[Vector Calculus MAT226 Spring 2025]
[Professor Sormani]
Lesson 20: Method of Lagrange Multipliers

You will cut and paste the photos of your notes and completed classwork in a googledoc entitled:

MAT226S25-lesson20-lastname-firstname

Then share editing of that document with me sormanic@gmail.com. You will also put photos of your homework in this googledoc. If you work with any classmates, be sure to write their names on the problems you completed together.

This topic is an essential application of Vector Calculus! In this lesson we present three examples.

If you missed class watch the [Playlist 226F21-19-ip123]

The Method of Lagrange Multipliers

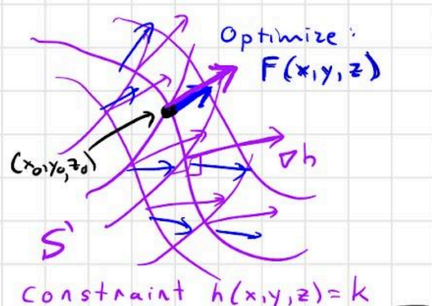
Optimize a function $F(x,y,z)$
subject to a constraint $h(x,y,z)=k$

Old method: Use $h(x,y,z)=k$ to solve for $z = \text{----}$
then sub in formula for z into $F(x,y,z)$
to get a function of only two variables.
which you then maximize over a domain.

Example 1: Maximize Volume of a Rectangular Block
under a plane $6x+2y+3z=6$

Example 2: Maximize distance from a point $P=\begin{pmatrix}3\\0\\7\end{pmatrix}$
to a surface $x^2+\left(\frac{y}{3}\right)^2+\left(\frac{z}{2}\right)^2=1$

Difficulties with old method:
Solving for z may be impossible
or have a very messy formula.
or the domain is difficult to find.



Key Idea of Lagrange:
$\nabla F \perp S$ at a max or min

Lagrange's Theorem

If $(x_0,y_0,z_0)\in S$ achieves the maximum value of F on S
$\max_S F = F(x_0,y_0,z_0)$
and $\nabla h_{(x_0,y_0,z_0)} \neq 0$

Then $\exists \lambda \in \mathbb{R}$ called the Lagrange multiplier
such that $\nabla F_{(x_0,y_0,z_0)} = \lambda \nabla h_{(x_0,y_0,z_0)}$

same theorem holds for a minimum as well.

Method of Lagrange Multipliers:

Find critical points by solving: $\boxed{\begin{gathered}\nabla F_{(x,y,z)} = \lambda \nabla h_{(x,y,z)} \\ h(x,y,z)=k\end{gathered}}$
which is four equations
with four unknowns: x, y, z, λ

$$\boxed{\begin{aligned} f_x(x,y,z) &= \lambda h_x(x,y,z) \\ f_y(x,y,z) &= \lambda h_y(x,y,z) \\ f_z(x,y,z) &= \lambda h_z(x,y,z) \\ h(x,y,z) &= k \end{aligned}}$$

Proof of Lagrange's Thm for Maxima: $\boxed{\text{HW}}$ Prove it for minima

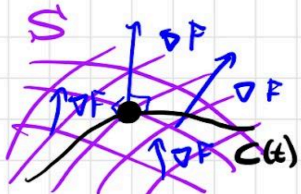
Given: $(x_0,y_0,z_0) \in S$ which means $h(x_0,y_0,z_0)=k$

the maximum value of F on S: $\max_S F = F(x_0,y_0,z_0)$

and $\nabla h_{(x_0,y_0,z_0)} \neq 0$

Show: $\exists \lambda \in \mathbb{R}$ such that $\nabla F_{(x_0,y_0,z_0)} = \lambda \nabla h_{(x_0,y_0,z_0)}$ this means Show $\nabla F \perp S$

Proof:



① $F(x_0,y_0,z_0) \geq F(x,y,z)$ for all $\forall (x,y,z) \in S$
① by given and defn of max

② Let $C(t) = (x(t),y(t),z(t)) \in S$ such that $C(0) = (x_0,y_0,z_0)$
So $h(x(t),y(t),z(t)) = k$ ② by defn of S

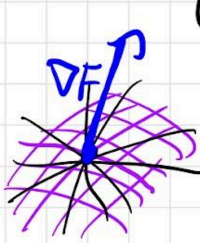
③ $F(x(0),y(0),z(0)) \geq F(x(t),y(t),z(t))$ for all t
③ by steps 1 and 2 together.

