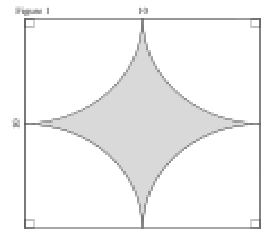
Solutions to Aleph-3 Middle School Problem-Solving Round

1. Given that Figure 1 depicts a square of side length 10 and four circular arcs centered at its vertices, what is the area of the shaded region?



Answer: $100 - 25\pi$

Notice how there are four quarter circles that are removed from the total area of the square, which together form a complete circle of radius 5. Therefore, the area of the shaded region is $10^2 - 5^2 \times \pi = 100 - 25\pi$

2. Mary says that she is thinking of two prime numbers that sum up to 55. What is the absolute difference between these two prime numbers?

Answer: 51

Since the sum of the two prime numbers is odd, the two prime numbers must have opposite polarity, i.e., one must be even and the other must be odd. There is only 1 even prime number: 2. Thus, the other prime must be 55-2=53, so their absolute difference is 53-2=51.

3. A die is weighted so that the number on each face is proportional to the probability of rolling that number (e.g. Rolling a 3 is three times as likely as rolling a 1. A 6 is two times as likely as rolling a 3). What is the probability of rolling a prime number?

Answer: $\frac{10}{21}$

Since the probability of rolling a given number is proportional to its value, let the probability of rolling a 1 be x. Thus, the probability of rolling a 2 is 2x, a 3 is 3x, etc…
Thus, the sum of the probabilities of rolling any of the numbers must be x+2x+3x+4x+5x+6x=21x.
We want to find the probability of rolling a prime number: 2, 3, or 5. Thus, the probability of rolling any of these values must be $\frac{2x+3x+5x}{21x} = \frac{10}{21}$.

4. There are some number of cardinals and goldfinches sitting on a branch. When 14 goldfinches and 3 cardinals land on the branch, 2/5 of the birds are cardinals. After 6 cardinals leave and 16 goldfinches arrive, 5/7 of the birds are now goldfinches. How many cardinals were there originally?

Answer: 28

Let the original number of cardinals and goldfinches be c and g, respectively. Therefore, we can say: $\frac{c+3}{g+14} = \frac{2}{3}$*, which represents the ratio of cardinals to goldfinches after the first round of birds comes. Note that since* $\frac{2}{5}$ *of the birds are cartinals,* $\frac{3}{5}$ *of the birds are goldfinches, so the overall ratio of cartinals to goldfinches is* $\frac{\frac{2}{5}}{\frac{3}{5}} = \frac{2}{3}$*. Also,* $\frac{c-3}{g+30} = \frac{2}{5}$*, based on the second round of birds arriving and leaving.*

$\frac{c+3}{g+14} = \frac{2}{3} \Rightarrow 2g + 28 = 3c + 9 \Rightarrow 2g = 3c - 19$
$\frac{c-3}{g+30} = \frac{2}{5} \Rightarrow 2g + 60 = 5c - 15 \Rightarrow 2g = 5c - 75$
$5c - 75 = 3c - 19 \Rightarrow 2c = 56 \Rightarrow c = 28$. *Thus, there are 28 cartinals initially.*

5. $\frac{\sqrt[3]{x^{10}}+\sqrt[3]{x}}{\sqrt[3]{x}} = 9$. Find x.

Answer: 2 OR $2, \; -1 \pm i\sqrt{3}$

LHS: $\frac{\sqrt[3]{x^{10}}+\sqrt[3]{x}}{\sqrt[3]{x}} = \frac{\sqrt[3]{x^{9}\cdot x}+\sqrt[3]{x}}{\sqrt[3]{x}} = \frac{\sqrt[3]{x^{9}}\sqrt[3]{x}+\sqrt[3]{x}}{\sqrt[3]{x}} = \frac{\sqrt[3]{x}(\sqrt[3]{x^{9}}+1)}{\sqrt[3]{x}} = \frac{\sqrt[3]{x}(x^{3}+1)}{\sqrt[3]{x}}$

Case 1: When x=0, $\frac{\sqrt[3]{x}(x^{3}+1)}{\sqrt[3]{x}} = \frac{\sqrt[3]{0}(0^{3}+1)}{\sqrt[3]{0}} = \frac{0(0+1)}{0} = \frac{0}{0}$

This produces an indeterminate form, so we discard this case.

