1.2 Completing the square

Another method we can use for solving quadratic equations is **completing the square**.

The method of completing the square aims to rewrite a quadratic expression using only one occurrence of the variable, making it an easier expression to work with.

If we expand the expressions $(x+d)^2$ and $(x-d)^2$, we obtain the results:

$$(x+d)^2 = x^2 + 2dx + d^2 \quad \text{and} \quad (x-d)^2 = x^2 - 2dx + d^2$$

Rearranging these gives the following important results:

KEY POINT 1.1

$$x^2 + 2dx = (x+d)^2 - d^2 \quad \text{and} \quad x^2 - 2dx = (x-d)^2 - d^2$$

To complete the square for $x^2 + 10x$, we can use the first of the previous results as follows:

$$10 \div 2 = 5$$

$$x^2 + 10x = (x+5)^2 - 5^2$$

$$x^2 + 10x = (x+5)^2 - 25$$

To complete the square for $x^2 + 8x - 7$, we again use the first result applied to the $x^2 + 8x$ part, as follows:

$$8 \div 2 = 4$$

$$x^2 + 8x - 7 = (x+4)^2 - 4^2 - 7$$

$$x^2 + 8x - 7 = (x+4)^2 - 23$$

To complete the square for $2x^2 - 12x + 5$, we must first take a factor of 2 out of the first two terms, so:

$$2x^2 - 12x + 5 = 2(x^2 - 6x) + 5$$

$$6 \div 2 = 3$$

$$x^2 - 6x = (x-3)^2 - 3^2, \text{ giving}$$

$$2x^2 - 12x + 5 = 2[(x-3)^2 - 9] + 5 = 2(x-3)^2 - 13$$

We can also use an algebraic method for completing the square, as shown in Worked example 1.5.

WORKED EXAMPLE 1.5

Express $2x^2 - 12x + 3$ in the form $p(x - q)^2 + r$, where p, q and r are constants to be found.

Answer

$$2x^2 - 12x + 3 = p(x - q)^2 + r$$

Expanding the brackets and simplifying gives:

$$2x^2 - 12x + 3 = px^2 - 2pqx + pq^2 + r$$

Comparing coefficients of x^2, coefficients of x and the constant gives

$2 = p$ (1) $\quad -12 = -2pq$ (2) $\quad 3 = pq^2 + r$ (3)

Substituting $p = 2$ in equation (2) gives $q = 3$

Substituting $p = 2$ and $q = 3$ in equation (3) therefore gives $r = -15$

$$2x^2 - 12x + 3 = 2(x - 3)^2 - 15$$

WORKED EXAMPLE 1.6

Express $4x^2 + 20x + 5$ in the form $(ax + b)^2 + c$, where a, b and c are constants to be found.

Answer

$$4x^2 + 20x + 5 = (ax + b)^2 + c$$

Expanding the brackets and simplifying gives:

$$4x^2 + 20x + 5 = a^2x^2 + 2abx + b^2 + c$$

Comparing coefficients of x^2, coefficients of x and the constant gives

$4 = a^2$ (1) $\quad 20 = 2ab$ (2) $\quad 5 = b^2 + c$ (3)

Equation (1) gives $a = \pm 2$.

Substituting $a = 2$ into equation (2) gives $b = 5$.

Substituting $b = 5$ into equation (3) gives $c = -20$.

$$4x^2 + 20x + 5 = (2x + 5)^2 - 20$$

1.7 Solving quadratic inequalities

We already know how to solve linear inequalities.

The following text shows two examples.

Solve $2(x+7) < -4$.	Expand brackets.
$2x + 14 < -4$	Subtract 14 from both sides.
$2x < -18$	Divide both sides by 2.
$x < -9$	

Solve $11 - 2x \geqslant 5$.	Subtract 11 from both sides.
$-2x \geqslant -6$	Divide both sides by -2.
$x \leqslant 3$	

The second of the previous examples uses the important rule that:

KEY POINT 1.4

If we multiply or divide both sides of an inequality by a negative number, then the inequality sign must be reversed.

Quadratic inequalities can be solved by sketching a graph and considering when the graph is above or below the x-axis.

WORKED EXAMPLE 1.15

Solve $x^2 - 5x - 14 > 0$.

Answer

Sketch the graph of $y = x^2 - 5x - 14$.

When $y = 0$, $x^2 - 5x - 14 = 0$

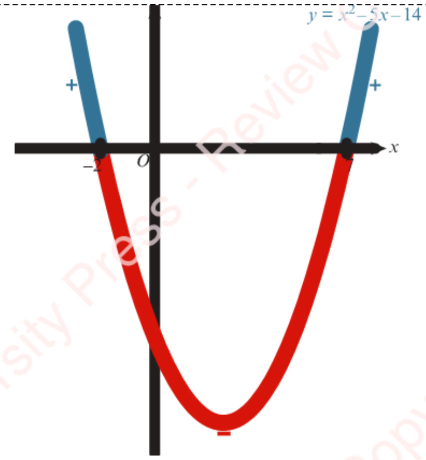
$$(x + 2)(x - 7) = 0$$

$$x = -2 \quad \text{or} \quad x = 7$$

So the x-axis crossing points are -2 and 7.

For $x^2 - 5x - 14 > 0$ we need to find the range of values of x for which the curve is positive (above the x-axis).

