}\right) = \frac{R^4}{8}$$

By applying the parallel axis theorem for product of inertia, we can obtain the product of inertia of the quarter circle about the centroidal axes:

$$I_{xy} = I_{base} - A\overline{xy}$$

$$= \frac{R^4}{8} - \frac{\pi R^2}{4} \times \frac{4R}{3\pi} \times \frac{4R}{3\pi}$$

$$\therefore I_{xy} = \frac{R^4}{8} - \frac{4R^4}{9\pi} = -\ 0.016R^4$$

Example. 4.7. *Find the product of inertia of the channel section shown in the Fig. 4.34 with respect to centroidal axis.*

Solution.

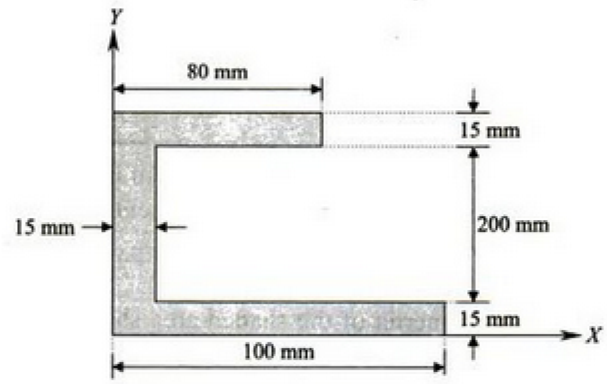


Fig. 4.34

S.No.	Element	Area (A_i) (mm^2)	x_i (mm)	y_i (mm)	$A_i x_i$	$A_i y_i$
1.	Rectangle-1	$100 \times 15 = 15$ $200 \times 15 = 30$	$\frac{100}{2} = 50$ $\frac{15}{2} = 7.5$	$\frac{15}{2} = 7.5$ $15 + \frac{200}{2} =$ $215 + \frac{15}{2} =$	$1500 \times 50 =$ $3000 \times 7.5 =$	$1500 \times 7.5 = 1$ $3000 \times 115 = 3$
2.	Rectangle-2	$80 \times 15 = 120$	$\frac{80}{2} = 40$		$1200 \times 40 =$	$1200 \times 222.5 =$
3.	Rectangle-3					
$\Sigma =$		5700			145500	623250

$$\therefore \bar{x} = \frac{\Sigma a_i x_i}{\Sigma a_i} = \frac{145500}{5700} = 25.53\ mm$$

$$\therefore \bar{y} = \frac{\Sigma a_i y_i}{\Sigma a_i} = \frac{623250}{5700} = 109.34\ mm$$

Product of inertia calculations:

S.No.	$\left(I_{xy}\right)_i$	$A_i\left(x_i - \bar{x}\right)(y_i - \bar{y})$
1.	0	$1500(50 - 25.53)(7.5 - 109.34) =- 3$
2.	0	$3000(7.5 - 25.53)(115 - 109.34) =-$
3.	0	$1200(40 - 25.53)(222.5 - 109.34) =$
$\Sigma=$	0	$- 2079276.4$

$$\therefore I_{xy} = \Sigma\left(I_{xy}\right)_i + \Sigma A_i\left(x_i - \bar{x}\right)(y_i - \bar{y})$$

$$I_{yy} =- 2079276.4\ cm^4$$

Example. 4.8. *Find the product of inertia of the shaded area shown in the Fig. 4.35 with respect to centroidal axes.*

Solution.

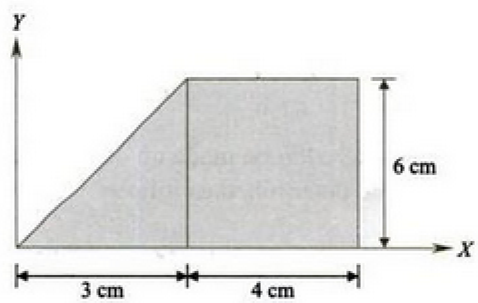


Fig. 4.35

S.No.	Element	Area (A_i) (cm^2)	x_i (cm)	y_i (cm)	$A_i x_i$	$A_i y_i$

1.	Triangle	$\frac{1}{2}\times3\times6\ =\ 9$	$\frac{2}{3}\times3\ =\ 2$	$\frac{6}{3}\ =\ 2$	$9\times2\ =\ 18$	$9\times2\ =\ 18$
2.	Rectangle	$6\times4\ =\ 24$	$3\ +\frac{4}{2}\ =\ 5$	$\frac{6}{2}\ =\ 3$	$24\times5\ =\ 12$(	$24\times3\ =\ 72$
$\Sigma=$		33			138	90

$$\therefore \bar{x} = \frac{\Sigma a_i x_i}{\Sigma a_i} = \frac{138}{33} = 4.18\ cm$$

$$\therefore \bar{y} = \frac{\Sigma a_i y_i}{\Sigma a_i} = \frac{90}{33} = 2.73\ cm$$

Product of inertia calculations:

S.No.	$\left(I_{xy}\right)_i$	$A_i\left(x_i - \bar{x}\right)(y_i - \bar{y})$
1.	$-\frac{3^2 6^2}{72} =-\ 4.5$	$9(2\ -\ 4.18)(2\ -\ 2.73) =\ 14.32$
2.	0	$24(5\ -\ 4.18)(3\ -\ 2.73) =\ 5.31$
$\Sigma=$	$-\ 4.5$	19.63

$$\therefore I_{xy} = \ \Sigma\left(I_{xy}\right)_i + \Sigma A_i\left(x_i - \bar{x}\right)(y_i - \bar{y})$$

$$I_{yy} = 15.13\ cm^4$$

Example. 4.9. *Find the product of inertia of the shaded area shown in the Fig. 4.36 with respect to centroidal axes.*

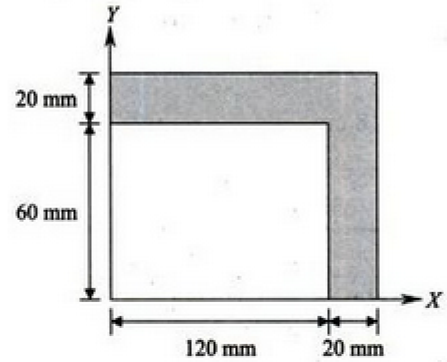


Fig. 4.36

Solution.