Case 2: When $x \neq 0$*,* $\frac{\sqrt[3]{x}(x^{3}+1)}{\sqrt[3]{x}} = x^{3} + 1$

Thus, we have $x^{3} + 1 = 9$

$x^{3} = 8$

The only real solution to this equation is 2; however, we also find imaginary solutions as well. Using the Roots of Unity, we can find three solutions to this equation in the complex plane, evenly spaced about the origin at a distance of 2. Since one solution is at $2 + 0i$ *(i.e., at 0 radians measured counterclockwise from the positive x-axis), the other solutions must be at* $\frac{2\pi}{3}$ *and* $\frac{4\pi}{3}$ *radians.*

The two other complex solutions are:

$2[cos(\frac{2\pi}{3}) + i \cdot sin(\frac{2\pi}{3})] = -1 + i\sqrt{3}$

$2[cos(\frac{4\pi}{3}) + i \cdot sin(\frac{4\pi}{3})] = -1 - i\sqrt{3}$

Thus, $x = 2, \; -1 \pm i\sqrt{3}$

(Note that 2 as an answer was also accepted due to the lack of specificity about complex solutions).

6. What is the smallest positive integer N which the following relation holds true: $\frac{1}{d_1} + \frac{1}{d_2} + \ldots \frac{1}{d_n} = 2$ where $d_1, d_2, d_3 \ldots d_n$ are all of the positive divisors of N, including N?

Answer: 6

We can test small numbers for N to determine what will work.

If N=1, $\frac{1}{1} = 1$

If N=2, $\frac{1}{1} + \frac{1}{2} = 1.5$

Notice how all prime numbers A will be equal to $1 + \frac{1}{A} \neq 2$ *for A≠1, so we only have to consider composite numbers:*

If N=4, $\frac{1}{1} + \frac{1}{2} + \frac{1}{4} = 1.75$

If N=6, $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 2$*. Thus, the smallest value for N is 6.*

7. How many positive integers less than 1001 are divisible by either 3,5, or 8?

Answer: 534

We will consider how many numbers less than 1000 are divisible by each of the following: 3, 5, 8, 15, 24, 40, 120.

Divisible by 3: 333
Divisible by 5: 200
Divisible by 8: 125
Divisible by 15: 66
Divisible by 24: 41
Divisible by 40: 25
Divisible by 120: 8

First, we will add together the numbers divisible by 3, 5, and 8 separately, without considering overlap: 333+200+125=658.
Next, we will consider the overlap between each of these categories, so we will subtract the numbers divisible by 15, 24, and 40 since those numbers were counted under at least two of the previous categories: 658-(66+41+25)=526.
Lastly, we will add back the numbers that are divisible by 120 since they were added thrice and subtracted thrice: 526+8=534. Thus, the answer is 534.

8. Define $a \ mod \ b$ to give the remainder when a is divided by b. For example, $5 \ mod \ 3 \ = \ 2$. Find the smallest possible value of the number n, if $n \ mod \ 8 \ = \ 7$, $n \ mod \ 7 \ = \ 6$, and $n \ mod \ 6 \ = \ 5$.

Answer: 167

Notice that the pattern for the value of n mod 8 repeats every 8 numbers, for mod 7 repeats every 7 numbers, and for mod 6 every 6 numbers. Thus, the overall pattern of all of these values repeats every LCM(6, 7, 8)=168 numbers.

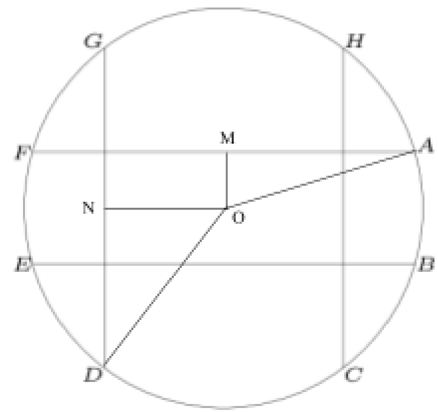
Additionally, notice how we can extend the mod pattern to the negative numbers: $7 \ mod \ 8 \equiv \ -1$, $6 \ mod \ 7 \equiv \ -1$, *and* $5 \ mod \ 6 \equiv \ -1$.

Finally, 168 is divisible by 6, 7, and 8, so it has a remainder of $0 \ mod \ 6, \ 7, \ or \ 8$. *Therefore, 168-1=167 will be equivalent to* $-\ 1 \ mod \ 6, \ 7, \ or \ 8$, *which we have already shown to be equivalent to* $n \ mod \ 8 \ = \ 7$, $n \ mod \ 7 \ = \ 6$, *and* $n \ mod \ 6 \ = \ 5$. *Thus, we have n=167.*

Checking this answer, $167 \ mod \ 8 \equiv 7$, $167 \ mod \ 7 \ = \ 6$, *and* $167 \ mod \ 6 \ = \ 5$.

