tab1

NOTE TO STUDENTS Spring 2026: These are the notes that I prepare for use in each class - they are provided for your reference (feel free to ask me about anything you see here). You'll notice more stuff gets added as we go along. Sometimes there are typos or other errors - I correct these as I find them.

Desmos TO DO: General tool for: given curve, draw rectangle (horizontal or vertical), rotate around given axes (include arbitrary angle of rotation - animate "rotation" to show shells vs washers?)

IDEA: Get students to explore the difference between getting AI to do a problem for you, and spending time learning how the AI is doing the problem. Is there a way to demonstrate the difference between "copying a problem" and "doing a problem"? Maybe a simple assignment with two different sets of instructions? Alternatively, What about giving some folks the answer, and others not?

Comment from Felix - if you want folks to actually do the WeBWorK, give them more than 1 week to do it!

Day 1: Antiderivatives

OpenLab Discussion: Introductions & Mathography

Welcome
Course Policies
OpenLab discussion assignment
Review of Antiderivatives assignment (WW guest access)

2-minute intros (works best in-person)

Review of derivatives

To test your knowledge of derivatives, take a look at the following problem sets in WeBWorK (you can access them without an account by clicking the "Guest Login" button)

- Power rule
- Trig functions
- Chain rule
- Product rule

- Quotient rule

Example: find the derivative of the given function y=f(x)

a. $f(x) = 5x^2 + 7x + 3$
b. $f(x) = \sin x + e^x + \ln x$
c. $f(x) = x^2 \cos x$
d. $f(x) = \sqrt{x^3 + 11}$

Antiderivatives

Definition. A function F is an **antiderivative** of a function f if $F'(x) = f(x)$ for all x in the domain of f.

General form of an antiderivative. Let F be an antiderivative of f on an interval I. Then:

i. For any constant C, the function $F(x) + C$ is also an antiderivative of f over I, and
ii. Every antiderivative of f over I has the form $F(x) + C$ for some constant C.

In other words, the **most general form of the antiderivative** of f over I is $F(x) + C$.

EXAMPLE: Find all antiderivatives of each function

- $f(x) = 3x^2$
- $f(x) = 1/x$
- $f(x) = \cos x$
- $f(x) = e^x$

Day 2: The Definite Integral and the Fundamental Theorem of Calculus

1. The definite integral
    a. Geometric definition
    b. Group work
2. Fundamental theorem of calculus 1.
    a. Define function F using x as upper limit in def integral of f
    b. Thm: derivative of F is f

    c. Ex: define such an F, take the deriv

3. Fun

[See Classwork: The Definite Integral and the Fundamental Theorem of Calculus]

Definition: The Definite Integral.

If $f(x)$ is a function defined on an interval $[a, b]$, the definite integral of f from a to b is given by

$$
\int_a^b f(x) d x=\lim {n \rightarrow \infty} \sum_{i=1}^n f\left(x_i^*\right) \Delta x
$$

provided the limit exists. If this limit exists, the function $f(x)$ is said to be integrable on $[a, b]$, or is an integrable function.

NOTE: On the course hub, the definite integral is defined as *signed area* - think about this, Jonas! Might be a good strategy to introduce it this way (provided I'm clear that the rigorous definition is coming later)

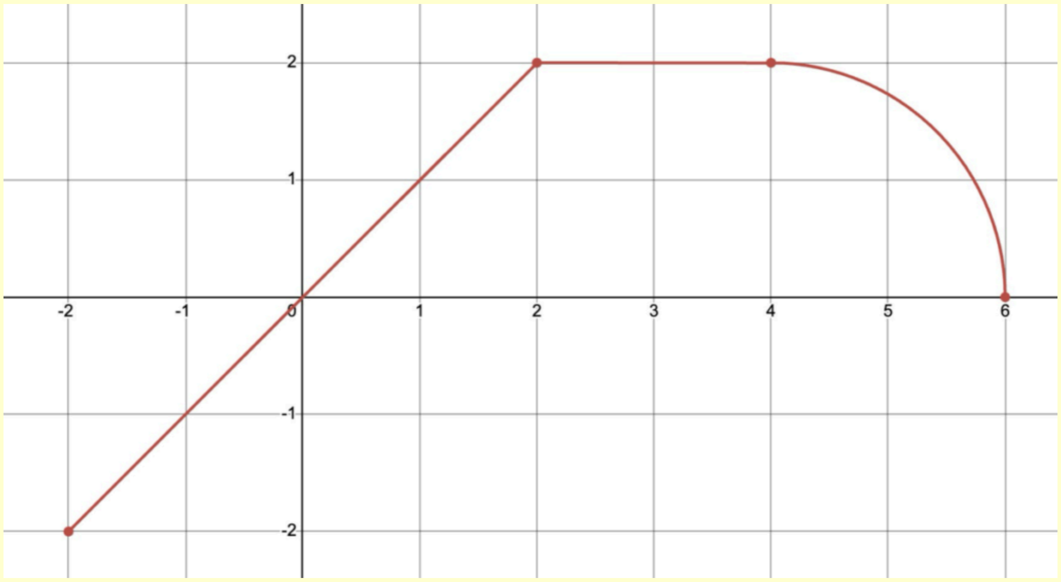
NOTE: you can use anything for the *variable of integration* without changing the value, so $\int_a^b f(x)dx = \int_a^b f(u)du = \int_a^b f(t)dt$.

Example 1 (warmup):

Use the graph of f(x) above to evaluate the following:

1. $\int_0^4 f(x)$
2. $\int_4^6 f(x)$

GROUP WORK 1



TThe graph of the function f(x) shown above consists of two straight lines and a quarter-circle. Use the graph to calculate the following values.

1. $\int_{0}^{4} f(x)dx$
2. $\int_{-2}^{6} f(x)dx$

NOTE: When we reverse the two endpoints, or limits of integration, the affect is to multiply the integral by -1.

3. $\int_{4}^{0} f(x)dx$
4. $\int_{2}^{-1} f(x)dx$
5. $\int_{4}^{5} f(x)dx$

THEOREM 1.4: Fundamental Theorem of Calculus, Part 1

If \$f(x)\$ is continuous over an interval \$[a, b]\$, and the function \$F(x)\$ is defined by

$$
F(x)=\int_a^x f(t) d t
$$

then $F^{\prime}(x)=f(x)$ over $[a, b]$

NOTE: constant is BOTTOM bound, variable is UPPER bound.

NOTE: observe difference between variable of integration and variable of function F(x)

Example 2: $\int_3^x \sqrt{x^2 + 45} dx$

NOTE: With a little cleverness, we can use this fact to calculate the value of definite integrals.

Example 3 (proof of FTC2 - optional): $\int_2^5 3x^2-4x dx$.

Let $G(x) = \int_a^x 3t^2-4t dt$ for some constant a

Notice that $\int_2^5 3x^2-4x dx = G(5)-G(2)$ (geometric argument - BUT still don't know how to calculate G(5) or G(2)!!)

Also notice that $G'(x)=3x^2-4x$ by the FTC part 1, so $G(x)$ is an antiderivative of $3x^2-4x$

Let $F(x)$ be any antiderivative of $G(x)$ -- give me one! e.g. $F(x)=\frac{3x^4}{4}-\frac{5x^2}{2}$. We know how to calculate $F(2)$ and $F(5)$.

BUT $F(x)$ and $G(x)$ are both antiderivatives of $3x^2-4x$, and we know that these differ only by a constant, so $G(x)=F(x)+C$.

Thus $G(5)=F(5)+C$ and $G(2)=F(2)+C$

so $\int_2^5 3x^2-4x dx = G(5)-G(2)=F(5)+C-(F(2)+C)=F(5)-F(2)$.

THEOREM 1.5: The Fundamental Theorem of Calculus, Part 2

If f is continuous over the interval $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then

$$
\int_a^b f(x) d x=F(b)-F(a)
$$

Example 4 (warmup): $\int_2^5 3x^2-4x dx$

GROUP WORK 2

Evaluate the following integrals (using the Fundamental Theorem of Calculus Part 2):

1. $\int_0^10 x^3+7x-1 dx$
2. $\int_{-1}^1 e^t dt$
3. $\int_\pi^{2\pi} -\sin x dx$
4. $\int_2^16 \sqrt{x} dx$

Day 3: The Substitution Method
(also Exponential and Logarithmic Integrals)

OpenLab Discussion: Write a Problem for the First Exam

NOTE: This lecture ran long, didn't have time for group work - I think it might be better to cut some of the conceptual stuff at the top & get right into the examples?

What does Integral mean?
There are actually two related terms, both often referred to as "the integral":

1. The **indefinite integral** $\int f(x)dx$ means the (most general) Antiderivative of f(x).

   It is equal to a *family of functions*, all of which have derivative equal f(x).

2. The **definite integral** $\int_a^b f(x)dx$ comes from the area problem, and means the limit of the sums of rectangles of height f(x) on [a,b].
   It is equal to a *real number*, representing the "signed area bounded by f(x) on [a,b]"

Now we know how antiderivatives (indefinite integrals) can help us compute definite integrals. Our next goal is to figure out how to find antiderivatives of many different functions.

The Substitution Rule
The first and most common/widely applicable rule for integration

Example:

1. Find the derivative $\sin(x^3)$.
   What rule do we use? Chain rule!

2. Find the integral $\int \sin(x^3)3x^2dx$.

   HINT: Use the previous example!

Now, how do we do this in general? This is one of the rare cases where we have an integration rule that works as "just the opposite of a derivative rule". The substitution rule is just the opposite of the chain rule.

Chain rule: $\frac{d}{dx}F(g(x)) = F'(g(x)) \cdot g'(x)$

Substitution rule: $\int F'(g(x)) \cdot g'(x)dx = F(g(x)) + C$

Using this rule is largely a matter of “visually” identifying the inner function g(x) (often written as u=g(x)) whose derivative g’(x) also appears, and the outer function F’(x) (often written as f(x)=F’(x))

Definition (The Substitution Rule). Let $u=g(x)$, where $g'(x)$ is continuous over an interval, let $f(x)$ be continuous over the corresponding range of g, and let $F(x)$ be an antiderivative of $f(x)$.