S.No.	Element	Area (A_i) (cm^2)	x_i *(mm)*	y_i *(mm)*	$A_i x_i$	$A_i y_i$
1.	Rectangle-1	$6\times2 = 12$	$12 + \frac{2}{2} = 13$	$\frac{6}{2} = 3$	$12\times13 = 156$	$12\times7 = 84$
2.	Rectangle-2	$2\times14 = 28$	$\frac{14}{2} = 7$	$6 + \frac{2}{2} = 7$	$28\times7 = 196$	$28\times7 = 196$
$\Sigma=$		40			352	232

$$\therefore \bar{x} = \frac{\Sigma a_i x_i}{\Sigma a_i} = \frac{352}{40} = 8.8\ cm$$

$$\therefore \bar{y} = \frac{\Sigma a_i y_i}{\Sigma a_i} = \frac{232}{40} = 5.8\ cm$$

Product of inertia calculations:

S.No.	$(y)_i$	$A_i\left(x_i - \bar{x}\right)(y_i - \bar{y})$
1.	0	$12(13 - 8.8)(3 - 5.8) =- 141.12$

2.	0	$28(7.8.8)(7 - 5.8) = - 60.48$
$\Sigma=$	0	$- 201.6$

$$\therefore I_{xy} = \Sigma\left(I_{xy}\right)_i + \Sigma A_i\left(x_i - \bar{x}\right)(y_i - \bar{y})$$

$$I_{yy} = - 201.6\ cm^4 = - 2.02\times10^6 mm^4$$

Mass moment of inertia:

Consider a body of mass M as shown in Fig. 4.37.

Let x = Distance of the centre of gravity of mass M from Y-axis

y = Distance of centre of gravity of mass M from X-axis

Then moment of the mass about the Y-axis = $M . x$

The above equation is known as first moment of mass about Y-axis.

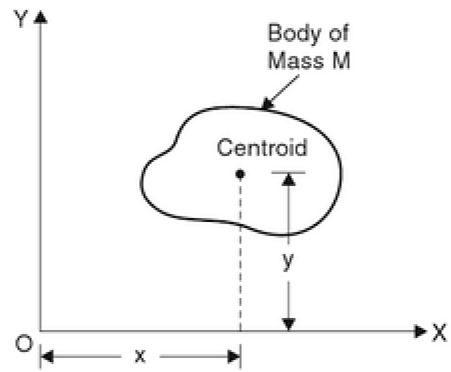


Fig. 4.37

If the moment of mass given by above equation is again multiplied by the perpendicular distance between the C.G. of the mass and Y-axis, then the quantity$(M.x).x = M.x^2$ is known as second moment of mass about Y-axis. This second moment of the mass is known as mass moment of inertia about Y-axis.

Similarly, the second moment of mass or mass moment of inertia about X-axis = $(M.y).y = M.y^2$.

Hence the product of the mass and the square of the distance of the centre of gravity of the mass from an axis is known as the mass moment of inertia about that axis. Mass moment of inertia is represented by I_m. Hence mass moment of inertia about X-axis is represented by $\left(I_m\right)_{xx}$ whereas about Y-axis is represented by $\left(I_m\right)_{yy}$.

Consider a body which is split up into small masses $m_1, m_2, m_3\ldots$ etc. Let the C.G. of the small areas from a given axis be at a distance of $r_1, r_2, r_3\ldots.$ etc. as shown in Fig. 4.38. then mass moment of inertia of the body about the given axis is given by

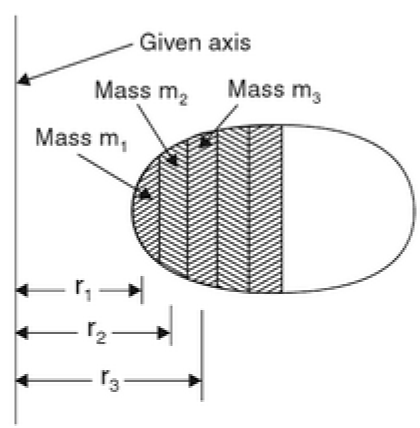


Fig. 4.38

$$I_m = m_1 r_1^2 + \ m_2 r_2^2 + m_3 r_3^2 + \ ... = \sum mr^3$$

If small masses are large in number then the summation in the above equation can be replaced by integration. Let the small masses are replaced by dm instead of m, then the above equation can be written as

$$I_m = \int r^2 dm$$

Radius of gyration:

Radius of gyration is the distance which when squared and multiplied with the total mass of the body gives the mass moment of inertia of the body.

Thus if I_m is the mass moment of inertia of a body of mass M about an axis then its radius of gyration k about that axis is given by the relation

$$I_m = k^2 \times M$$

$$\therefore k = \sqrt{\frac{I_m}{M}}$$

Mass moment of inertia of bodies:

i) Mass moment of inertia of a rectangular plate :
About its centroidal axes:

Consider a rectangular plate of width b and depth d and uniform thickness t. Consider a small element of width b and depth dy at a distance of y from X-X axis.

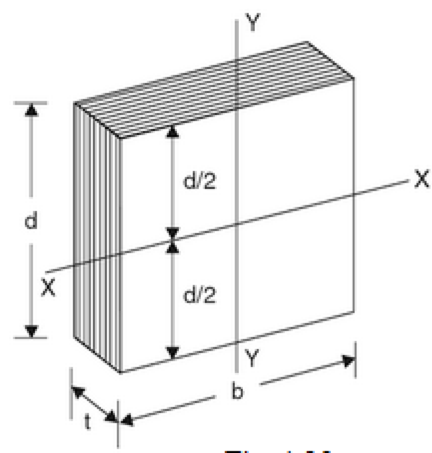


Fig. 4.39

Area of the strip, $dA = b. dy$

Volume of the strip, $dV = dA. t = b. t. dy$

Mass of the strip, $dM = density \times volume\ of\ the\ strip$

$= \rho(b. t. dy)$

Mass moment of inertia of the strip about X-X axis is

$$\left(dI_m\right)_{xx} = \rho(b. t. dy) \times y^2$$

∴ Mass moment of inertia of the plate will be obtained by integrating the above equation between the limits $-\frac{d}{2} to \frac{d}{2}$.