The solution is $x < -2$ or $x > 7$.



TIP

For the sketch graph, you only need to identify which way up the graph is and where the x-intercepts are: you do not need to find the vertex or the y-intercept.

WORKED EXAMPLE 1.16

Solve $2x^2 + 3x \leqslant 27$.

Answer

Rearranging: $2x^2 + 3x - 27 \leqslant 0$

Sketch the graph of $y = 2x^2 + 3x - 27$.

When $y = 0$, $2x^2 + 3x - 27 = 0$

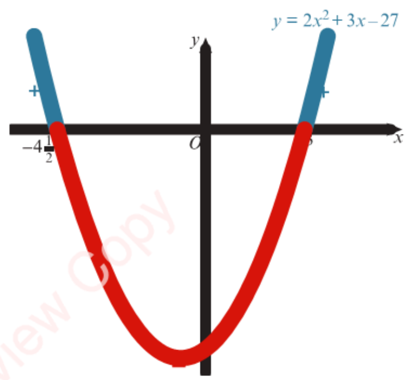
$$(2x + 9)(x - 3) = 0$$

$$x = -4\tfrac{1}{2} \text{ or } x = 3$$

So the x-axis intercepts are $-4\tfrac{1}{2}$ and 3.

For $2x^2 + 3x - 27 \leqslant 0$ we need to find the range of values of x for which the curve is either zero or negative (below the x-axis).

The solution is $-4\tfrac{1}{2} \leqslant x \leqslant 3$.



1.8 The number of roots of a quadratic equation

If $f(x)$ is a function, then we call the solutions to the equation $f(x) = 0$ the **roots** of $f(x)$.

Consider solving the following three quadratic equations of the form $ax^2 + bx + c = 0$ using the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$x^2 + 2x - 8 = 0$	$x^2 + 6x + 9 = 0$	$x^2 + 2x + 6 = 0$
$x = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times (-8)}}{2 \times 1}$	$x = \frac{-6 \pm \sqrt{6^2 - 4 \times 1 \times 9}}{2 \times 1}$	$x = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 6}}{2 \times 1}$
$x = \frac{-2 \pm \sqrt{36}}{2}$	$x = \frac{-6 \pm \sqrt{0}}{2}$	$x = \frac{-2 \pm \sqrt{-20}}{2}$
$x = 2$ or $x = -4$	$x = -3$ or $x = -3$	no real solution
two distinct real roots	**two equal real roots**	**no real roots**

The part of the quadratic formula underneath the square root sign is called the **discriminant**.

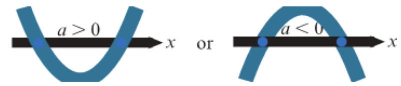
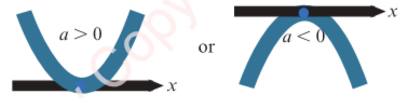
KEY POINT 1.5

The discriminant of $ax^2 + bx + c = 0$ is $b^2 - 4ac$.

The sign (positive, zero or negative) of the discriminant tells us how many roots there are for a particular quadratic equation.

$b^2 - 4ac$	Nature of roots
> 0	two distinct real roots
$= 0$	two equal real roots (or 1 repeated real root)
< 0	no real roots

There is a connection between the roots of the quadratic equation $ax^2 + bx + c = 0$ and the corresponding curve $y = ax^2 + bx + c$.

$b^2 - 4ac$	Nature of roots of $ax^2 + bx + c = 0$	Shape of curve $y = ax^2 + bx + c$
> 0	two distinct real roots	The curve cuts the x-axis at two distinct points. 
$= 0$	two equal real roots (or 1 repeated real root)	The curve touches the x-axis at one point. 
< 0	no real roots	The curve is entirely above or entirely below the x-axis. 

Find the values of k for which the equation $4x^2 + kx + 1 = 0$ has two equal roots.

Answer

For two equal roots: $b^2 - 4ac = 0$

$k^2 - 4 \times 4 \times 1 = 0$

$k^2 = 16$

$k = -4$ or $k = 4$

WORKED EXAMPLE 1.18

Find the values of k for which $x^2 - 5x + 9 = k(5 - x)$ has two equal roots.

Answer

$x^2 - 5x + 9 = k(5 - x)$ Rearrange the equation into the form $ax^2 + bx + c = 0$.

$x^2 - 5x + 9 - 5k + kx = 0$

$x^2 + (k - 5)x + 9 - 5k = 0$

For two equal roots: $b^2 - 4ac = 0$

$(k - 5)^2 - 4 \times 1 \times (9 - 5k) = 0$

$$k^2 - 10k + 25 - 36 + 20k = 0$$
$$k^2 + 10k - 11 = 0$$
$$(k + 11)(k - 1) = 0$$
$$k = -11 \quad \text{or} \quad k = 1$$

WORKED EXAMPLE 1.19

Find the values of k for which $kx^2 - 2kx + 8 = 0$ has two distinct roots.

Answer

$kx^2 - 2kx + 8 = 0$

For two distinct roots:

$$b^2 - 4ac > 0$$
$$(-2k)^2 - 4 \times k \times 8 > 0$$
$$4k^2 - 32k > 0$$
$$4k(k - 8) > 0$$

Critical values are 0 and 8.

Note that the critical values are where $4k(k - 8) = 0$.

Hence, $k < 0$ or $k > 8$.