Then, $\int f(g(x))g'(x)dx = \int f(u)du = F(u) + C = F(g(x)) + C$

Let’s work through the example above, using this idea:

Example 1, part 2. Find the integral $\int \sin(x^3)3x^2dx$ using the substitution rule.

STEPS FOR USING THE SUBSTITUTION RULE:

1. Identify one part of the expression as the inner function *u=g(x)*.
   *This is the **substitution** we are making (substituting u for g(x))*
2. Find the derivative of u, written du=g’(x)dx.
3. Now substitute: use the two equations u=g(x) and du=g’(x)dx to replace every occurrence of x with an expression of u
4. Integrate with respect to u.
5. Substitute u=g(x) to get the answer in terms of x.

Example: $\int (x^2 - 3)^5 2xdx$

Example: $\int x(3x^2 + 4)^4dx$

GROUP WORK DAY 2

Use Substitution to find the indefinite integral:

1. $\int 7x^{3}\cos(x^{4})dx$

2. $\int z\sqrt{z^{2} - 5}dz$

3. $\int 3x^{2}e^{2x^{3}}dx$

4. $\int \frac{\sin t}{\cos^3 t} dt$

5. $\int \frac{\sin t}{\cos^2 t} dt$

6. $\int \frac{x}{\sqrt{x-1}} dx$

Example (BOARD): Use substitution to evaluate the definite integral $\int_0^1 x^2\left(1 + 2x^3\right)^5 dx$

NOTE: In Step 3, substitution, use the substitution u=g(x) to convert the limits of integration from x-values a and b to u-values g(a) and g(b)

GROUP WORK DAY 2 Cont'd

Use Substitution to find the definite integral:

7. $\int_{-1}^{0} x(2x^2 - 3)^5 dx$

8. $\int_{0}^{1} xe^{4x^2+3} dx$

9. $\int_{1}^{2} \frac{5}{3y+4} dy$

Day 4: Integration by Parts

Motivating example: $\int x \sin x dx$

RECALL: Product rule $\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$

Integrate: $\int \frac{d}{dx} f(x)g(x)dx = \int f'(x)g(x) + f(x)g'(x)dx$

Simplify: $f(x)g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx$

Rearrange: $\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$

This is it - but let's replace (substitute) to make it easier to read,
$v = g(x),\ u = f(x),\ dv = g'(x)dx,\ du = f'(x)dx$

Definition (Integration by parts). Let $u=f(x)$ and $v=g(x)$ be functions with continuous derivatives. Then, the integration-by-parts formula for the integral involving these two functions is:

$$\int u\, dv = uv - \int v\, du$$

STEPS FOR USING INTEGRATION BY PARTS:

1. Break up your integral into two parts, u= and dv=
   How do we choose u and dv? TALK ABOUT STRATEGIES
2. Differentiate u to get du=, and integrate dv to get v=
3. Use the Integration by parts formula to rewrite the integral
4. Finally, find the integral of vdu.

Return to our motivating example (solve using integration by parts): $\int x \sin x dx$

Example: $\int \frac{\ln x}{x^3} dx$

GROUP WORK:

Use Integration by Parts to evaluate the following:

1. $\int x \ln x \, dx$

2. $\int xe^{2x} dx$

3. $\int x^2 e^{3x} dx$

4. $\int t^3 e^{t^2} dt$

Day 5: Trigonometric Integrals

Recall: Trig Identities

Trig Identities	Double angle identities (use these to reduce the power of sine or cosine)
• $\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\quad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$ • $\sec\theta = \frac{1}{\cos\theta}$ $\quad \csc\theta = \frac{1}{\sin\theta}$ • $\sin^2\theta + \cos^2\theta = 1$ • $\sec^2\theta = 1 + \tan^2\theta$	• $\sin^2 x = \frac{1-\cos(2x)}{2}$ • $\cos^2 x = \frac{1+\cos(2x)}{2}$

Derivatives and Integrals of Trig Functions

Derivatives of Trig Functions	Integrals of Trig Functions
• $\frac{d}{dx}(\sin x) = \cos x$ • $\frac{d}{dx}(\cos x) = -\sin x$ • $\frac{d}{dx}(\tan x) = \sec^2 x$ • $\frac{d}{dx}(\cot x) = -\csc^2 x$ • $\frac{d}{dx}(\sec x) = \sec x \tan x$ • $\frac{d}{dx}(\csc x) = -\csc x \cot x$	• $\int \cos x dx = \sin x + C$ • $\int \sin x dx = -\cos x + C$ • $\int \sec^2 x dx = \tan x + C$ • $\int \sec x \tan x dx = \sec x + C$ • $\int \tan x dx = \ln\lvert\sec x\rvert + C$ • $\int \sec x dx = \ln\lvert\sec x + \tan x\rvert + C$

Lecture: today is about integrating trig functions!
Require Knowledge (MEMORIZE!)
Trig identities
Trig derivatives
Trig integrals

Using Trig identities.
Example: integral of tanx

EXAMPLE: $\int \cos^3 x \ \sin x \, dx$

Using Trig integrals and derivatives:
Example: integral secx secx tanx dx

Group Work 1

Evaluate the following integrals:

1. $\int \sin^4 x \cos x dx$
2. $\int \cos^2 x \sin^3 x dx$
3. $\int \sec^3 x \tan x dx$
4. $\int \tan^6 x \sec^4 x dx$
5. $\int \tan^3 x dx$
6. $\int \sin^2 x dx$

Group Work:

Evaluate the following integrals:

1. $\int \sin^4 x \cos x dx$
2. $\int \cos^2 x \sin^3 x dx$
3. $\int \sec^3 x \tan x dx$

4. $\int \tan^6 x \sec^4 x dx$

5. $\int \tan^3 x dx$

6. $\int \sin^2 x dx$

Day 6: Trigonometric Substitution

Fall 2024: I started today with an extended discussion of, first, Polynomials (as the class including id and constant fcns, closed under +,-,*) -- asking if they are closed under differentiation and integration. Then rationals fcns (also closed under /) -- are they closed under differentiation? Integration?
Ex: 2x/(x^2+8)^3, 1/x, 1/(1+x^2)
NOTE: This might have run a little long - consider cutting it down a bit.

NOTE: Today's goal is to discover the following integration formulas (note the first one is incorrect in the textbook - should have an inverse sine (u/a) on the RHS):

Example: $\int \frac{1}{\sqrt{4-x^2}} dx$

Can we integrate it another way?
Try substituting x=2sin(theta).

Example: $\int \frac{1}{9+x^2} dx$

Substitute x=3tan(theta)

Example: $\int \frac{7}{\sqrt{27-3x^2}} dx$

Factor sqrt(3) out of denom
Substitute x=3sin(theta)

Integration rules (that result in Inverse Trig Functions)

1. $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin \frac{u}{a} + C$ (for a>0)
2. $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1} \frac{u}{a} + C$ (for a>0)
3. $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1} \frac{|u|}{a} + C$ (for a>0)

Day 7: Trigonometric Substitution

WeBWorK: Integration - Trigonometric Substitution

Today, we'll need the following trig identities:

Double-angle identities

$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$

$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$

$\sin(2\theta) = 2\sin\theta\cos\theta$

$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$

Recall three "inverse trig function" integration rules from last time:

Integration rules that result in Inverse Trig Functions

1. $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C$ (for a>0)
2. $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$ (for a>0)
3. $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{|u|}{a} + C$ (for a>0)

You can use these rules whenever you come upon a function that matches the pattern. However, the idea behind these rules - trigonometric substitution - is useful in a lot more circumstances.

Trigonometric Substitution

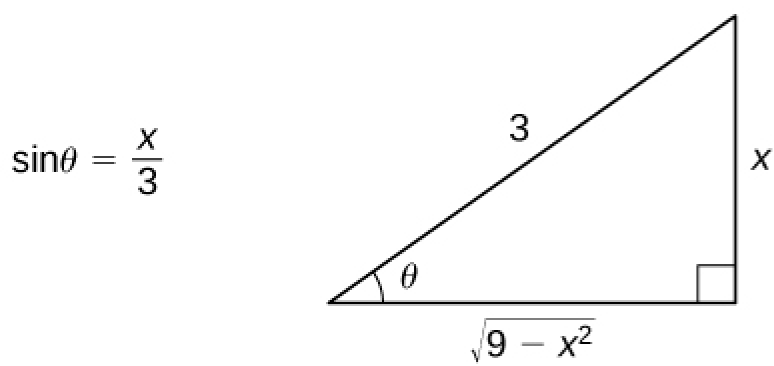
For integrals involving:	Use this substitution:
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$

Example: $\int \sqrt{9 - x^2}dx$

Hint: to integrate cosine squared, use the double-angle identity
$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$

After integrating, convert from 2theta back to theta using the double-angle identity:
$\sin(2\theta) = 2\sin\theta\cos\theta$

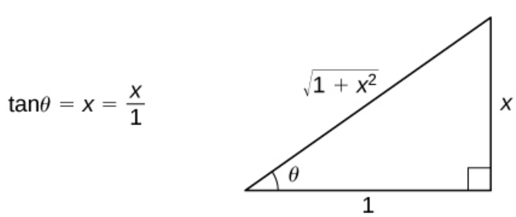
Finally, to convert from theta back to x, draw a reference triangle:



ANS: $\frac{9}{2}\sin^{-1}(\frac{x}{3}) + \frac{x\sqrt{9-x^2}}{2} + C$

Example. $\int \frac{dx}{\sqrt{1+x^2}}$

Use $x = \tan\theta$



Recall $\int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C$

ANS: $\ln|\sqrt{1 + x^2} + x| + C$

Example. $\int \frac{1}{\sqrt{x^2+4x-12}} dx$

PROBLEM: Doesn't fit any of our patterns!
We can rewrite the bottom without the "x" term using COMPLETING THE SQUARE:

Complete the square: $x^2 + 4x - 12$
$= x^2 + 4x + 4 - 4 - 12$
$= (x^2 + 4x + 4) - 16$
$= (x + 2)^2 - 16$

Replace in the original to obtain:

$\int \frac{1}{\sqrt{(x+2)^2-16}} dx$
Now we have something that looks a little like "x^2-a^2".
Substitute $x + 2 = 4\sec\theta$

Use reference triangle to convert back to x.
ANS: $\ln|\frac{x+2}{4} + \frac{\sqrt{(x+2)^2-16}}{4}| + C$

Day 8: Partial Fractions

WeBWorK: Integration - Partial Fractions distinct linear factors

Overall goal: be able to integrate any rational function P(x)/Q(x), where P,Q are polynomials

Essential Calculus facts:

- ln(x)
- arctanx
- substitution

Essential algebra techniques:

- Long division (polynomials) - rewrite a fraction so degree(Q(x))<degree(P(x))
- Partial fractions - rewrite a fraction as a sum of simpler fractions ("break a fraction up into simpler pieces)
- Completing the square - rewrite a trinomial as a binomial
- Factoring* - rewrite a sum as a product

* This one is hard, not always possible in practice (but it will be in THIS class!)