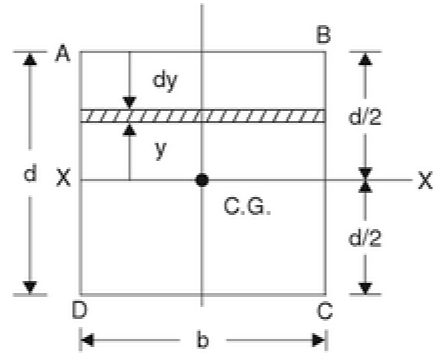


Fig. 4.40

$$\left(I_m\right)_{xx} = \int_{-\frac{d}{2}}^{\frac{d}{2}} \rho(b.t.dy)\times y^2$$

$$= \rho.b.t \int_{-\frac{d}{2}}^{\frac{d}{2}} y^2 dy$$

$$\rho.b.t\left[\frac{y^3}{3}\right]_{-d/2}^{d/2}$$

$$= \frac{\rho.b.t}{3}\left[\frac{d^3}{8} + \frac{d^3}{8}\right] = \frac{\rho.b.td^3}{12}$$

$$= \rho.b.d.t.\frac{d^2}{12}$$

$$\therefore \left(I_m\right)_{xx} = \frac{Md^2}{12}$$

Similarly the mass moment of inertia of the rectangular plate about Y-Y axis passing through C.G. of the plate is given by

$$\left(I_m\right)_{yy} = \frac{Mb^2}{12}$$

About the base:

Consider a rectangular plate of width *b* and depth *d* and uniform thickness *t*. Consider a small element of width *b* and depth *dy* at a distance of *y* from line CD.

Area of the strip, $dA = b.dy$

Volume of the strip, $dV = dA.t = b.t.dy$

Mass of the strip, $dM = density \times volume\ of\ the\ strip$

$$= \rho(b.t.dy)$$

Mass moment of inertia of the strip about line CD is

$$\left(dI_m\right)_{CD} = \rho(b.t.dy)\times y^2$$

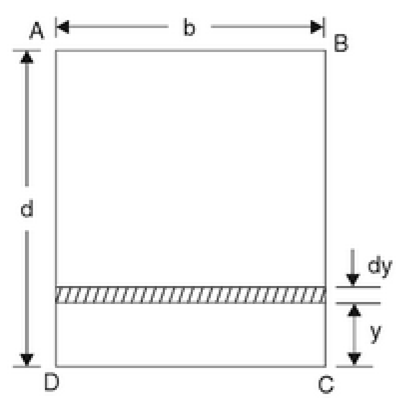


Fig. 4.41

∴ Mass moment of inertia of the plate will be obtained by integrating the above equation between the limits $0 \; to \; d$.

$$\left(I_m\right)_{CD} = \int_0^d \rho(b.t.dy)\times y^2$$

$$= \rho.b.t\int_0^d y^2 dy$$

$$\rho.b.t\left[\frac{y^3}{3}\right]_0^d$$

$$= \frac{\rho.b.t}{3}\left[d^3\right] = \frac{\rho.b.td^3}{3}$$

$$= \rho.b.d.t.\frac{d^2}{3}$$

$$\therefore \left(I_m\right)_{CD} = \frac{Md^2}{3}$$

Similarly the mass moment of inertia of the rectangular plate about the line AD is given by

$$\left(I_m\right)_{AD} = \frac{Mb^2}{3}$$

Mass moment of inertia of a hollow rectangular plate:

Fig. 4.42 shows a hollow rectangular plate in which ABCD is the main plate and EFGH is the cut-out section.

The mass moment of inertia of the main plate ABCD about X-X is given by equation

$$= \frac{1}{12}Md^2$$

The mass moment of inertia of the cut-out section EFGH about X-X axis

$$= \frac{1}{12}md_1^{\;2}$$

Where M = mass of main plate ABCD

$= \rho.b.d.t$

m = Mass of the cut-out section EFGH

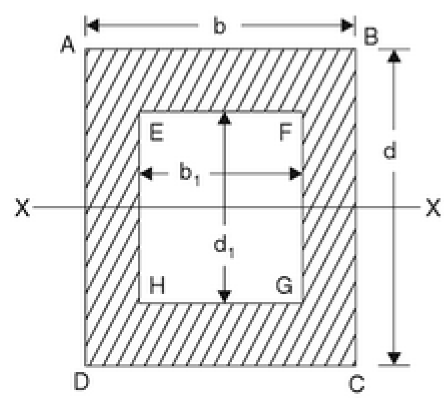


Fig. 4.42

$$= \rho . b_1 . d_1 . t$$

Then mass moment of inertia of hollow rectangular plate about X-X axis is given by

$$\left(I_m\right)_{xx} = \frac{Md^2}{12} - \frac{md_1^2}{12}$$

ii) Mass moment of inertia of circular plate:

Fig. 4.43 shows a circular plate of radius R and thickness *t* with O as centre. Consider an elementary circular ring of radius *r* and width *dr*.

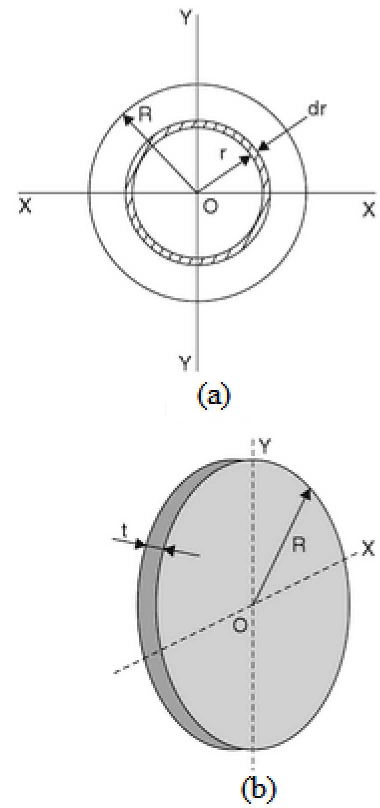


Fig. 4.43

Area of ring, $dA = 2\pi r . dr$

Volume of ring = Area of ring × t= $dA . t$

$$= 2\pi r . dr . t$$

Mass of ring, $dM = density \times volume\ of\ the\ ring$

$$= \rho(2\pi r . dr . t)$$

In this case first find the mass moment of inertia about an axis passing through O and perpendicular to the plane containing X-Y axis, i.e., about Axis Z-Z.