④ $f(t) = F(x(t),y(t),z(t))$ has a maximum at $t=0$. ④ by step 3 and defn of max.

⑤ $f'(0) = 0$ ⑤ by max $\Rightarrow$ critical pt in calc I.

⑥ $0 = f'(0) = \frac{d}{dt} F(x(t),y(t),z(t))\Big|_{t=0}$ ⑥ by defn of f in step 4

⑦ $0 = \nabla F_{(x_0,y_0,z_0)} \cdot C'(0)$ ⑦ by chain rule



⑧ $\nabla F_{(x_0,y_0,z_0)} \perp C'(0)$ for any curve $C(t) \in S$ s.t $C(0) = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$
⑧ by $\vec{v} \cdot \vec{w} = 0 \Rightarrow \vec{v} \perp \vec{w}$

⑨ $\nabla F_{(x_0,y_0,z_0)} \perp S$ ⑨ by $C(t)$ is any curve in S

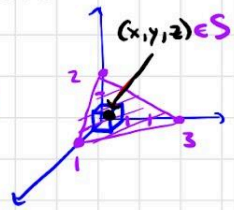
Since $\nabla h \neq 0$ we have $\nabla F_{(x_0,y_0,z_0)} = \lambda \nabla h_{(x_0,y_0,z_0)}$ QED.

Classwork 1

Classwork

Example 1: Maximize Volume of a Rectangular Block under a plane $6x+2y+3z=6$

Solve using the Method of Lagrange Multipliers



$F(x,y,z) = x \cdot y \cdot z$ (volume of a rectangular block

$h(x,y,z) = 6x+2y+3z=6$ (constraint : a plane)

$\exists \lambda \in \mathbb{R}$ s.t $\nabla F_{(x,y,z)} = \lambda \nabla h_{(x,y,z)}$

$h(x,y,z) = k$ ⟵ point in S

$$\nabla F = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x}(xyz) \\ \frac{\partial}{\partial y}(xyz) \\ \frac{\partial}{\partial z}(xyz) \end{pmatrix} = \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix}$$

$$\nabla h = \begin{pmatrix} h_x \\ h_y \\ h_z \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x}(6x+2y+3z) \\ \frac{\partial}{\partial y}(6x+2y+3z) \\ \frac{\partial}{\partial z}(6x+2y+3z) \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix} \longleftarrow \text{normal to the plane } S$$

$$\boxed{\begin{matrix} \nabla F = \lambda \nabla h \\ h(x,y,z) = k \end{matrix}} \Leftrightarrow \boxed{\begin{matrix} \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix} = \lambda \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix} \\ 6x+2y+3z=6 \end{matrix}} \Leftrightarrow \boxed{\begin{matrix} yz = \lambda 6 \\ xz = \lambda 2 \\ xy = \lambda 3 \\ 6x+2y+3z=6 \end{matrix}}$$

Must solve this system (not linear)

We want to solve the system for (x,y,z) don't care about λ

so get rid of λ first if possible.

$\frac{yz}{6} = \lambda \quad \frac{xz}{2} = \lambda \quad \frac{xy}{3} = \lambda$ Thus: $\frac{yz}{6} = \frac{xz}{2} = \frac{xy}{3}$ How to use this?