9. In the diagram to the right, a circle of radius 25 is intersected with 4 lines. CH || DG, AF || BE, and AF ⊥DG. If CH = DG = 40 and AF = BE = 48, what is the area of the rectangle in the center?

Answer: 420



Refer to the figure to the right in which we constructed two right triangles. *Note that in the original problem, NO, OM, OA, and OD were not drawn.*

Statement	*Explanation*
OD = OA = 25	*OD and OA are radii of the circle of which radius=25 is given.*
ND = 20 and MA = 24	$\overline{ON}$ and $\overline{OM}$ *are perpendicular bisectors to chords* $\overline{GD}$ and $\overline{AF}$ *respectively(If you cannot see this, then notice* $\triangle OND \cong \triangle ONG$*)*
NO = 15	*Pythagorean theorem or recognizing that is similar to a 3-4-5 triangle.* $\sqrt{25^2 - 20^2} = 15$
MO = 7	*Pythagorean theorem:* $\sqrt{25^2 - 24^2} = 7$
Area of the inner rectangle given by corners M,O,N and an unlabeled vertices = 105	*Area of rectangle formula. Base x height = 15 * 7 = 105*
*Area of rectangle is **420***	*4- way symmetry. Four other rectangles can be constructed to make the big rectangle, which is the same as the first one. 105 * 4 = 420*

If this solution looks intimidating, it's not! It's just that this two-column explanation was used for clarity, and it happens to take a lot of space. Usually with this circle problems, it's just a good idea to draw a perpendicular bisector to chords or radii to points.

Alternate solution: In an another line of reasoning, one could guess 69 and 420 for #8 and #9, respectively to obtain the correct answer for #9. Shoutout to that one person actually did this and got this question correct.

10. Call a 2-digit number ab spectacular if it is equal to $a^2 + 2b + 1$. For instance, 43 is not spectacular because $4^2 + (2)(3) + 1 = 23 \neq 43$. There are two spectacular 2-digit numbers. What is their sum?

Answer: 116

<u>ab</u> can be represented as 10a+b.

$10a + b = a^2 + 2b + 1$

$a^2 - 10a + b + 1 = 0$

$a = \frac{10 \pm \sqrt{100 - 4(b+1)}}{2} = 5 \pm \sqrt{25 - b - 1} = 5 \pm \sqrt{24 - b}$

a must be a real number, so $24 - b$ must be a perfect square. b must be a whole number between 0 and 9, inclusive, so $24 - b = 1, 4, 9, 16, 25, 36, \ldots$ Of these values, only $24 - b = 16$ is possible, so $b = 8$.

Thus, $a = 5 \pm \sqrt{24 - 8} = 5 \pm \sqrt{16} = 5 \pm 4 = 1, 9$.

Therefore, the two 2-digit spectacular numbers are 18 and 98, so the answer is 18+98 = 116.

11. TIEBREAKER: How many digits would $\lfloor e^{\pi^{e^{\pi}}} \rfloor$ have? Answer with an integer that does not have any operations on it (e.g. no exponents or logarithms). ⌊x⌋ is defined as the greatest integer smaller than x. This is an estimation question that is likely not possible to be solved by hand. It will not be used to affect your score except for the event of a tiebreaker.

Answer: 158,810,688,650

There is a way to reasonably estimate the answer to this question. Notice that this expression is a "power tower." Let us take an example of how power towers are evaluated. $2^{2^{2^2}} = 2^{2^4} = 2^{16} = 65536$. Notice that the exponents at the top are evaluated first, not from the bottom. Let us use the "fundamental theorem of engineering" that . This is because $\pi \approx 3.14$ and $e \approx 2.718$, which are both about 3. Thus, this can be evaluated to $3^{3^{3^3}} = 3^{3^{27}}$. Now we are going to use another approximation that $3^2 \approx 10$. Thus $3^{(3^2)^{13.5}} \approx 3^{10^{13.5}} \approx 3^{10000000000000}$. Let us again apply the approximation that $3^2 = 10$. Then, $(3^2)^{5 * 10^{12.5}} \approx 10^{5 * 10^{12.5}}$. Knowing the rule that 10^n will have $n + 1$ digits, this approximation has about $5 \times 10^{12.5} + 1$ digits. This answer is only about 1 order of magnitude away from the actual value. This overestimate resulted from using the $3^2 \approx 10$ approximation. Accounting for this overestimate and considering that it is better to undercount than overcount for the purposes of absolute differences, one should reduce their answer by a few orders of magnitude. So, 10^{11} or 100000000000 would be a more reasonable answer that is on the same order of magnitude as the actual value.