∴ Mass moment of inertia of the circular ring about axis Z-Z

$$= dM \times r^2$$

$$= \rho(2\pi r . dr . t) \times r^2 = \rho . t . 2\pi r^3 dr$$

The mass moment of inertia of the whole circular plate will be obtained by integrating the above equation between the limits *0* to *R*.

∴ Mass moment of inertia of circular plate about Z-Z axis is given by

$$\left(I_m\right)_{zz} = \int_0^R \rho . t . 2\pi r^3 dr = 2\pi\rho . t \int_0^R r^3 dr$$

$$= 2\pi\rho . t \left[\frac{r^4}{4}\right]_0^R = \pi\rho . t . \frac{R^4}{2}$$

$$\therefore \left(I_m\right)_{zz} = \rho . \pi R^2 . t . \frac{R^2}{2} = \frac{MR^2}{2}$$

From perpendicular axis theorem, we have $I_{zz} = I_{xx} + I_{yy}$ or $(I_m)_{zz} = (I_m)_{xx} + (I_m)_{yy}$

And due to symmetry, we have $(I_m)_{xx} = (I_m)_{yy}$

$$\therefore (I_m)_{xx} = (I_m)_{yy} = \frac{(I_m)_{zz}}{2} = \frac{MR^2}{4}$$

iii) Mass moment of inertia of a triangular plate:

Consider a triangle ABC of base width = 'b' and height = 'h'. Consider a small strip of thickness dy at a distance y from X-axis.

Area of the strip, dA = Length $DE \times dy$ .... (i)

Considering two similar triangles ADE and AOB,

$$\frac{DE}{OB} = \frac{AD}{AO}$$

Where OB = b, AO = h and AD = $(h - y)$

$$\therefore \frac{DE}{b} = \frac{(h-y)}{h}$$

$$DE = \frac{b\,(h-y)}{h}$$

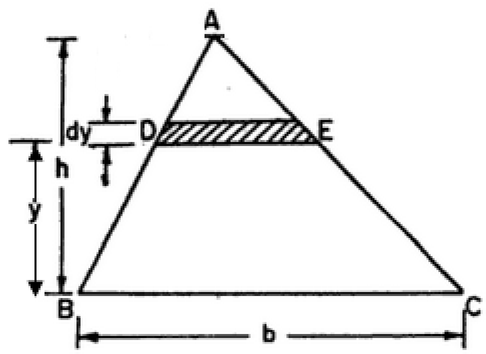


Fig. 4.44

Substituting the value of DE in equation (i), we get

Area of strip, $dA = \frac{b\,(h-y)}{h} \cdot dy$

Volume of strip, $dV = \frac{b\,(h-y)}{h} \cdot dy \cdot t$

Mass of strip, $dM = \rho\, \frac{b\,(h-y)}{h} \cdot dy \cdot t$

Mass moment of inertia of the strip about base is given by

$$(dI_m)_{BC} = \rho \cdot t \cdot \frac{b\,(h-y)}{h} \cdot y^2 dy$$

$\therefore$ Mass moment of inertia of the triangular plate about base is obtained by integrating above equation between the limits 0 to h

$$(I_m)_{xx} = \int_0^h \rho \cdot t \cdot \frac{b\,(hy^2 - y^3)}{h}\, dy$$

$$= \rho t \cdot \frac{b}{h}\left[\frac{hy^3}{3} - \frac{y^4}{4}\right]_0^h$$

$$= \rho t.\frac{b}{h}\left[\frac{h^4}{12}\right] = \frac{\rho t b h^3}{12}$$

$$\left(I_m\right)_{xx} = \frac{\rho bht}{2}.\frac{h^2}{6} = \frac{Mh^2}{6}$$

Mass moment of inertia of a triangular plate about parallel centroidal axis is obtained from parallel axis theorem as

$$\left(I_m\right)_{BC} = \left(I_m\right)_{xx} + My^2$$

$$\left(I_m\right)_{xx} = \left(I_m\right)_{BC} - My^2$$

$$= \frac{Mh^2}{6} - M\left[\frac{h}{3}\right]^2$$

$$\therefore \left(I_m\right)_{xx} = \frac{Mh^2}{18}$$

Similarly mass moment of inertia of a triangular plate about centroidal Y-Y axis is given as

$$\left(I_m\right)_{yy} = \frac{Mb^2}{24}$$

Moment of inertia of thin plates:

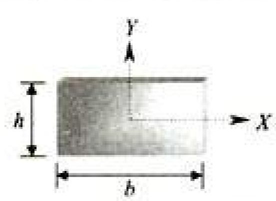
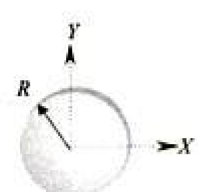
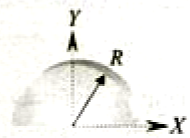
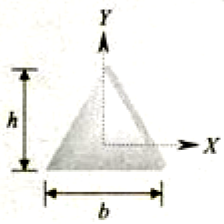
Plate	*Shape*	$\bar{I}_{xx}$	$\bar{I}_{yy}$
Rectangular		$\frac{Mh^2}{12}$	$\frac{Mb^2}{12}$
Circular		$\frac{MR^2}{4}$	$\frac{MR^2}{4}$

Plate	Shape	$\bar{I}_{xx}$	$\bar{I}_{yy}$
Semicircular		$\frac{MR^2}{4}$ (about base)	$\frac{MR^2}{4}$
Triangular		$\frac{Mh^2}{18}$	$\frac{Mb^2}{24}$

Mass moment of inertia of Solids:

i) Mass moment of inertia of solid cylinder:

Consider a cylinder of radius R, length L and density ρ. The coordinate axes are chosen about the centroid as shown in Fig. 4.45. Suppose we cut a circular disc of infinitesimal thickness dz perpendicular to Z-axis at a distance of z from the origin,

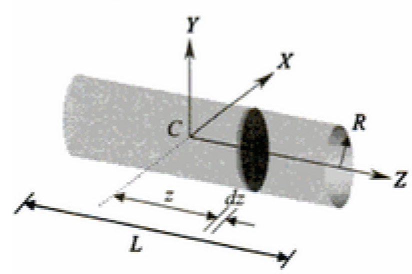


Fig. 4.45

The area of the disc is given as, $dA = \pi R^2$

The volume of the disc is given as, $dV = \pi R^2\, dz$

The mass of the disc is given as, $dM = \rho\pi R^2\, dz$

Therefore, mass moment of inertia of the disc about Z-axis is given as,

$$\left(dI_m\right)_{zz} = dM\frac{R^2}{2} = \rho\pi\frac{R^4}{2}\, dz \qquad \left[\because M.M.I.\ of\ circular\ disc = \frac{MR^2}{2}\right]$$

∴ Mass moment of inertia of the entire cylinder about Z-axis is obtained by integrating the above expression between the limits $-\frac{L}{2}\ to \frac{L}{2}$

$$\therefore \left(I_m\right)_{zz} = \int_{-\frac{L}{2}}^{\frac{L}{2}} \rho\pi\frac{R^4}{2}\, dz$$

$$= \rho\pi\frac{R^4}{2}[z]_{-L/2}^{L/2} = \rho\pi\frac{R^4}{2}L$$

$$\therefore \left(I_m\right)_{zz} = \rho\pi R^2 L\frac{R^2}{2} = \frac{MR^2}{2}$$

The mass moment of inertia of circular disc about an axis lying on its plane is given as,

$$\left(dI_m\right)_{xx} = dM\frac{R^2}{4} = \rho\pi\frac{R^4}{4}\,dz$$

By transfer theorem, the mass moment of inertia of the disc about the centroidal X-axis is given as

$$\left(dI_m\right)_{xx} = \rho\pi\frac{R^4}{4}\,dz + \rho\left(\pi R^2\right)dz.z^2$$

The mass moment of inertia of the entire cylinder about centroidal X-axis is obtained by integrating the above expression between the limits $-\frac{L}{2}\ to\frac{L}{2}$

$$\therefore\left(I_m\right)_{xx} = \int_{-\frac{L}{2}}^{\frac{L}{2}} \rho\pi\frac{R^4}{4}\,dz + \int_{-\frac{L}{2}}^{\frac{L}{2}} \rho\left(\pi R^2\right)dz.z^2$$

$$= \rho\pi\frac{R^4}{4}[z]_{-L/2}^{L/2} + \rho\left(\pi R^2\right)\left[\frac{z^3}{3}\right]_{-L/2}^{L/2}$$

$$= \rho\pi\frac{R^4}{4}\,L + \rho\left(\pi R^2\right)\frac{L^3}{12}$$

$$\therefore\left(I_m\right)_{xx} = \left(I_m\right)_{yy} = \frac{M}{12}\left[3R^2 + L^2\right]$$

Note:

For a slender rod, radius $R \ll L$,

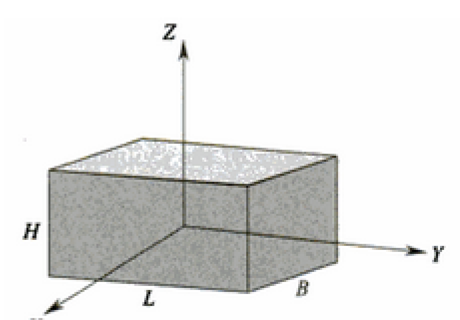
$$\therefore\left(I_m\right)_{xx} = \left(I_m\right)_{yy} = \frac{ML^2}{12}$$

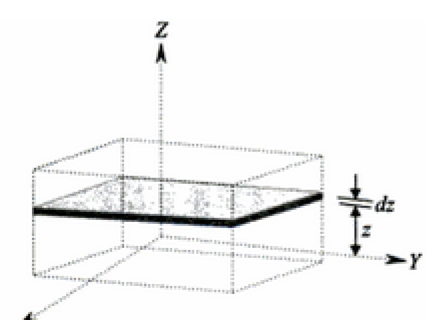
For a thin disc, length $L \ll R$,

$$\therefore\left(I_m\right)_{xx} = \left(I_m\right)_{yy} = \frac{MR^2}{4}$$

ii) Mass moment of inertia of a Prism:

Consider a prism of length L, breadth B, and height H and let its density be ρ.
Suppose we cut a thin plate of thickness dz at a height z from X-Y plane.
The area of the plate is given as, $dA = LB$
The volume of the disc is given as, $dV = LB.\,dz$
The mass of the disc is given as, $dM = \rho\,LB.dz$





Therefore, mass moment of inertia of the disc about Z-axis is given as,

$$\left(dI_m\right)_{zz} = \frac{\rho LB}{12}\left[L^2 + B^2\right]dz$$

∴ Mass moment of inertia of the entire prism about Z-axis is obtained by integrating the above expression between the limits $-\frac{H}{2}\ to\frac{H}{2}$

$$\therefore\left(I_m\right)_{zz} = \int_{-H/2}^{H/2}\frac{\rho LB}{12}\left[L^2 + B^2\right]dz$$

$$= \frac{\rho LB}{12}\left[L^2 + B^2\right][z]_{-H/2}^{H/2} = \frac{\rho LBH}{12}\left[L^2 + B^2\right]$$

$$\therefore\left(I_m\right)_{zz} = \frac{M}{12}\left[L^2 + B^2\right]$$

Similarly,

$$\therefore\left(I_m\right)_{xx} = \frac{M}{12}\left[L^2 + H^2\right]$$

$$\therefore\left(I_m\right)_{yy} = \frac{M}{12}\left[B^2 + H^2\right]$$

Note:

For a thin Plate, $H = 0$,

$$\therefore\left(I_m\right)_{xx} = \frac{M}{12}\left(L^2\right)$$

$$\therefore\left(I_m\right)_{yy} = \frac{M}{12}\left(B^2\right)$$

$$\therefore\left(I_m\right)_{zz} = \frac{M}{12}\left[L^2 + B^2\right]$$

iii) Mass moment of inertia of a Sphere:

Consider a sphere of radius R and density ρ. Suppose we cut a circular disc of radius r and infinitesimal thickness dz at a distance of z from X-Y plane.