So try something else: maybe notice symmetry

"λ lambda" Lagrange

$xyz = x\lambda 6$
$yxz = y\lambda 2$
$zxy = z\lambda 3$

now all three equations equal xyz

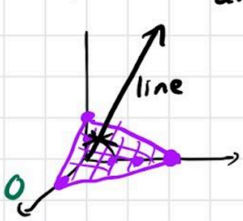
Thus $x\lambda 6 = y\lambda 2 = z\lambda 3$

$\boxed{\text{Assume } \lambda \neq 0}$ divide it out and get $6x = 2y = 3z$ which is a line

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} 1/6 \\ 1/2 \\ 1/3 \end{pmatrix} = \begin{pmatrix} t/6 \\ t/2 \\ t/3 \end{pmatrix}$$

$6x+2y+3z=6$

S



where does the line cross S?

$6\frac{t}{6} + 2\frac{t}{2} + 3\frac{t}{3} = 6 \Rightarrow t+t+t=6$ so $t=2$

So $x = \frac{2}{6}$ $y = \frac{2}{2}$ $z = \frac{2}{3}$

Must also check what happens when $\lambda = 0$

Maybe more critical pts!

What happens when $\lambda = 0$?

Check: $F\left(\frac{2}{6}, \frac{2}{2}, \frac{2}{3}\right) = \left(\frac{2}{6}\right)\left(\frac{2}{2}\right)\left(\frac{2}{3}\right) = \boxed{\frac{2}{9}}$

$\frac{yz}{6} = 0$ AND $\frac{xz}{2} = 0$ AND $\frac{xy}{3} = 0$

constraint $6x + 2y + 3z = 6$

$\begin{pmatrix} y=0 \\ \text{OR} \\ z=0 \end{pmatrix}$ AND $\begin{pmatrix} x=0 \\ \text{OR} \\ z=0 \end{pmatrix}$ AND $\begin{pmatrix} x=0 \\ \text{OR} \\ y=0 \end{pmatrix}$

This leads to 8 overlapping case.

$(0,0,z), (0,y,0), (x,0,0), (0,0,0)$

$6\cdot 0 + 2\cdot 0 + 3z = 6$
$z = 2$ $\Longrightarrow F(0,0,2) = \boxed{0}$

$6\cdot 0 + 2y + 3\cdot 0 = 6$
$y = 3$ $\Longrightarrow F(0,3,0) = \boxed{0}$

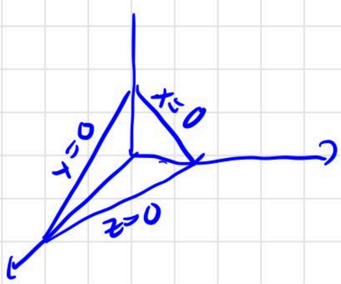
$6\cdot x + 2\cdot 0 + 3\cdot 0 = 6$
$x = 1$ $\Longrightarrow F(1,0,0) = \boxed{0}$

on boundary

$F(0,0,0) = 0$
$6\cdot 0 + 2\cdot 0 + 3\cdot 0 = 0 \neq 6$
not even on surface

Is there a boundary?

Domain
$x \geq 0, y \geq 0, z \geq 0$



boundary is a triangle

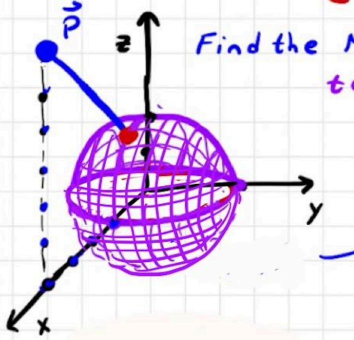
but on the boundary
$F = \boxed{0}$ because $F(x,y,z) = xyz$

So max $F = \boxed{\frac{2}{9}}$ at $\left(\frac{2}{6}, \frac{2}{2}, \frac{2}{3}\right)$

min $F = \boxed{0}$ on boundary.