Example. Integrate the following:

a) $\int \frac{1}{x+1} + \frac{2}{x-2} dx$ b) $\int \frac{3x}{x^2-x-2} dx$

NOTE: They are equal! How do we rewrite b) as a)?

What if degree of numerator is >= degree of denominator?

STEP 1: If degree of numerator is >= degree of denominator, use polynomial long division.

Example: $\int \frac{x^2+3x+5}{x+1} dx$

(NOTE: $= \int x + 2 + \frac{3}{x+1} dx$)

Example: $\int \frac{3x+2}{x^3-x^2-2x} dx$

STEP 2: factor the denominator.

What we do next will depend on the type(s) of factors that appear in the denominator.

Step 3: re-write as partial fractions using the factors of the denominator.

RULE: Distinct (non-repeated) linear factors:
Given $\frac{P(x)}{Q(x)}$, if Q(x) can be factored as $(a_1x + b_1)(a_2x + b_2)...(a_nx + b_n)$ where each linear factor is distinct, then
$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + ... + \frac{A_n}{a_nx+b_n}$ for some constants $A_1, A_2, ..., A_n$.

Step 4: Find the values of any constants in the partial fraction expansion.

Step 5: Integrate the resulting expression.

ANS: $= -\ln|x| + \frac{4}{3}\ln|x-2| - \frac{1}{3}\ln|x+1| + C$

GROUP WORK:

$$\int \frac{x^2+3x+1}{x^2-4}dx$$

$$\int \frac{\cos x}{\sin^2 x - \sin x}dx$$

HINT: use substitution first.

GROUP WORK SOLNS:

A. $= x + \frac{11}{4}\ln|x-2| + \frac{1}{4}\ln|x+2| + C$

B. $= -\ln|\sin x| + \ln|\sin x - 1| + C$

Day 10: Partial Fractions II

WeBWorK: Integration - Partial Fractions repeated and irreducible quadratic factors

Goal for today: To be able to integrate any rational function.

General Strategy for integrating rational functions

1. If the degree of the numerator is greater or equal to the degree of the denominator, do polynomial long division.
2. Factor the denominator into linear and irreducible quadratic elements.
3. Set up Partial Fractions expansion based on factors in denominator.
    a. Distinct Linear Factors: *One fraction for each factor. Each numerator is a constant A, B, C, etc.*
    b. Repeated Linear Factors: *One fraction for each power, up to the power of the factor. Each numerator is a constant A, B, C, etc.*
    c. Distinct Quadratic Factors: *One fraction for each factor. Each numerator is a linear function Ax+B, Cx+D, etc*
    d. Repeated Quadratic Factors: *One fraction for each power, up to the power of the factor. Each numerator is a linear function Ax+B, Cx+D, etc.*
4. Solve for all constants.
5. Integrate the resulting expression.
    a. Use ln(x), arctan(x), u-substitution.
    b. Complete the square if other strategies don't work.

Example 1:

$\int \frac{x^2+3x+5}{x+1} dx$

Example 2: Repeated linear Factors

$$\int \frac{x-2}{(2x-1)^2(x-1)} dx.$$

Partial Fraction expansion for a repeated root: *One fraction for each power, up to the power of the factor. Each numerator is a constant A, B, C, etc.*

ANS: $\ln|2x-1| - \frac{3}{2(2x-1)} - \ln|x-1| + C.$

GROUP WORK

Example 3:

$$\int \frac{2x-3}{x^3+x}\,dx.$$

Definition: a quadratic $ax^2 + bx + c$ is called **irreducible** if it cannot be factored into linear factors with real coefficients (only happens when the discriminant is negative, $b^2 - 4ac < 0$)

Partial Fraction expansion for

- Irreducible quadratic factors: *One fraction for each factor. Each numerator is a linear function Ax+B, Cx+D, etc*
- Repeated Quadratic Factors: *One fraction for each power, up to the power of the factor. Each numerator is a linear function Ax+B, Cx+D, etc.*

ANS: $\frac{3}{2}\ln|x^2+1| + 2\tan^{-1}x - 3\ln|x| + C.$

Example 4: $\int \frac{x^3+8x^2+14x+16}{(x^2+8)(x^2+6x+14)}\,dx$

Partial Fraction expansion: $\frac{x}{x^2+8} + \frac{2}{(x+3)^2+5}$

ANS: $\frac{1}{2}\ln(x^2+8) + \frac{2}{\sqrt{5}}\arctan\left(\frac{x+3}{\sqrt{5}}\right) + C$

Day 11: Improper Integrals

WeBWorK: Integration - Improper Integrals

SPRING 2026: Class could maybe use more scaffolding before launching into the group work - do more examples at board?

Integrating over an Infinite Interval

Example: Find the area between the graph of $f(x) = \frac{1}{x}$ and the x-axis over the interval $[1, +\infty)$. Is it finite or infinite?

Set up an integral! What does it mean to have one of the limits of the integral be infinity?

Definition. Suppose f(x) is continuous over an interval of the form $[a, +\infty)$ for some constant a. Then: $\int_a^{\infty} f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$

QUESTION: What about intervals of the form $(-\infty, b]$?
What about intervals of the form $(-\infty, +\infty)$?
Examples:

Integrating over an interval where the function is undefined at one endpoint

Example. $\int_0^4 \frac{1}{\sqrt{4-x}}dx$

RULE: Replace the problematic endpoint with t, and take limit a t-> value

GROUP WORK:
State whether the improper integral converges or diverges. If it converges, find the value.

Example A. $\int_{-\infty}^0 \frac{1}{x^2+4}dx.$

Example B. $\int_{-\infty}^{\infty} xe^x dx.$

Example C. $\int_{-1}^{1} \frac{1}{x^3}\, dx$

A COMPARISON THEOREM.

A Comparison Theorem

Let $f(x)$ and $g(x)$ be continuous over $[a, +\infty)$. Assume that $0 \le f(x) \le g(x)$ for $x \ge a$.

i. If $\int_a^{+\infty} f(x)\, dx = \lim_{t \to +\infty} \int_a^t f(x)\, dx = +\infty$, then $\int_a^{+\infty} g(x)\, dx = \lim_{t \to +\infty} \int_a^t g(x)\, dx = +\infty$.

ii. If $\int_a^{+\infty} g(x)\, dx = \lim_{t \to +\infty} \int_a^t g(x)\, dx = L$, where L is a real number, then $\int_a^{+\infty} f(x)\, dx = \lim_{t \to +\infty} \int_a^t f(x)\, dx = M$ for some real number $M \le L$.

Example. Use the comparison theorem to show that $\int_1^{\infty} \frac{1}{xe^x}\, dx$ converges.

Day 12: Taylor Polynomials

WeBWorK: Taylor and MacLaurin Polynomials
Book Homework: find the Taylor polynomials of degree two approximating the given function centered at the given point.

BIG IDEA: We can create polynomials that look like (approximate) basically ANY function. How do we do it?

One option would be to "match multiple points" (Lagrange Interpolation - Desmos demo) -- if you have a finite number of points, you can create a polynomial that passes through them. Unfortunately, as you add more & more points, the function tends to get crazier and crazier!

How? We use Calculus!

INTRO: Consider the cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$. This polynomial is entirely determined by the four coefficients a,b,c,d.
QUESTION: Which of the coefficients determines the value of f(0)?

Notice: I can change the value of f(0) to whatever I want, just by changing d.

QUES: Find the values of f'(0), f''(0), f'''(0). Which coefficient(s) determine the value of each?

OBSERVATION: In a polynomial, the value of the nth derivative at x=0 is determined entirely by the coefficient of x^n.

QUESTION: What is cos(0.7)? Can you find it without using any trig functions on your calculator, only +/-/*/?

Start with any function, y=cosx.
Pick a point on the graph, say x=0 -- this gives the point (0,1).
I am going to try to make a polynomial that matches my function at that point.
Here goes:
First try: y=1
Second try: y=1-1/2 x^2
Third try: y=1 - 1/2 x^2 + 1/24 x^4-1/720 x^6
BASIC IDEA: (Desmos demonstration)

QUESTION: What is cos(0.7)?
Find it without using any trig functions on your calculator! Plug into our third approximation.

NOTE: cos(0.7) == 0.764842187
Third approx gives y(0.7)=0.764841

These polynomials are called the Partial MacLaurin Polynomials for f(x). How do we find the correct coefficients to use in them? They follow a pattern.

Definition. If f(x) is a function with derivatives of all orders at x=0, then the MacLaurin polynomial of f(x) of order n is:

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + ... + a_n x^n$$

Where:

$$a_0 = f(0)$$
$$a_1 = \frac{f'(0)}{1!}$$
$$a_2 = \frac{f''(0)}{2!}$$
$$a_3 = \frac{f'''(0)}{3!}$$
...
$$a_n = \frac{f^{(n)}(0)}{n!}$$

NOT SURE WHETHER TO DO THIS EXAMPLE, OR MOVE DIRECTLY TO sin(x)??