The area of the disc is given as, $dA = \pi r^2$

The volume of the disc is given as, $dV = \pi r^2\, dz$

The mass of the disc is given as, $dM = \rho\pi r^2\, dz$

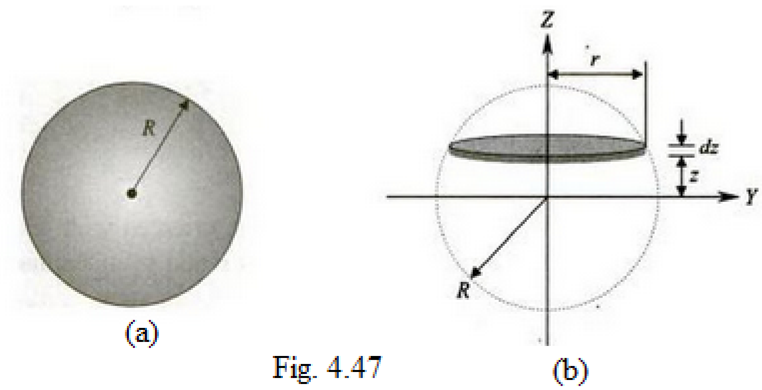


Fig. 4.47

Therefore, mass moment of inertia of the disc about Z-axis is given as,

$$\left(dI_m\right)_{zz} = dM\frac{r^2}{2} = \frac{\rho\pi r^4}{2}dz$$

∴ Mass moment of inertia of the entire sphere about Z-axis is obtained by integrating the above expression between the limits $-R$ to R

$$\therefore \left(I_m\right)_{zz} = \int_{-R}^{R} \frac{\rho\pi r^4}{2}dz$$

Since R is the radius of the sphere, $r^2 = R^2 - z^2$

$$\therefore \left(I_m\right)_{zz} = \int_{-R}^{R} \frac{\rho\pi\left(R^2 - z^2\right)^2}{2}dz$$

$$= \frac{\rho\pi}{2}\int_{-R}^{R}\left(R^4 + z^4 - 2R^2z^2\right)dz$$

$$= \frac{\rho\pi}{2}\left[R^4z + \frac{z^5}{5} - 2R^2\frac{z^3}{3}\right]_{-R}^{R}$$

$$= \frac{8}{15}\rho\pi R^5$$

$$= \rho\, \frac{4}{3}\pi R^3 \times \frac{2}{5}R^2$$

$$\therefore \left(I_m\right)_{zz} = \frac{2}{5}MR^2$$

Due to symmetry, the moment of inertia remains the same for any axis passing through the centroid. Hence in general, we express

$$I_m = \frac{2}{5}MR^2$$

iv) Mass moment of inertia of a cone:

Consider a cone of base radius *R*, height *H,* and density ρ. Suppose we cut a circular disc of radius *r* and infinitesimal thickness *dz* at a distance *z* from the origin.

The area of the disc is given as, $dA = \pi r^2$

The volume of the disc is given as, $dV = \pi r^2\, dz$

The mass of the disc is given as, $dM = \rho\pi r^2\, dz$

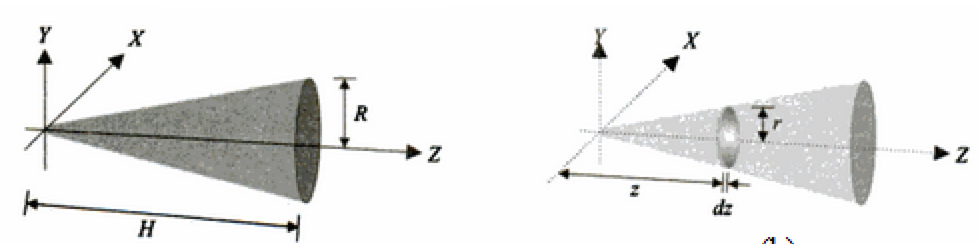


Fig. 4.48

Therefore, mass moment of inertia of the disc about Z-axis is given as,

$$\left(dI_m\right)_{zz} = dM\frac{r^2}{2} = \frac{\rho\pi r^4}{2}dz$$

∴ Mass moment of inertia of the entire cone about Z-axis is obtained by integrating the above expression between the limits $0\ to\ H$

$$\therefore \left(I_m\right)_{zz} = \int_0^H \frac{\rho\pi r^4}{2}dz$$

By similar triangles, we know,

$$\frac{r}{R} = \frac{z}{H}$$

$$\therefore r = \frac{R}{H}.z$$

$$\therefore \left(I_m\right)_{zz} = \int_0^H \frac{\rho\pi}{2}\frac{R^4}{H^4} z^4 dz$$

$$= \frac{\rho\pi}{2}\frac{R^4}{H^4}\int_0^H z^4 dz$$

$$= \frac{\rho\pi}{2}\frac{R^4}{H^4}\times\frac{H^5}{5} = \frac{1}{10}\rho\pi R^4 H$$

$$= \rho\frac{1}{3}\pi R^2 H\times\frac{3}{10}R^2$$

$$\therefore \left(I_m\right)_{zz} = \frac{3}{10}MR^2$$

The mass moment of inertia of the disc about an axis lying on its plane is given as,

$$\left(dI_m\right)_{xx} = dM\frac{r^2}{4} = \frac{\rho\pi r^4}{4}dz$$

By transfer theorem, the mass moment of inertia of the disc about the centroidal X-axis is given as

$$\left(dI_m\right)_{xx} = \frac{\rho\pi r^4}{4}dz + \rho\pi r^2\, dz\, .\, z^2$$

The mass moment of inertia of the entire cone about centroidal X-axis is obtained by integrating the above expression between the limits $0\ to H$