Classwork 2

Classwork 2



Find the Minimum distance from a $P=(5,0,6)$ to a surface a sphere of radius 2

$$d(\vec{p},\vec{x}) = |\vec{p}-\vec{x}| = \sqrt{(5-x)^2+(0-y)^2+(6-z)^2}$$

derivatives of square roots are difficult, so instead minimize $\boxed{d^2 = (5-x)^2+y^2+(6-z)^2} = F(x,y,z)$

constraint: sphere of radius 2

$$x^2+y^2+z^2 = 2^2 = h(x,y,z)$$

Solve using the Method of Lagrange Multipliers $\nabla F = \lambda \nabla h$

$$\nabla F = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} (5-x)^2+y^2+(6-z)^2 \\ \frac{\partial}{\partial y} (5-x)^2+y^2+(6-z)^2 \\ \frac{\partial}{\partial z} (5-x)^2+y^2+(6-z)^2 \end{pmatrix} = \begin{pmatrix} 2(5-x)(-1) \\ 2y \\ 2(6-z)(-1) \end{pmatrix}$$

by chain rule

$$\nabla h = \begin{pmatrix} h_x \\ h_y \\ h_z \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} x^2+y^2+z^2 \\ \frac{\partial}{\partial y} x^2+y^2+z^2 \\ \frac{\partial}{\partial z} x^2+y^2+z^2 \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

$$\boxed{\begin{array}{c} \begin{pmatrix} -2(5-x) \\ 2y \\ -2(6-z) \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} \\ x^2+y^2+z^2 = 4 \end{array}}$$

λ Lambda "Lagrange Multiplier"

↑ Next solve this system.

$$\boxed{\begin{aligned}-10+2x&=\lambda 2x\\ 2y&=\lambda 2y\\ -12+2z&=\lambda 2z\\ x^2+y^2+z^2&=4\end{aligned}}$$

4 equations
4 unknown

Notice the second equation $2y=\lambda 2y$

Case I: $y=0$ OR Case II $\lambda=1$

$$\boxed{\begin{aligned}x^2+0^2+z^2&=4\\ -10+2x&=\lambda\cdot 2x\\ -12+2z&=\lambda\cdot 2z\end{aligned}}$$

3 equations and 3 unknowns

Solve for $x=\dfrac{10}{2-2\lambda}$

OR $\lambda=1$

Solve for $z=\dfrac{12}{(2-2\lambda)}$

OR $\lambda=1$

Case Ia: $y=0$ and $\lambda=1$

Covered by $\lambda=1$ in Case II above

Case Ib: $y=0$ and $x=\dfrac{10}{2-2\lambda}$ and $z=\dfrac{12}{2-2\lambda}$

and $x^2+z^2=4$

plug into $\left(\dfrac{10}{2-2\lambda}\right)^2+\left(\dfrac{12}{2-2\lambda}\right)^2=4$

$$10^2+12^2=4(2-2\lambda)^2$$

$$\frac{100+144}{4}=(2-2\lambda)^2$$

$$61=(2-2\lambda)^2$$

$$\pm\sqrt{61}=2-2\lambda$$

$x=\dfrac{10}{\pm\sqrt{61}}$ and $z=\dfrac{12}{\pm\sqrt{61}}$

4 critical pts

$$\left(\frac{\pm 10}{\sqrt{61}},\ 0,\ \frac{\pm 12}{\sqrt{61}}\right)$$

Case II $\lambda = 1$

$$\boxed{\begin{aligned} -10 + 2x &= \lambda 2x \\ 2y &= \lambda 2y \\ -12 + 2z &= \lambda 2z \\ x^2+y^2+z^2 &= 4 \end{aligned}}$$

$$\boxed{\begin{aligned} -10 + 2x &= 2x \\ \cancel{2y = 2y} \\ -12 + 2z &= 2z \\ x^2+y^2+z^2 &= 4 \end{aligned}}$$

3 equations
3 unknowns

$-10 = 0$ impossible

So Case II never happens

What about boundary?

No worries because S goes smoothly around with no edges.