***Example 1.** Find the MacLaurin polynomial of order 6 for f(x)=cos(x). Use it to approximate cos(0.7).*

Example 2. Find the MacLaurin Polynomial of order 8 for f(x)=sin(3x). Use it to approximate f(-.22)

One key idea of MacLaurin Polynomials is that they always approximate a function *near x=0*. We can approximate a function near another point, provided we can find the derivatives of the function at that point:

Definition. If f(x) is a function with derivatives of all orders at x=a, then **the Taylor polynomial of f(x) at x=a of order n** is:

$$p_n(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + a_4(x-a)^4 + ... + a_n(x-a)^n$$

Where:

$$a_0 = f(a)$$
$$a_1 = \frac{f'(a)}{1!}$$
$$a_2 = \frac{f''(a)}{2!}$$

$a_3 = \frac{f'''(a)}{3!}$

...

$a_n = \frac{f^{(n)}(a)}{n!}$

Example 3. Find the Taylor polynomial of order 4 at x=1 for $f(x) = \ln x$.

Day 13: Taylor Polynomials

WeBWorK: Taylor Polynomials with Remainder

Book Homework: Taylor's Theorem with Remainder.

- *verify that the given choice of n in the remainder estimate yields Rn <= 1/1000, and find the value of Taylor polynomial pn of f at the point. If Rn is not less than 1/1000, determine it's value*
- *find the smallest value of n such that the remainder estimate Rn <= 1/1000 on the indicated interval*

GOALS:
Compute error of a Taylor Polynomial on an interval.
?? (might skip this? Just talk about the remainder function, etc?). Use the remainder theorem to bound the error of a Taylor Polynomial on an interval.

2026: I did two examples on the board (rather than group work) - I think they needed the modeling & reassurance.

Example 1:
A. Find the Taylor polynomial of order 4 centered at x=1 for $f(x) = \ln x$.
B. Use it to approximate the value of ln(0.8). How close is your approximation to the correct value?
C. For which values of x is your polynomial within 0.001 of the correct value of f(x)?

Example 2:
Find a Taylor polynomial that approximates the function $f(x) = \sqrt{x}$ the interval [0.5, 1.5] to within 0.01 of the correct value.

Day 14: Sequences

WeBWorK: Series – Sequences

Book homework: Chp 5.1 #1, 3, 7, 9, 12, 13–15 odd, 23–37 odd, 47–51 odd
- find terms of a sequence given the formula
- arithmetic and geometric sequences
- find explicit formulas for sequences from recursive defn
- limits of sequences, limit rules
- L'Hopital's rule
- determine if a sequence is bounded, monotone, increasing, decreasing

Sequences

Definition: An **infinite sequence** $\{a_n\}_{n=1}^{\infty}$ is an ordered list of numbers of the form

$$a_1,\ a_2,\ a_3,\ \ldots$$

- the subscript n is called the **index variable** of the sequence
- each number a_n is a term of the sequence

Example 1: 2, 4, 6, 8, 10, 12, 14, 16, 18, ...

What is a_1? a_2? a_7?
If I know a_20, how do I find a_21?

There are two fundamental ways of giving the terms in a sequence:

- a **recurrence relation**, which tells you how to calculate the next term in the sequence based on the current (or previous) terms. One or more initial terms must also be given explicitly.
  Example 1 recurrence relation: $a_{n+1} = a_n + 2$, with $a_1 = 2$
- **explicit formulas**, which tell you exactly how to calculate a_n as a function of n:
  $a_n = f(n)$.
  Example 1 explicit formula: $a_n = 2n$

Two important kinds of sequences

Example 2: Consider the sequence $10, 13, 16, 19, 22, 25, ...$

A. Find the next 5 terms of the sequence.
B. Find a recurrence relation. *Do we need any initial values?*
C. Find an explicit formula that gives a_n as a function of *n*.

Arithmetic sequences

In an arithmetic sequence, the *difference* between any two consecutive terms is the same. If this common difference is given by *b*, then:

- We can describe an arithmetic sequence with the recurrence relation $a_{n+1} = a_n + b$.
- We can give an explicit formula for the sequence by: $a_n = a_1 + b(n - 1)$

Example 3: Consider the sequence $3, 9, 27, 81, ...$

A. Find the next 5 terms of the sequence.
B. Find a recurrence relation. Do we need any initial values?
C. Find an explicit formula that gives a_n as a function of *n*.

Geometric

In a geometric sequence, the *ratio* of every two consecutive terms is the same. If this common ratio is given by q, then:

- We can describe a geometric sequence with the recurrence relation $a_{n+1} = q \cdot a_n$.
- We can give an explicit formula for the sequence by: $a_n = a_1 \cdot q^{n-1}$

BONUS Example: Consider the sequence $1, 1, 2, 3, 5, 8, 13, 21, ...$

A. Find the next 5 terms of the sequence.
B. Find a recurrence relation. Do we need any initial values?
C. Find an explicit formula that gives a_n *as a function of n.*

Graphing sequences: When we graph a sequence, we plot each term (n, a_n) (*using n as the x-value and a_n as the y-value)*

Three important properties of sequences:

- Increasing. A sequence $\{a_n\}$ is increasing if $a_{n+1} \geq a_n$ for all n (each term is ≥ the previous term)

- Decreasing. A sequence $\{a_n\}$ is increasing if $a_{n+1} \leq a_n$ for all n (each term is ≤ the previous term)
- Bounded. A sequence $\{a_n\}$ is bounded above if there is a constant M such that every term each term $a_n \leq M$. (it is bounded below if there is a constant M s.t. each term is $a_n \geq M$).

Limit of a Sequence

Definition. If the terms a_n of a sequence become arbitrarily close to a number L as n becomes sufficiently large, we say $\{a_n\}$ **is a convergent sequence**, and **L is the limit of the sequence**. We write:

$$\lim_{n \to \infty} a_n = L$$

If a sequence is not convergent, we say it is a **divergent sequence**

NOTE: If a sequence is divergent, it can diverge in different ways (for example, the limit could be infinity, or -infinity, or nothing at all DNE)

How do we find the limit of a sequence?

We will need to build up a toolbox of ideas -- limits of some basic sequences, plus rules about how we combine limits.

Example 4: Find the limit of the sequence:

i. $\lim_{n \to \infty} \left(\frac{1}{3}\right)^n$

ii. $\lim_{n \to \infty} 5^n$

Theorem: The limit of the geometric sequence $\{r^n\}$ (for r>0) :

- if $0 < r < 1$ then $\lim_{n \to \infty} r^n = 0$
- If r=1 then $\lim_{n \to \infty} r^n = 1$
- If $r > 1$ then $\lim_{n \to \infty} r^n = \infty$ (the sequence **diverges**)

NOTATION: We sometimes write $a_n \to L$ as shorthand for $\lim_{n \to \infty} a_n = L$

EXAMPLE: $\left\{\frac{2^n}{3^n}\right\}$

Find the limit of each sequence

Example 5: $\left\{5 - \frac{3}{n^2}\right\}$

Example 6: $\left\{\frac{3n^4 - 7n^2 + 5}{6 - 4n^4}\right\}$

Example 7: $\left\{\frac{e^n}{n^2 + 2n + 1}\right\}$

Example 8: $\{\ln(2n + 1) - \ln(n)\}$

Example 9: $a_n = \sqrt{5 - \frac{3}{n^2}}$

Example 10: $a_n = \sqrt{\frac{2n+1}{3n+5}}$ (L'Hospital's Rule)

LIMIT LAWS - in solving the above, we make use of the following limit laws:

THEOREM (sequences given by functions): Suppose the explicit formula for a sequence is given by a function f(x), so $a_n = f(n)$. If $\lim_{x \to \infty} f(x) = L$ for some number L, then

$\lim_{n \to \infty} a_n = L$.

> *Side question - what if the limit of f(x) DNE? Does this mean that the limit of the sequence DNE? NOPE! Consider the sequence* $a_n = \sin(n \cdot \pi + \frac{1}{n})$

THEOREM: Limit Laws (algebraic)

Given sequences $\{a_n\}$ and $\{b_n\}$ with $\lim_{n \to \infty} a_n = A$ and $\lim_{n \to \infty} b_n = B$, and given any real number *c*, then:

1. $\lim_{n \to \infty} c = c$
2. $\lim_{n \to \infty} ca_n = cA$
3. $\lim_{n \to \infty} (a_n \pm b_n) = \lim_{n \to \infty} a_n \pm \lim_{n \to \infty} b_n = A + B$
4. $\lim_{n \to \infty} (a_n \cdot b_n) = \left(\lim_{n \to \infty} a_n\right) \cdot \left(\lim_{n \to \infty} b_n\right) = A \cdot B$

5. $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\left(\lim_{n\to\infty} a_n\right)}{\left(\lim_{n\to\infty} b_n\right)} = \frac{A}{B}$, provided $B \neq 0$ and each $b_n \neq 0$

THEOREM: Limit Laws (composition)

If $\lim_{n\to\infty} a_n = A$ and $f(x)$ is continuous at x=A. Then $\lim_{n\to\infty} f(a_n) = f(A)$

Day 15: Infinite Series

WeBWorK: Series – Infinite Series

TODAY: How do we add up infinitely many numbers?

Example: Suppose an oil leak occurs in a pipeline near a mountain stream. On the first day, 1 gallon of oil leaks into the stream. Each subsequent day, half as much oil leaks into the stream as the day before.
QUESTION: If this continues forever, how much oil will enter the stream altogether?

Let a_n be the amount of oil that leaks into the stream on day n.
Find a1, a2, a3, a4, a5, ...

Find a formula for an.

ANS: $a_n = \left(\frac{1}{2}\right)^{n-1}$

What is the total amount of oil leaked into the stream (for all time)?