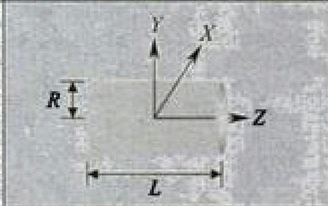
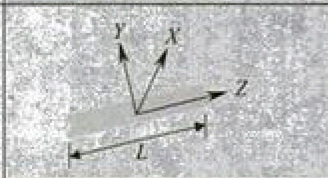
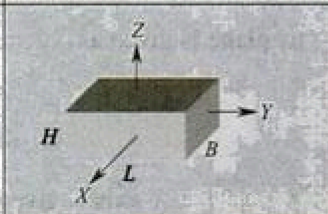
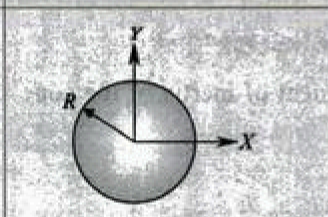
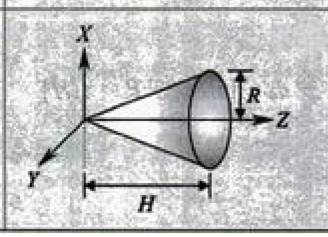
$$\therefore \left(I_m\right)_{xx} = \int_0^H \frac{\rho\pi r^4}{4}dz + \int_0^H \rho\pi r^2 . z^2 dz$$

$$= \frac{\rho\pi}{4}\frac{R^4}{H^4}\int_0^H z^4 dz + \frac{\rho\pi R^2}{H^2}\int_0^H z^4 dz$$

$$= \frac{1}{20}\rho\pi R^4 H + \frac{1}{5}\rho\pi R^2 H^3$$

$$\therefore \left(I_m\right)_{xx} = \left(I_m\right)_{yy} = \frac{3}{5}M\left[\frac{R^2}{4} + H^2\right]$$

Mass moment of inertia of solids:

Shape	Figure	I_{xx}	I_{yy}	I_{zz}
Cylinder		$\frac{M}{12}[3R^2 + L^2]$	$\frac{M}{12}[3R^2 + L^2]$	$\frac{MR^2}{2}$
Slender rod		$\frac{ML^2}{12}$	$\frac{ML^2}{12}$	
Prism		$\frac{M}{12}[L^2 + H^2]$	$\frac{M}{12}[B^2 + H^2]$	$\frac{M}{12}(L^2 + B^2)$
Sphere		$\frac{2}{5}MR^2$ (about any diametric axis)		
Cone		$\frac{3}{5}M\left[\frac{R^2}{4} + H^2\right]$	$\frac{3}{5}M\left[\frac{R^2}{4} + H^2\right]$	$\frac{3}{10}MR^2$

Example. 4.10. *Find the mass moment of inertia of thin plate bent as shown in Fig. 4.49 about A-A axis. Density of the plate is 7850 kg/m³and thickness is 5 mm.*

Solution.

Given:

Density $\rho = 7850\ kg/m^3$

Thickness, t = 5 mm = 0.005 m.

We know that the mass of thin plate is ρAt.

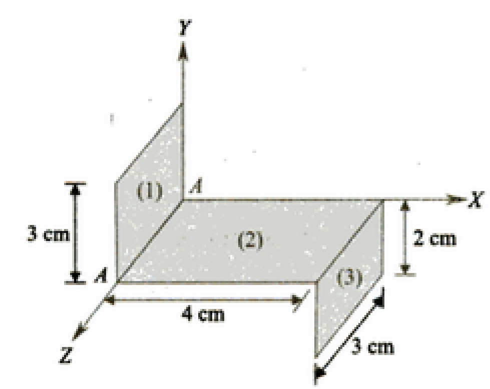


Fig. 4.49

$$M_1 = 7850\times(0.03\times0.03)\times0.005 = 0.0353\ kg$$

$$M_2 = 7850\times(0.04\times0.03)\times0.005 = 0.0471\ kg$$

$$M_3 = 7850\times(0.03\times0.02)\times0.005 = 0.0236\ kg$$

Calculation of mass moment of inertia:

For plate-I

The mass moment of inertia about its centroidal axis parallel to the Z-axis is

$$\left(I_{zz}\right)_1 = \frac{M_1 h_1^2}{12}$$

$$= \frac{0.0353\times0.03^2}{12} = 2.65\times10^{-6} kg.m^2$$

∴ Mass moment of inertia about A-A axis is obtained as,

$$\left(I_{AA}\right)_1 = \left(I_{zz}\right)_1 + M_1 d_1^2$$

$$= 2.65\times10^{-6} + \left[0.0353\times(0.015)^2\right] = 1.06\times10^{-5} kg.m^2$$

For plate-II

The mass moment of inertia about its centroidal axis parallel to the Z-axis is

$$\left(I_{zz}\right)_2 = \frac{M_2 h_2^2}{12}$$

$$= \frac{0.0471\times0.04^2}{12} = 6.28\times10^{-6} kg.m^2$$

∴ Mass moment of inertia about A-A axis is obtained as,

$$\left(I_{AA}\right)_2 = \left(I_{zz}\right)_2 + M_2 d_2^2$$

$$= 6.28\times10^{-6} + \left[0.0471\times(0.02)^2\right] = 2.51\times10^{-5} kg.m^2$$

For plate-III

The mass moment of inertia about its centroidal axis parallel to the Z-axis is

$$\left(I_{zz}\right)_3 = \frac{M_3 h_3^2}{12}$$

$$= \frac{0.0236\times0.02^2}{12} = 7.87\times10^{-7} kg.m^2$$

∴ Mass moment of inertia about A-A axis is obtained as,

$$\left(I_{AA}\right)_3 = \left(I_{zz}\right)_3 + M_3 d_3^2$$

$$where\ d_3 = \sqrt{(0.04)^2 + (0.04)^2} = \sqrt{1.7\times10^{-3}} m$$

$$= 7.87\times10^{-7} + \left[0.0236\times\left(\sqrt{1.7\times10^{-3}}\right)^2\right] = 4.09\times10^{-5} kg.m^2$$

∴ Mass moment of inertia of the composite plate is given as,

$$I_{AA} = \Sigma\left(I_{AA}\right)_i$$

$$= 1.06\times10^{-5} + 2.51\times10^{-5} + 4.09\times10^{-5}$$

$$= 7.66\times10^{-5} kg.m^2$$

Example. 4.11. *Determine mass moment of inertia of a thin rectangular plate of thickness 5 mm in which a semicircular portion is cut as shown in Fig. 4.50 about the base. The density of the material of the plate is 7850 kg/m³.*

Solution.