So check the four points: 4 critical pts $\left(\frac{\pm 10}{\sqrt{61}}, 0, \frac{\pm 12}{\sqrt{61}}\right)$

$F\left(\frac{10}{\sqrt{61}}, 0, \frac{12}{\sqrt{61}}\right) = \_\_\_\_$ $\qquad$ $F\left(\frac{10}{\sqrt{61}}, 0, \frac{-12}{\sqrt{61}}\right) = \_\_\_\_$

$F\left(\frac{-10}{\sqrt{61}}, 0, \frac{12}{\sqrt{61}}\right) = \_\_\_\_$ $\qquad$ $F\left(\frac{-10}{\sqrt{61}}, 0, \frac{-12}{\sqrt{61}}\right) = \_\_\_\_$

Find Max + Min.

The Method of Lagrange Multipliers works with any number of variables

Maximize $F(x,y)$ subject to constraint $h(x,y)=k$

HW $\longrightarrow$ then solve 3 eqns $\left\{ \boxed{\begin{array}{l} \nabla F = \lambda \nabla h \\ h(x,y) = k \end{array}} \right.$ 3 eqns $\boxed{\begin{array}{l} F_x = \lambda h_x \\ F_y = \lambda h_y \\ h(x,y) = k \end{array}}$ for 3 unknowns x, y, k

Maximize $F(x_1, x_2 \ldots x_m)$ subject to $h(x_1, \ldots x_m) = k$

$$\nabla F = \begin{pmatrix} F_{x_1} \\ F_{x_2} \\ \vdots \\ F_{x_m} \end{pmatrix} = \lambda \begin{pmatrix} h_{x_1} \\ h_{x_2} \\ \vdots \\ h_{x_m} \end{pmatrix}$$

m+1 eqns for m+1 unknowns

Very difficult to solve when many variables.

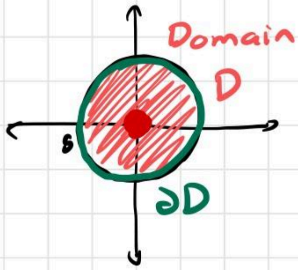
(Next meeting we will have 2 constraints.)

Classwork 3

Classwork 3

Maximize $f(x,y) = e^{-xy/4}$ over the disk $\boxed{x^2+y^2 \le 25}$

D= Domain



step I: $\underline{\text{Find}}$ and $\underline{\text{Check}}$ Critical Points of f on the interior of D

$$\text{int}(D) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \; x^2+y^2 < 25 \right\}$$

step II: Check for max on boundary

$$\partial D = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \; x^2+y^2 = 25 \right\}$$

$\boxed{\text{Step I}}$

Look for $\nabla f = \vec{0}$ inside $\text{int}(D)$

$$\nabla f = \begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} e^{-xy/4} \\ \frac{\partial}{\partial y} e^{-xy/4} \end{pmatrix} = \begin{pmatrix} e^{-xy/4} \frac{\partial}{\partial x}\left(\frac{-xy}{4}\right) \\ e^{-xy/4} \frac{\partial}{\partial y}\left(\frac{-xy}{4}\right) \end{pmatrix} = \begin{pmatrix} \frac{-y}{4} e^{-xy/4} \\ \frac{-x}{4} e^{-xy/4} \end{pmatrix}$$

only 2 variables

$$\nabla f = \vec{0} \qquad \left.\begin{array}{l} \frac{-y}{4} e^{-xy/4} = 0 \Rightarrow \frac{-y}{4} = 0 \Rightarrow y = 0 \\ \quad\text{AND} \qquad\qquad\qquad\qquad\qquad \text{AND} \\ \frac{-x}{4} e^{-xy/4} = 0 \Rightarrow \frac{-x}{4} = 0 \Rightarrow x = 0 \end{array}\right\} \begin{array}{l}(0,0) \text{ is} \\ \text{the only} \\ \text{critical pt.}\end{array}$$

Is $(0,0) \in \text{int}(D)$? $0^2+0^2 < 25$ ✓

Check $f(0,0) = e^{-0\cdot 0/4} = e^0 = \boxed{1}$

$\boxed{\text{Step II}}$ Check for max on ∂D

$\boxed{\text{Step II}}$ Check for max on ∂D

Maximize $f(x,y) = e^{-xy/4}$ over $\partial D = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \ x^2+y^2=25 \right\}$