$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...$

In order to find the sum of this infinite series, we start by adding up only finitely many of the terms - we call these partial sums:

Whats the total amount of oil that has been leaked into the stream after day 1? What about after day 2? What about after day 3? Etc up to day 5 1, 2, 3, 4, 5 (S1, S2, ..., S5)

NOTE: $S_k = 1 + \frac{1}{2} + \frac{1}{4} + ... + \frac{1}{2^{k-1}}$

The total amount of oil over all time will be $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$

Definition: If the sequence $\{S_k\}$ of partial sums **converges to a limit L**, we say that the infinite sum converges and $\sum_{n=1}^{\infty} a_n = L$. If the sequence of partial sums diverges we say the infinite series diverges.

Let's see if we can figure out whether the partial sums in our "oil" example converge. Calculate the value of the first 5 or 6 (desmos demo). Use a calculator to find S_10 and S_20

ANS: S10=1.998, S20=1.999998

Definition. An infinite series is an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ...$$

For each positive integer k, we call the sum of the first k elements

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + ... + a_k$$

the kth partial sum. If the sequence of partial sums $\{S_k\}$ converges to a limit L, we call L the **sum of the series**, and we write:

$$\sum_{n=1}^{\infty} a_n = L$$

If the sequence of partial sums diverges, we say the series diverges.

EXAMPLE (SKIP?): use the sequence of partial sums to determine if the series converges

A. $\sum_{n=1}^{\infty} \frac{n}{n+1}$

B. $\sum_{n=1}^{\infty} (-1)^n$

C. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

HINTS:

For a), write out some partial sums. Observe the individual terms are getting closer to 1.

FACT: For the series to converge, the a_n must approach 0.

For b), partial sums alternate. Does the sequence of partial sums converge?

For c), find S1 through S5 (in fraction form). We can see that the kth partial sum is given by the explicit formula:

$$S_k = \frac{k}{k+1}$$

Does the sequence of partial sums converge? To what? What is the sum of the series?

Re-indexing series to start at 1
Example: harmonic series??
Express repeating decimal as ratio of integers??

Geometric series

Definition. A geometric series is any series that can be written in the form:

$$a + ar + ar^2 + ar^3 + ... = \sum_{n=1}^{\infty} ar^{n-1}$$

(we call *a* the initial term, and *r* the ratio)

QUESTION: When does a geometric series converge (and what does it converge to)?
Start with a geometric series (assume a>0):

$$\sum_{n=1}^{\infty} ar^{n-1}$$

The partial sums are given by:

$$S_k = \sum_{n=1}^{k} ar^{n-1} = a + ar + ar^2 + ar^3 + ... + ar^{k-1}$$

First, let's consider what happens if r=1. What is Sk? What happens to the sequence of partial sums {Sk}? *Diverges.*

What if r is not equal to 1? Let's start with the partial sum Sk, and see if we can get an explicit formula ("closed form" we call it).

$$S_k = a + ar + ar^2 + ar^3 + ... + ar^{k-1}$$

Multiply both sides by (1-r):

$$(1 - r)S_k = a(1 - r)(1 + r + r^2 + r^3 + ... + r^{k-1})$$

Expand RHS: $= a\left[(1 + r + r^2 + r^3 + ... + r^{k-1}) - (r + r^2 + r^3 + ... + r^{k-1} + r^k)\right]$

Cancel terms on the RHS:

$$(1 - r)S_k = a(1 - r^k)$$

Now divide by 1-r:

$$S_k = \frac{a(1-r^k)}{(1-r)}$$

Great! Now, what happens to the partial sums as k->infinity?

FACT: In a geometric series $\sum_{n=1}^{\infty} ar^{n-1}$,

- If $|r| \geq 1$ then the series diverges
- If $|r| < 1$ then the series converges to $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$

Example: Find the sum of the series, or determine that the sum does not exist.

A. $3 + \left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^4 + ...$

B. $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n+2}$

C. $\sum_{n=3}^{\infty} \frac{3^{n-1}}{10^{2n}}$

D. $\sum_{n=1}^{\infty} \frac{7+4^{n-1}}{9^{n-1}}$

NOTE: For D, will need:

THEOREM: Algebraic properties of convergent series

Suppose $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$. Then

1. $\sum_{n=1}^{\infty} (a_n + b_n)$ converges and equals A+B
2. $\sum_{n=1}^{\infty} (a_n - b_n)$ converges and equals A-B
3. For any real number c, $\sum_{n=1}^{\infty} ca_n$ converges and equals cA

Telescoping Series

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. We saw at the start of class that this series converges to 1 (by looking at some partial sums and making a guess). Let's see if we can prove it!

STEP 1: Rewrite 1/n(n+1) using partial fractions

STEP 2: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$

STEP 3: Write out the first few partial sums. Cancel the "telescoping" terms. What is a formula for Sk?

STEP 4: What happens to Sk as k->infinity? What is the sum of the series?

Example: Determine if the series converges and, if so, find the sum: $\sum_{n=1}^{\infty} \left(\cos\left(\frac{1}{n}\right) - \cos\left(\frac{1}{n+1}\right)\right)$

Writing a repeating decimal as a fraction

Example: Is the number 3.26262626... rational or irrational? If rational, write it as a ratio of integers.

HINT: Write it as an infinite sum using powers of 10^2 on the bottom. The part after 3 is a geometric series, so we can find the sum to get a fraction. Add 3 to this fraction to express the number as a ratio of integers.

Day 16: The Divergence and Integral Tests

WeBWorK: Series – Integral Test
Series – Divergence Test

Desmos partial sums demo

FROM LAST TIME:

Writing a repeating decimal as a fraction
Telescoping Series
Algebraic Properties of convergent series (+, -, constant multiple)

Today: 7(?) examples (finished up geometric series, telescoping series, divergence test, integral test, p-series)

1. $\sum_{n=3}^{\infty} \frac{3^{n-1}}{10^{2n}}$
2. $\sum_{n=1}^{\infty} \frac{7+4^{n-1}}{9^{n-1}}$
3. Write as a fraction: 3.26262626... **Geometric Series**
4. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$ **Telescoping Series**

   Algebraic rules for convergent series(?)
5. $\sum_{n=1}^{\infty} \frac{n}{n+1}$ **Divergence test**
6. $\sum_{n=1}^{\infty} \frac{1}{n}$ Harmonic Series. **Integral test**
7. $\sum_{n=1}^{\infty} \frac{1}{n^5}$. **p-series.**

FROM LAST TIME: Geometric series, telescoping series, algebraic properties of series

Geometric Series:

1. $\sum_{n=3}^{\infty} \frac{3^{n-1}}{10^{2n}}$

2. $\sum_{n=1}^{\infty} \frac{7+4^{n-1}}{9^{n-1}}$

Writing a repeating decimal as a fraction

Example 3: Is the number 3.26262626… rational or irrational? If rational, write it as a ratio of integers.

HINT: Write it as an infinite sum using powers of 10^2 on the bottom. The part after 3 is a geometric series, so we can find the sum to get a fraction. Add 3 to this fraction to express the number as a ratio of integers.

3.262626… = 3 + .26 + .0026 + .000026 + … =

Telescoping Series

Example 4. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. Let's see if we can determine if it converges, and if so find the limit.
STEP 1: Rewrite 1/n(n+1) using partial fractions
STEP 2: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$
STEP 3: Write out the first few partial sums. Cancel the "telescoping" terms. What is a formula for Sk?
STEP 4: What happens to Sk as k->infinity? What is the sum of the series?

??(SKIP?) Example 4a: Determine if the series converges and, if so, find the sum:
$\sum_{n=1}^{\infty} \left(\cos\left(\frac{1}{n}\right) - \cos\left(\frac{1}{n+1}\right)\right)$

THEOREM: *Algebraic properties of convergent series*

Suppose $\sum_{n=1}^{\infty} a_n = A$ *and* $\sum_{n=1}^{\infty} b_n = B$. *Then*

1. $\sum_{n=1}^{\infty} (a_n + b_n)$ *converges and equals A+B*

2. $\sum_{n=1}^{\infty} (a_n - b_n)$ *converges and equals A-B*

3. *For any real number c,* $\sum_{n=1}^{\infty} ca_n$ *converges and equals cA*

Testing series for convergence/divergence

It's going to be important in looking at these tests that we pay attention to two things:
1. Does the test apply? (there are always some conditions that must be met to see if we can apply the test).
2. What conclusion does the test give? (sometimes we can conclude a series converges, sometimes we conclude it diverges, and sometimes the test doesn't tell us anything).

Example 5: sum n/(n+1)

Theorem: Divergence Test - If $\lim_{n\to\infty} a_n = c \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then the series $\sum_{n=1}^{\infty} a_n$ diverges

BEWARE: The converse (the "reverse conclusion") is NOT TRUE. Just because $\lim_{n\to\infty} a_n = 0$ doesn't mean that the series $\sum_{n=1}^{\infty} a_n$ converges!! (in this case, the test is inconclusive)

Example. For each series, use the divergence test. If it the test proves that the series diverges, say so. If not, say the test is **inconclusive**.

A. $\sum_{n=1}^{\infty} \frac{n}{3n-1}$ B. $\sum_{n=1}^{\infty} \frac{1}{n^3}$ C. $\sum_{n=1}^{\infty} e^{1/n^2}$

Start next section with harmonic series example?

Example 6: sum 1/n

Theorem: Integral Test. Suppose $\sum_{n=1}^{\infty} a_n$ is a series with positive terms a_n. Suppose *f(x)* is a continuous, decreasing function and above some integer N, we have $a_n = f(n)$ (for all $n \geq N$). **Then** $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)dx$ either both converge, or both diverge.

Example. Use the integral test to determine whether the series converges or diverges (assume all conditions for the integral test are met):

A. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{x^5}}$

B. $\sum_{n=1}^{\infty} \frac{1}{2n+1}$

P-Series

Definition. For any real number *p*, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called a **p-series**.

FACT. A p-series converges if $p > 1$, and diverges if $p \leq 1$.

Important Example: The harmonic series.

Does the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} +$... converge?

FACT: The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Example 7: Determine whether the series converges or diverges.