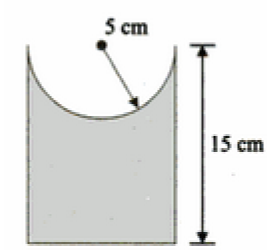


Fig. 4.50

Mass calculations:

Rectangular plate, $M_1 = \rho_1 A_1 t_1$

$$= 7850\times(0.1\times0.15)\times0.005 = 0.589\ kg$$

Semicircular plate, $M_2 = \rho_2 A_2 t_2$

$$= 7850\times\left(\pi\times\frac{0.05^2}{2}\right)\times0.005 = 0.154\ kg$$

Mass moment of inertia calculations:

For rectangular plate, we know that mass moment of inertia of a thin rectangular plate about its centroidal axis is

$$I_1 = \frac{M_1 h^2}{12}$$

Therefore, mass moment of inertia of the plate about its base is

$$I_1 = \frac{M_1 h^2}{12} + M_1\left[\frac{h}{2}\right]^2 = \frac{M_1 h^2}{3}$$

$$= \frac{(0.589)(0.15)^2}{3} = 4.418\times10^{-3} kg.m^2$$

For semicircular plate, we know that mass moment of inertia of a thin semicircular plate about its centroidal axis is

$$I_2 = 0.07 M_2 R^2$$

Therefore, mass moment of inertia of the plate about its base is

$$I_2 = 0.07 M_2 R^2 + M_2 d^2$$

$$= 0.07\times0.154\times(0.05)^2 + 0.154[0.15 - (4\times0.05)/(3\times\pi)]^2$$

$$= 2.58\times10^{-3} kg.m^2$$

Therefore, the mass moment of inertia of the composite section is given as

$$I = I_1 - I_2$$

$$= 4.418\times10^{-3} - 2.58\times10^{-3} = 1.838\times10^{-3} kg.m^2$$

Example. 4.12. *A cube of 250 mm side has mass density of 4000 kg/m³. Determine the mass moment of inertia of the cube about one of its edges.*

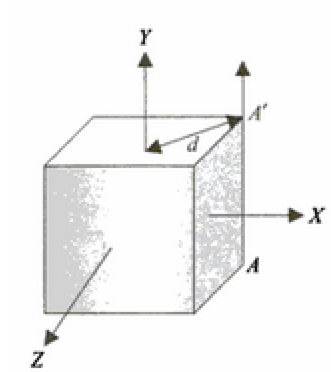


Fig. 4.51

Solution.

Given:

Side, $a = 0.25\ m$

Density, $\rho = 4000\ kg/m^3$

∴ Its mass, $M = \rho a^3 = 4000\times0.25^3 = 62.5\ kg$

Considering the cube as a prism, the mass moment of inertia about the centroidal Y-axis is obtained as

$$I_{yy} = M(a^2 + a^2)/12$$

$$= 62.5\frac{[(0.25)^2+(0.25)^2]}{12} = 0.651\ kg.m^2$$

By transfer theorem, moment of inertia about one of the edges AA' is obtained as

$$I_{AA'} = I_{yy} + Md^2$$

We know that $d^2 = \left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2 = 0.03125\ m^2$

$$I_{AA'} = 0.651 + 62.5(0.03125) = 2.6\ kg.m^2$$

Example. 4.13 *A brass cone having a base diameter of 40 cm and a height of 25 cm is placed on top of a vertical cylinder of same diameter and a height of 20 cm. Determine the mass moment of the composite body about the vertical geometric axis. Take density of brass and steel to be 8400 kg/m³and 7850 kg/m³respectively.*

Solution.

Given:

For brass cone

$$R_1 = 20\ cm = 0.2\ m$$

$$H_1 = 25cm = 0.25m$$

$$\rho_1 = 8400\ kg/m^3$$

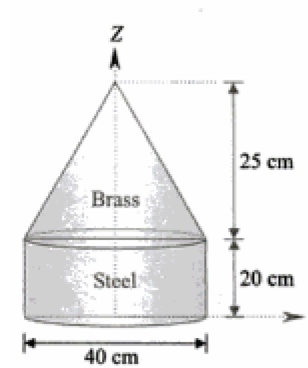


Fig. 4.52

For steel cylinder

$$R_2 = 20\ cm = 0.2\ m$$

$$H_2 = 20cm = 0.2m$$

$$\rho_2 = 7850\ kg/m^3$$

Mass calculations

Cone,

Volume, $V_1 = \frac{1}{3}\pi R_1^2 H_1$

$$= \frac{1}{3}\pi\times0.2^2\times0.25 = 10.47\times10^{-3}m^3$$

Mass, $M_1 = \rho_1 V_1$

$$= 8400\times10.47\times10^{-3} = 87.95\ kg$$

Cylinder,

Volume, $V_2 = \pi R_2^2 H_2$

$$= \pi\times0.2^2\times0.2 = 25.13\times10^{-3}m^3$$

Mass, $M_2 = \rho_2 V_2$

$$= 7850\times25.13\times10^{-3} = 197.27\ kg$$

Mass moment of inertia calculations

Cone,

$$\left(I_{zz}\right)_1 = \frac{3}{10}M_1R_1^2$$

$$= \frac{3}{10}\times87.95\times0.2^2 = 1.055\ kg.m^2$$

Cylinder,

$$\left(I_{zz}\right)_2 = \frac{1}{2}M_2R_2^2$$

$$\frac{1}{2}\times197.27\times0.2^2 = 3.95\ kg.m^2$$

∴ Mass moment of inertia of the composite body about the vertical geometric axis is obtained as

$$I_{zz} = \left(I_{zz}\right)_1 + \left(I_{zz}\right)_2$$

$$= 1.055 + 3.95 = 5.005\ kg.m^2$$