Use Lagrange Multipliers

constraint $h(x,y) = x^2+y^2 = 25$

$$\nabla f = \begin{pmatrix} -\frac{y}{4} e^{-xy/4} \\ -\frac{x}{4} e^{-xy/4} \end{pmatrix} = \lambda \nabla h \quad \text{where } \nabla h = \begin{pmatrix} h_x \\ h_y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

Solve system:

$$\boxed{\begin{aligned} -\tfrac{y}{4} e^{-xy/4} &= \lambda 2x \\ -\tfrac{x}{4} e^{-xy/4} &= \lambda 2y \\ x^2+y^2 &= 25 \end{aligned}}$$

Try and Solve

(Many tries)

hard to handle $e^{-xy/4}$ So I decide to solve for it

Case I $y=0$

$x^2+0^2=25$

$x=\pm 5$

plug into first eqn

$-\frac{0}{4}e^0=\lambda\cdot 2\cdot(\pm 5)$

$0=\pm 10\lambda$

so $\lambda=0$

plug into 2nd eqn

$\frac{\pm 5}{4}e^0=0\cdot 2\cdot 0$

impossible

So Case I fails

Case II $x=0$

$0^2+y^2=25$

$y=\pm 5$

plug into 1st eqn

plug into 2nd eqn

Case II also fails.

Case III x,y both not 0

$$\boxed{\begin{aligned} e^{-xy/4} &= -\frac{8\lambda x}{y} \text{ from 1st eqn}\\ e^{-xy/4} &= \frac{-8\lambda y}{x} \text{ from 2nd eqn}\\ x^2+y^2 &= 25\end{aligned}}$$

$\frac{-8\lambda x}{y}=\frac{-8\lambda y}{x}$ from 1st eqn + 2nd eqn

$-8\lambda x^2=-8\lambda y^2$

$x^2=y^2$ or $\lambda=0$

Case IIIa $\lambda=0$

$e^{-xy/4}=0$

but an exponential is never

so this case fails.

Case IIIb $\lambda\neq 0$

$x^2=y^2$

$x^2+y^2=25$

$x^2+x^2=25$

$x^2=\frac{25}{2}$

$x=\frac{\pm 5}{\sqrt{2}},\ y=\frac{\pm 5}{\sqrt{2}}$

4 points on ∂D

$\left(\frac{5}{\sqrt{2}},\frac{5}{\sqrt{2}}\right)$

$\left(\frac{-5}{\sqrt{2}},\frac{5}{\sqrt{2}}\right)$

$\left(\frac{5}{\sqrt{2}},\frac{-5}{\sqrt{2}}\right)$

$\left(\frac{-5}{\sqrt{2}},\frac{-5}{\sqrt{2}}\right)$

So we have only four boundary critical points to check in

$f(x,y)=e^{-xy/4}$

$f\left(\frac{5}{\sqrt{2}},\frac{5}{\sqrt{2}}\right)=e^{-\left(\frac{5}{\sqrt{2}}\right)\left(\frac{5}{\sqrt{2}}\right)/4}=e^{-25/8}$

$f\left(\frac{-5}{\sqrt{2}},\frac{5}{\sqrt{2}}\right)=e \qquad =\boxed{e^{25/8}}$

$f\left(\frac{5}{\sqrt{2}},\frac{-5}{\sqrt{2}}\right)= \qquad =\boxed{e^{25/8}}$

$f\left(\frac{-5}{\sqrt{2}},\frac{-5}{\sqrt{2}}\right)= \qquad =e^{-25/8}$

$\max_D f=e^{25/8}$ at $(x,y)\in\partial D$ $\left(\frac{5}{\sqrt{2}},\frac{-5}{\sqrt{2}}\right)$ and $\left(\frac{-5}{\sqrt{2}},\frac{5}{\sqrt{2}}\right)$

Also inside D $f(0,0)=1$ not max

Check that you have watched the complete [Playlist 226F21-19-ip123](#).