A. $\sum_{n=1}^{\infty} \frac{1}{n^4}$

B. $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$

GROUP WORK

A. Write as a ratio of integers: 17.5261261261261...

B. Determine if the series converges and, if so, find the sum: $\sum_{n=1}^{\infty}\left(\cos\left(\frac{1}{n}\right) - \cos\left(\frac{1}{n+1}\right)\right)$

C. Use the divergence test. If the test proves that the series diverges, say so. If not, say the test is **inconclusive**. I. $\sum_{n=1}^{\infty}\frac{1}{n^3}$ II. $\sum_{n=1}^{\infty} e^{1/n^2}$

GROUP WORK 2

D. Use the integral test to determine whether the series converges or diverges (assume all conditions for the integral test are met): I. $\sum_{n=1}^{\infty} \frac{1}{n^3}$ II. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n-1}}$

E. Determine whether each p-series converges or diverges. I. $\sum_{n=1}^{\infty} \frac{1}{n^4}$ II. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^5}}$

Day 17: Comparison Tests

WeBWorK: Series – Comparison Tests

Start today with GROUP WORK from last time

Comparison test:

IDEA: What information can we get about a series by comparing it to another series?

Theorem: Comparison Test: Only works for positive series $\sum_{n=1}^{\infty} a_n$ with each $a_n > 0$.

If you can find a series that is BIGGER than yours and CONVERGES, then your series CONVERGES

I. If $\sum_{n=1}^{\infty} b_n$ is a series with $0 \le a_n \le b_n$ for all n (**or at least, for all $n > N$ for some integer N) and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

If you an find a series that is SMALLER than yours and DIVERGES, then your series DIVERGES

II. I. If $\sum_{n=1}^{\infty} b_n$ is a series with $a_n \ge b_n \ge 0$ for all n (**or at least, for all $n > N$ for some integer N) and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Example. Use the comparison test to determine whether the series converges or diverges.

A. $\sum_{n=1}^{\infty} \frac{1}{n^3+3n+1}$ B. $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ C. $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$

HINTS:
A. 1/n^3
B. (1/2)^n
C. (1/n)

Limit Comparison Test

Sometimes the comparison test doesn't quite work, because we can't find a good series to compare to -- for example:

$$\sum_{n=1}^{\infty} \frac{1}{n^2-1}$$

(We want to compare with 1/n^2, but this is a. Always smaller, and b. Converges -- so we can't conclude anything).

However, there is a more powerful way of comparing two series - by looking at the ratio of their terms, and taking the limit.

Theorem. Limit Comparison Test.

Suppose $a_n, b_n \geq 0$ for all $n \geq 1$.

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges

Example. Using the Limit Comparison Test

For each of the following series, use the limit comparison test to determine whether the series converges or diverges. If the test does not apply, say so.

A. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$ B. $\sum_{n=1}^{\infty} \frac{2^n+1}{3^n}$ C. $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$

HINTS:
A. compare 1/sqrt(n)
B. (2/3)^n
C. 1/n (use L\Hopitals for the limit!)

GROUP WORK

A. Use the comparison test to determine whether the series converges or diverges.

I. $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ II. $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$

B. Use the limit comparison test to determine whether the series converges or diverges. If the test does not apply, say so.

I. $\sum_{n=1}^{\infty} \frac{2^n+1}{3^n}$ II. $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$

Day 19: Alternating Series

WeBWorK: Series – Alternating Series

Post-Spring-Break Review

Example: Does the series converge? Explain why or why not.

a. $\sum_{n=1}^{\infty} \frac{1}{n^4}$ *p-series*

b. $\sum_{n=1}^{\infty} \frac{3^{n-1}}{5^n}$ *geometric series*

c. $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ *comparison with 1/2^n*

d. $\sum_{n=1}^{\infty} \frac{1}{n^2-1}$ *limit comparison with 1/n^2*

e. $\sum_{n=1}^{\infty} \frac{2x^2-7}{3x^2+3x+1}$ *divergence test (or limit comparison)*

Alternating Series

An alternating series is a series whose terms alternate between positive and negative values.

Defn. Any series whose terms alternate between positive and negative values is called an **alternating series**. Any alternating series can be written as:

A. $\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + ...$

B. $\sum_{n=1}^{\infty} (-1)^{n} b_n = - b_1 + b_2 - b_3 + b_4 - ...$

Where $b_n > 0$ for all n.

Theorem: Alternating Series Test. An alternating series converges if:

i. $0 < b_{n+1} < b_n$ for all $n \geq 1$

ii. $\lim_{n->\infty} b_n = 0$

Example. Determine whether the following series converge:

A. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ B. $\sum_{n=1}^{\infty} \frac{(-1)^{n} \cdot n}{n+1}$ C. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot n}{2^n}$

Absolute and Conditional Convergence

Let's consider two series, $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} |a_n|$. Which one is more likely to converge?

Example: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ vs $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right|$. Which converges?

Harmonic series diverges (p-series). Alternating harmonic series converges (alternating series).

Definition. We say a series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ converges.

The series $\sum_{n=1}^{\infty} a_n$ is **conditionally convergent** if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

Theorem: Absolute convergence implies conditional convergence. If $\sum_{n=1}^{\infty} \left|a_n\right|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Example. Determine whether each series converges absolutely, converges conditionally, or diverges.

A. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n+1}$ B. $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$ C. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n^3+1}$

HINTS: A. Absolute value series diverges by limit comparison with harmonic series. Original series converges by Alternating Series test. Thus the series converges conditionally.
B. Absolute value series converges by comparison with 1/n^2. Thus the series converges absolutely.
C. Absolute value series converges by comparison with n/2n^3.

Day 20: Ratio and Root Tests

WeBWorK: Series – Ratio and Root Tests

We saw before that we can compare two series by looking at the ratio of their terms. Here we look at a test that does something similar - but we look instead at the ratio of successive terms of the same series:

Theorem: Ratio Test. Let $\sum_{n=1}^{\infty} a_n$ be a series with nonzero terms. Let

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

i. If $0 \le \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely

ii. If $\rho > 1$ or $\rho = \infty$, then then $\sum_{n=1}^{\infty} a_n$ diverges

iii. If $\rho = 1$, the test gives no information (test fails)

Example. Use the ratio test to determine whether the series converges or diverges.

A. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ B. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ C. $\sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2}{(2n)!}$ D. $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$

NOTE: B relies on limit defn of e: $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$

One more test - this time, based on the idea of geometric series.

We know a series $\sum_{n=1}^{\infty} r^n$ converges when $|r| < 1$ (Geometric series). Note that if we take the nth root of the terms, it cancels the nth power and we are left just with the ratio r:

$\sqrt[n]{r^n} = r$

The idea behind the root test is that even if a series is not exactly a geometric series, if it behaves like one as we take the limit then we can think about it like we think about geometric series.

Root Test. Let $\sum_{n=1}^{\infty} a_n$ be a series. Let

$$\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

i. If $0 \le \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely

ii. If $\rho > 1$ or $\rho = \infty$, then then $\sum_{n=1}^{\infty} a_n$ diverges

iii. If $\rho = 1$, the test gives no information (test fails)

Example. For each of the following series, use the root test to determine whether the series converges or diverges:

A. $\sum_{n=1}^{\infty} \frac{(n^2+3n)^n}{(4n^2+5)^n}$ B. $\sum_{n=2}^{\infty} \frac{n^n}{(\ln(n))^n}$ C. $\sum_{n=1}^{\infty} \frac{1}{n^n}$

Choosing a Convergence Test: Check out this handy checklist!
Also, take a look at this table of all the convergence tests!

GROUP WORK

Does the alternating series below converge conditionally? Does it converge absolutely? Be sure to state any tests you use along the way, and the results of those tests.

A. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n^3+1}$

GROUP WORK

Use the Ratio Test to determine whether the series converges or diverges (be sure to state any other tests you use along the way, and the results of those tests).

B. $\sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2}{(2n)!}$

GROUP WORK

Use the Ratio Test to determine whether the series converges or diverges (be sure to state any other tests you use along the way, and the results of those tests).

C. $\sum_{n=1}^{\infty} \frac{1}{n^n}$

GROUP WORK

Determine whether the series converges or diverges. State any tests you use along the way, and the results of those tests.

D. $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$

ALTERNATING:

A. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n+1}$ B. $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$ C. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n^3+1}$

Example. Use the ratio test to determine whether the series converges or diverges.

A. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ B. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ C. $\sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2}{(2n)!}$ D. $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$

Example. For each of the following series, use the root test to determine whether the series converges or diverges:

A. $\sum_{n=1}^{\infty} \frac{(n^2+3n)^n}{(4n^2+5)^n}$ B. $\sum_{n=2}^{\infty} \frac{n^n}{(\ln(n))^n}$ C. $\sum_{n=1}^{\infty} \frac{1}{n^n}$

Day 21: Power Series

WeBWorK: Series – Power Series

Find radius of convergence/interval of convergence for various power series.
Differentiating and integrating power series

Today we take the idea of infinite series and use it to create a function, based on the idea of a Polynomial but with infinitely many terms.

Example: $f(x) = 1 + x + x^2 + x^3 + ... = \sum_{n=0}^{\infty} x^n$

Question: If I plug in a number for x, does it even make sense? For which x will the series converge?

If we replace x with a number, will the series converge?
For which x will it converge?
What type of series is this?
NOTE: geometric series with ratio r=x, so converges when |x|<1.
What function is this equal to?

Power Series

A power series is a series whose terms involve a variable - in particular, x^n.

Definition. A power series is a series of the form:

A. $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + ...$ (this is a power series centered at x=0)

B. $\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 ...$ (this is a power series centered at x=a)

The idea behind a power series is that it defines a function -- you put in an x, and if the series converges then it gives back a number f(x). If we define the function $f(x) = \sum_{n=0}^{\infty} x^n$, what is the domain of this function?

This power series has **interval of convergence** (-1,1), and **radius of convergence** 1.