Extra Credit: prove Lagrange Theorem for Minima

Lehman Official HW (do before next lesson): 13.10/ 1, 3, 5, 7,
and problems about max vol, min cost, refraction of light,
production level, putnam challenge, or one related to your major

Homework for our class:

First finish the classwork above then do the problems below:

Selected homework for our class:

Use Method of Lagrange Multipliers to solve:

(1) Maximize $f(x,y)=xy$ subject to constraint $x+y=10$

hint find ∇f

hint find ∇h where $h(x,y)=k$ is the constraint

hint write equations $\boxed{\begin{array}{l}\nabla f = \lambda \nabla h \\ h(x,y)=k\end{array}}$ 3 equations

and solve for x, y and λ.

hint check value of f at each solution

(3) Minimize $f(x,y)=x^2+y^2$ subject to constraint $x+y-4=0$

(5) Minimize $f(x,y)=x^2-y^2$ subject to constraint $x-2y+6=0$

(7) Maximize $f(x,y)=2x+2xy+y$ subject to constraint $2x+y=100$

Solutions:

Use Method of Lagrange Multipliers to solve:

(1) Maximize $f(x,y)=xy$ subject to constraint $x+y=10$

hint find ∇f

hint find ∇h where $h(x,y)=k$ is the constraint

hint write equations $\boxed{\begin{matrix}\nabla f = \lambda \nabla h \\ h(x,y)=k\end{matrix}}$ 3 equations

and solve for x,y and λ.

hint check value of f at each solution

$$\nabla f = \begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} xy \\ \frac{\partial}{\partial y} xy \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$$

$$\nabla h = \begin{pmatrix} h_x \\ h_y \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x}(x+y) \\ \frac{\partial}{\partial y}(x+y) \end{pmatrix} = \begin{pmatrix} 1+0 \\ 0+1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\boxed{\begin{matrix}\nabla f = \lambda \nabla h \\ h(x,y)=k\end{matrix}} \Longrightarrow \boxed{\begin{matrix}\begin{pmatrix} y \\ x \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ x+y=10\end{matrix}} \Longrightarrow \boxed{\begin{matrix} y = \lambda \cdot 1 \\ x = \lambda \cdot 1 \\ x+y=10\end{matrix}} \Rightarrow \boxed{\begin{matrix} x=\lambda \\ y=\lambda \\ x+y=10\end{matrix}}$$

Solving $10 = x+y = \lambda+\lambda$ so $2\lambda = 10$ so $\lambda = 5$

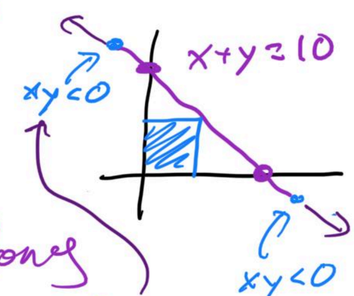
so $x=\lambda=5$ and $y=\lambda=5$

Check $f(5,5) = 5\cdot 5 = 25$

There are no other critical points

Is this a max? Yes

because if we travel along the constraint xy becomes negative.



So the final answer for

(1) the max value is 25 achieved at (x,y)=(5,5).

For (3) you should show all work as above and reach but the final answer:

the min value is 8 achieved at (x,y)=(2,2)

Why? Because (2,2) is the only critical point and you can see f gets larger as you move far down the line in either direction.

(5) When you start this you should eventually get to the following system of three equations:

$2x=\lambda$

$-2y=-2\lambda$

$x-2y+6=0$

So

$x=\lambda/2$

$y=\lambda$

and sub in

$\lambda/2-2\lambda+6=0$

so

$\lambda=4$

so

$x=2$

and

$y=4$

So there is a critical point at (2,4)

To see this is a min check

$f(2,4)=-12$

and check f on either side along the constraint. So for example try $y=0$ and solve for x on the constraint and check the value of f there, and then try $x=0$ and so on. Then write a final conclusion.

No solution for (7)