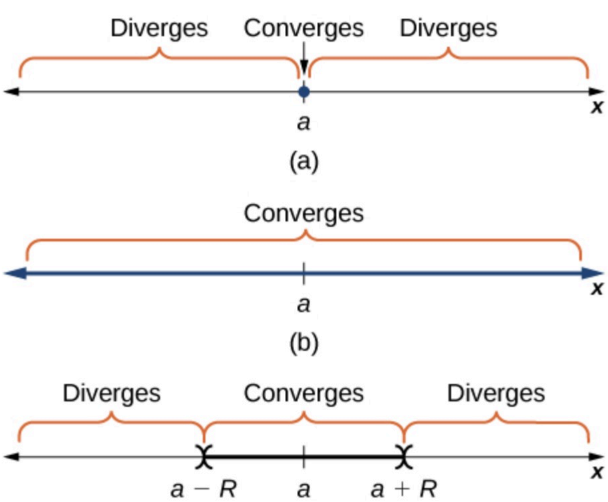
Convergence of a Power Series

Defn. Given a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ centered at x=a.

I. The set of real numbers x for which the power series converges is called the **interval of convergence**.

II. The **radius of convergence** is the distance from x=a to the edge of the interval of convergence. If a power series converges for all x, then we say the radius of convergence is $R = \infty$.

FACT: For every power series, one of three things always happens:
1. Converges at only the center point x=a.
2. Converges for all x.
3. Converges on an interval (a-R, a+R) (may or may not include the endpoints)



Example. Find the interval and radius of convergence of each power series:

A. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ B. $\sum_{n=0}^{\infty} n!\, x^n$ C. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)3^n}$

A. all x,   B. only x=0,   C. test endpoints! [-1,5)

Combining Power Series

We'll be talking more next time about finding power series for various functions. But for now, it's helpful to see how we can work with an existing power series and modify it in various ways.

Theorem. Combining Power Series

Suppose that the two power series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge to $f(x)$ and $g(x)$ respectively on a common interval I.

1. The power series $\sum_{n=0}^{\infty} \left(c_n x^n \pm d_n x^n\right)$ converges to $f \pm g$ on I. (you can **add or subtract power series**)
2. If $m \geq 0$ is an integer and b is a real number, the power series $\sum_{n=0}^{\infty} bx^m c_n x^n$ converges to $bx^m f(x)$ on I. (you can **multiply a power series by a constant or power of x**)
3. If $m \geq 0$ is an integer and b is a real number, the power series $\sum_{n=0}^{\infty} c_n \left(bx^m\right)^n$ converges to $f(bx^m)$ for all x such that bx^m is in I. (you can replace x with bx^m in a power series -- that is, you can compose f(x) with bx^m)

Example. Consider the function $f(x) = \frac{1}{1-x}$. We know from above that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.

A. Find the power series for the derivative $f'(x)$ and the antiderivative $F(x)$ of f(x)
B. What is the power series for the function $\ln(1 - x)$?
C. What is the power series for the function $\frac{3}{1-x^2}$?

HINTS: A. differentiate/integrate term-by-term.
B. look at the power series for the integral of f(x) -- is that the function we're looking for?
C. Substitute x^2 for x in the original power series

Day 22: Taylor and MacLaurin Series

WeBWorK: Series – Taylor and Maclaurin Series

FALL 2024: Started with prior material, managed to get through the definition of Taylor Series only. Will pick it up next time

6.3 Find Taylor series of the function at the given point (power fcn, polynomial, sine/cosine, exponential, natural log, 1/x) - using the defn of taylor series
6.4 MacLaurin/Taylor Series of common functions

Deriving M/T Series by using know series for common functions
Find the MacLaurin series of the function (2^x, sin(x^2), e^(x^3), (sinx)^2 using half-angle identity.
Compute first 3 nonzero terms of MacLaurin series (tanx, ln(cosx),e^x cosx, e^(sinx), (sec x)^2)

Definition. If f(x) is a function with derivatives of all orders at x=a, then **the Taylor series for f(x) at x=a** is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{2!}(x-a)^3 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n + ...$$

The Taylor series for f at 0 is known as the **MacLaurin series for f.**

NOTE: This is exactly like the definition of Taylor/MacLaurin polynomials, except we continue for all n instead of stopping at some finite number k.
Also note, the coefficients here are exactly the same as in Taylor/MacLaurin polynomials (it's the power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ centered at x=a, with coefficients:

$$a_0 = f(a) \qquad a_1 = \frac{f'(a)}{1!} \qquad a_2 = \frac{f''(a)}{2!} \qquad a_3 = \frac{f'''(a)}{3!} \qquad a_n = \frac{f^{(n)}(a)}{n!})$$

Example. Find the Taylor series for $f(x) = \frac{1}{x}$ at $x = 1$. Determine the interval of convergence.
HINT: Can do this the "long way", or by rewriting 1/x as 1/(1-(1-x)) and substituting into the series we found yesterday for 1/(1-x).

Example. Find the MacLaurin Series for $f(x) = \cos x$ and its interval of convergence.
HINT: We did this already for MacLaurin polynomials -- BUT doesn't hurt to do it again! This time focus on finding a formula for a_n.

Example. Find the MacLaurin Series for $f(x) = \cos(x^2)$ and its interval of convergence.
HINT: Substitute into the series found above!

Example. Find the MacLaurin Series for $f(x) = e^x$ and its interval of convergence.

Example. Find the MacLaurin Series for $f(x) = x^3 e^{4x}$ and its interval of convergence.
HINT: Write our function in terms of the previous function, then substitute into the power series appropriately.

Pro Tip: Want a handy reference of the Taylor Series of common functions? Take a look at Table 6.1 in the book:

Function	Maclaurin Series	Interval of Convergence
$f(x) = \frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$-1 < x < 1$
$f(x) = e^x$	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$-\infty < x < \infty$
$f(x) = \sin x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$-\infty < x < \infty$
$f(x) = \cos x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$-\infty < x < \infty$
$f(x) = \ln(1+x)$	$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$	$-1 < x \le 1$
$f(x) = \tan^{-1} x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$-1 \le x \le 1$
$f(x) = (1+x)^r$	$\sum_{n=0}^{\infty} \binom{r}{n} x^n$	$-1 < x < 1$

Table 6.1 Maclaurin Series for Common Functions

Day 23: *Taylor and MacLaurin Series*

WeBWorK: Series – Taylor and Maclaurin Series
(today we cover material from previous lecture - Day 23)

Day 24: Approximating Areas, Definition of Definite Integral

1.1 Approximating Areas (p. 5 – 20)
WeBWorK: *Applications – Approximation of Area*

Day 25: Exam 3

Day 26: Areas Between Two Curves

2.1 Areas Between Two Curves (p. 122 – 128)
WeBWorK: *Applications – Area Between Curves*

Recall definition of definite integral (of f(x) from x=a to x=b).
In particular, recall the picture - rectangles, width delta x, height f(x).

Finding the Area between Two Curves
Let f(x) and g(x) be continuous functions such that $f(x) \geq g(x)$ for all x in an interval [a,b]. Let R be the region bounded above by the graph of f(x), below by g(x), and on the left and right by x=a and x=b. Then the area of R is given by:

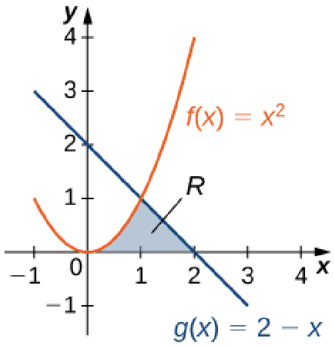
$$A = \int_a^b [f(x) - g(x)]dx$$

Example. If R is the region bounded above by the graph of $f(x) = x + 4$ and below by the graph of $g(x) = 3 - \frac{x}{2}$ over the interval [1,4], find the area of R.

Example. Find area of the region bounded above by the graph of $f(x) = 9 - \left(\frac{x}{2}\right)^2$ and below by $g(x) = 6 - x$.

Example. If R is the region between the graphs of $f(x) = \sin x$ and $g(x) = \cos x$ over the interval $[0, \pi]$, find the area of R.

Example. Consider the region R bounded below by the x-axis and above by the functions $f(x) = x^2$ and $g(x) = 2 - x$ (depicted below). Find the area of R.



Regions Defined with Respect to y

NOTE: It's annoying that the above example takes 2 integrals to calculate! We can do this with a single integral by thinking of it differently - instead of taking x as our "basic variable" and integrating along the x-axis, instead we will take y as our "basic variable" and integrate along the y-axis. How do we do this?

First, let's rewrite f(x) and g(x) as functions of y.

- Replace f(x) with y.
- Solve for y to obtain a new function u(y).
- This gives the x as a function of y -- that is, for any given value of y (like y=4), we can plug in to u(y) to get the corresponding value of x.

Do the same for g(x) to obtain a new function v(y).

$u(y) = \sqrt{y}$ and $v(y) = 2 - y$

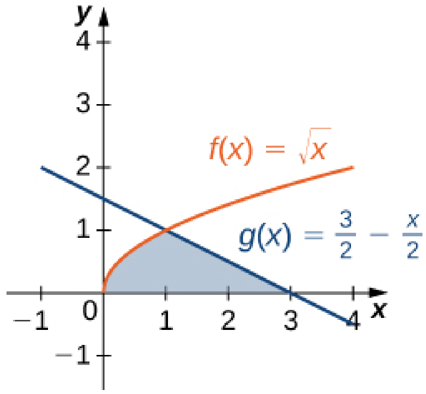
Now we integrate *along y-axis* from the "bottom" to the "top" of the region.

What are the bounds of integration (that is, the lowest and highest values of y)? y=0 to y=1. Divide the interval [0,1] along the y-axis into intervals. On each interval, draw a *horizontal* rectangle. This rectangle has high delta-y, and width v(y)-u(y).

Now set up the integral with respect to y and evaluate it to find the area of the region R:

$$A = \int_0^1 [v(y) - u(y)]dy$$

Example. Let R be the region shown below. Find the area of the region by setting up and evaluating an integral with respect to y.


y
4
3
2
1
−1
0
1
2
3
4
x
−1
$f(x) = \sqrt{x}$
$g(x) = \frac{3}{2} - \frac{x}{2}$

Day 25: Exam 3

Day 27: Volumes by slicing

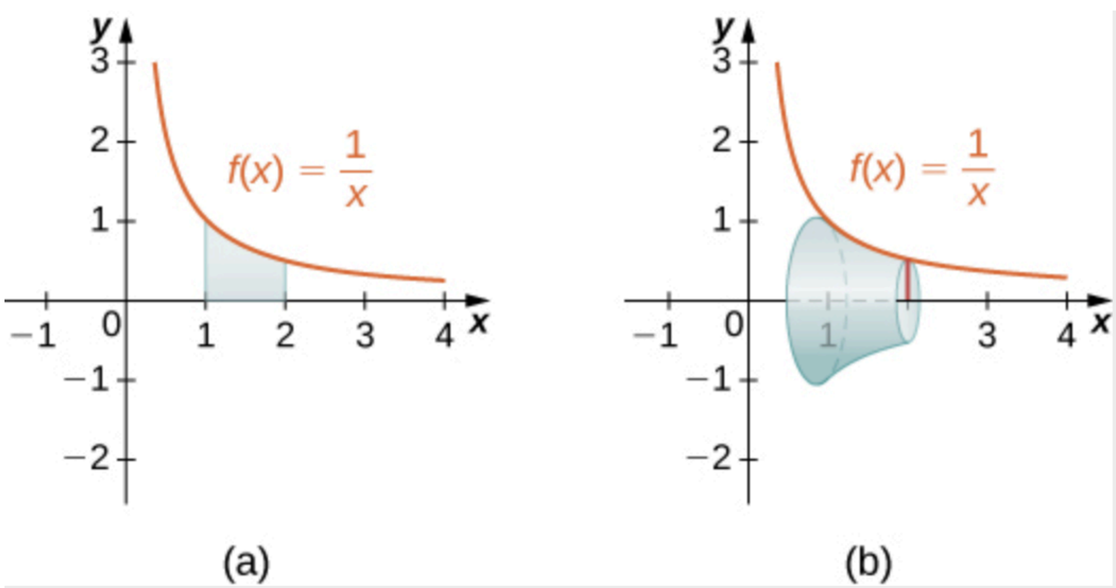
2.2 Determining Volumes by Slicing (p. 141 – 149)
WeBWorK: *Applications – Volumes of Revolution*
NOTE: For next time, try splitting this WW set (currently used for Days 27&28 combined) into 2 (disks/washers, and cylindrical shells), and incorporate some of the 'non-revolution' volumes by slicing as well from the previous (unused) WW set "Applications -- Volumes by Slicing"

Volumes of Revolution

If a region in a plane is revolved around a line in that plane, the resulting solid is called a solid of revolution, as shown in the following figure.

Example 1: Use the method of slicing to find the volume of the solid of revolution formed by revolving the region between the graph of the function $f(x) = \frac{1}{x}$ and the x-axis over the interval $[1,2]$ around the x-axis.

https://www.desmos.com/3d/fhdnouyrmi



We will find this volume by using the **disk method**.
Consider approximating the volume as follows:
Divide the interval from x=1 to x=2 into a finite number of sub-intervals.

On each subinterval, approximate the volume by a solid disk (sketch), with radius given by f(x).

Find the volume of the disk:

What is the width? Δx.

What is the area of one flat surface of the disk? $\pi[f(x_i)]^2$

If we add up all the volumes of these disks, we will get an (approximate) value of the volume.

To improve our value, we can divide the original interval into more subintervals (this gives more disks). When we take the limit as the number of subintervals approaches infinity, we end up with a definite integral that gives the exact volume:

$$\int_1^2 \pi\cdot\left(\frac{1}{x}\right)^2 dx$$

Example 2. Find the volume obtained by rotating the region bounded by $f(x) = 4 - 2x$, and the x and y axes, about the y-axis.

NOTE: Since we are rotating about the y-axis, need to integrate wrt y!

HINT: radius at height y is given by $u(y) = 2 - \frac{1}{2}y$. Bounds of integration along y-axis are y=0 to y=4. Volume = $\int_0^4 \pi(2 - \frac{1}{2}y)^2 dx$

Rule. The disk method. Suppose f(x) is continuous and non-negative on the interval [a,b]. Let R be the region bounded above by f(x), below by the x-axis, and left and right by x=a and x=b. Then the volume of revolution formed by revolving R around the x-axis is given by:

$$V = \int_a^b \pi[f(x)]^2 dx$$

Example 3. Consider the region R bounded by $f(x) = \sqrt{x}$ and the horizontal line y=1 on the interval [1,4]. Find the volume of the solid formed by rotating R around the x-axis.

NOTE: Cross-sections are not circles - they are pierced circles. How do we find the area?

Example 4. Find the volume of the solid obtained by rotating the region enclosed by $f(x) = 2x$ and $g(x) = x^2$ about the line x=-2.

Rule. The washer method

RULE: THE WASHER METHOD

Suppose $f(x)$ and $g(x)$ are continuous, nonnegative functions such that $f(x) \geq g(x)$ over $[a, b]$. Let R denote the region bounded above by the graph of $f(x)$, below by the graph of $g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$. Then, the volume of the solid of revolution formed by revolving R around the x-axis is given by

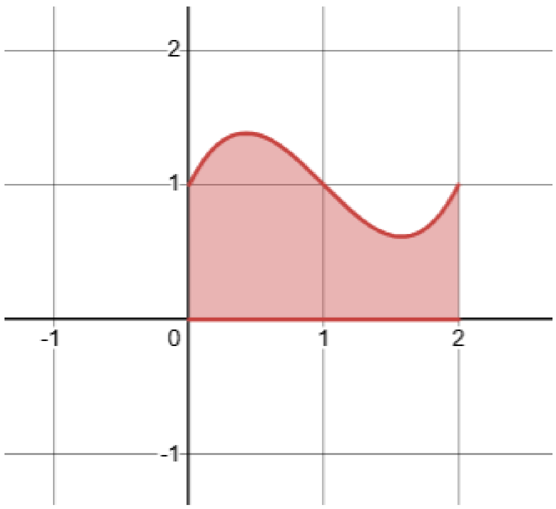
$$V = \int_a^b \pi \left[(f(x))^2 - (g(x))^2\right] dx. \tag{2.5}$$

Day 28: Volumes of Revolution: Cylindrical Shells

2.3 Volumes of Revolution: Cylindrical Shells (p. 156 – 165)
WeBWorK: *Applications – Volumes of Revolution*

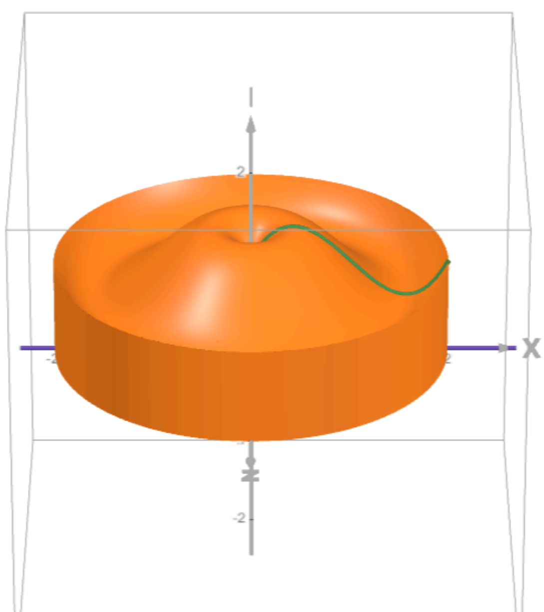
Consider solid obtained by rotating the region shown in the graph around the y-axis:

The region:



$f(x) = x^3 - 3x^2 + 2x + 1$

The volume:



Alternate approach:

Example: Find the volume of the solid obtained by rotating the region R enclosed by the function $f(x) = x^3 - 3x^2 + 2x + 1$ and the x-axis on the interval [0,2], about the y-axis.

QUESTION: How should we go about finding the volume? If we want to use the disk/washer method, we will be slicing horizontally (perp to y-axis). In this case, the cross-sections for all y on the upper part of the figure will be "double-washers", bounded on one side by the graph of f(x) and on the other side also by the graph of f(x).
THIS IS COMPLICATED - requires finding not one, but TWO inverse functions of f(x).

The Method of Cylindrical Shells

Instead, let us approach this example by integrating along the x-axis.
Divide the interval [0,2] into subintervals and look at a typical thin rectangle with height given by f(xi) and width Δx.

The result of rotating this rectangle about the y-axis will give what shape?
We call this shape a cylindrical shell. How "thick" is the shell? Δx
What is the volume of this shell?
Now, if we make a cut in this shape and unroll it, we get (basically) a rectangular solid.
This rectangular solid has dimensions Δx by f(xi) by 2pixi $\Delta x \times f(x_i) \times 2\pi x_i$, and the volume is thus $2\pi x_i f(x_i)\Delta x$.
If add up these volumes, we get an approximation of the volume we are after.
If we take the limit as the number of subintervals goes to infinity, we get a definite integral:

$$\int_a^b 2\pi x f(x)dx$$

In the case of our example, that's:

$$\int_0^3 2\pi x(x^3 - 3x^2 + 2x + 1)dx$$ (or approx 4.5132741)

Rule. The method of cylindrical shells. Suppose f(x) is continuous and non-negative. Define R as the region bounded above by the graph of f(x) and below by the x-axis over the interval [a,b]. Then the volume of the solid of revolution formed by revolving R around the y-axis is given by $V = \int_a^b 2\pi x f(x) dx$

Group Work

a. Let R be the region bounded by the graphs of $y = x^3$, $y = 2 - x^2$, and the y-axis. Find the volume of the solid obtained by rotating R about the y-axis

Group Work

The problems below use the same region R as in the previous example: Let R be the region bounded by the graphs of $y = x^3$, $y = 2 - x^2$, and the y-axis.

b. Set up, but do not evaluate, a definite integral giving the volume of the solid obtained by rotating R about the x-axis.

c. Set up, but do not evaluate, a definite integral giving the volume of the solid obtained by rotating R about the line x=-2.

d. Set up, but do not evaluate, a definite integral giving the volume of the solid obtained by rotating R about the line y=3