General Maths Unit 4

Core Topic: Matrices
Chapter 11

Content

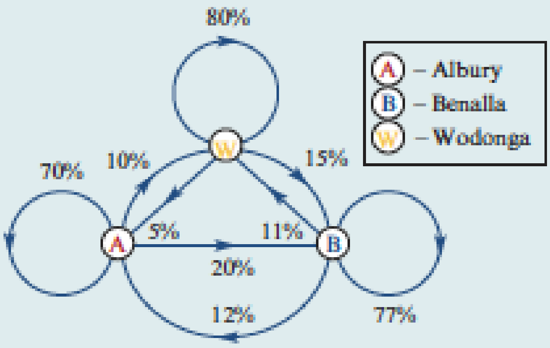
11a – Transition Matrices and setting up a transition matrix

- A transition Matrix T is a square matrix which can be used to investigate all possibilities when two events occur
- The columns in a Transition Matrix must add up to 1

Example – Setting up a Transition Matrix from a Network drawing

Example 1 Setting up a transition matrix

The diagram gives the weekly return rates of rental cars at three locations: Albury, Wodonga and Benalla. Construct a transition matrix that describes the week-by-week return rates at each of the three locations. Convert the percentages to proportions.



Example Setting up a Transition Matrix from worded information

Example 2 **Setting up a transition matrix**

A factory has a large number of machines. Machines can be in one of two states: operating or broken. Broken machines are repaired and come back into operation, and vice versa. On a given day:

- 85% of machines that are operational stay operating
- 15% of machines that are operating break down
- 5% of machines that are broken are repaired and start operating again
- 95% of machines that are broken stay broken.

Construct a transition matrix to describe this situation. Use the columns to define the situation at the 'Start' of the day and the rows to describe the situation at the 'End' of the day.

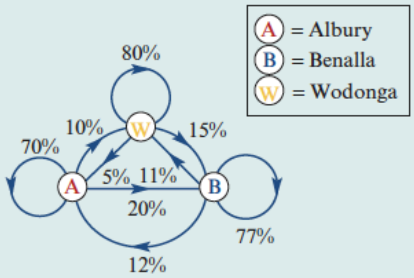
Ex 11a

Ex 11b – Interpreting Transition Matrices

Example

The following transition matrix, T, and its transition diagram can be used to describe the weekly pattern of rental car returns in three locations: Albury, Wodonga and Benalla.

$$T = \begin{array}{c} \begin{matrix} A & W & B \end{matrix} \\ \begin{bmatrix} 0.7 & 0.05 & 0.12 \\ 0.1 & 0.8 & 0.11 \\ 0.2 & 0.15 & 0.77 \end{bmatrix} \end{array} \begin{array}{l} \\ A \\ W \\ B \end{array} \text{ Returned to}$$



Use the transition matrix T and its transition diagram to answer the following questions.

a What percentage of cars rented in Wodonga each week are predicted to be returned to:

i Albury? **ii** Benalla? **iii** Wodonga?

b Two hundred cars were rented in Albury this week. How many of these cars do we expect to be returned to:

i Albury? **ii** Benalla? **iii** Wodonga?

c What percentage of cars rented in Benalla each week are *not* expected to be returned to Benalla?

d One hundred and sixty cars were rented in Albury this week. How many of these cars are expected to be returned to either Benalla or Wodonga?

Solution

Ex 11b

Ex 11c – Transition Matrices Using Recursion

Recurrence Relation

Recurrence relation

$$S_0 = \text{intial value}, \quad S_{n+1} = TS_n$$

Example – Using a Recurrence Relation with Transition Matrices

The factory has a large number of machines. The machines can be in one of two states: operating (O) or broken (B). Broken machines are repaired and come back into operation and vice versa.
At the start, 80 machines are operating and 20 are broken.
Use the recursion relation

$$S_0 = \text{intial value}, \; S_{n+1} = TS_n$$

where

$$S_0 = \begin{bmatrix} 80 \\ 20 \end{bmatrix} \text{ and } T = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix}$$

to determine the number of operational and broken machines after 1 day and after 3 days.

Harder examples……(Working Backwards)

We have a transition matrix $T = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$

and we know that the state matrix $S_4 = \begin{bmatrix} 25\,587 \\ 34\,413 \end{bmatrix}$.

Determine S_3 and S_2.

Rule for finding the state matrix after n steps

$$S_n = T^n S_0$$

Example

The factory has a large number of machines. The machines can be in one of two states: operating (O) or broken (B). Broken machines are repaired and come back into operation and vice versa.

Initially, 80 machines are operating and 20 are broken, so:

$$S_0 = \begin{bmatrix} 80 \\ 20 \end{bmatrix} \text{ and } T = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix}$$

Determine the number of operational and broken machines after 10 days.

Steady State

- As the matrix powers of T get higher and higher the values of the resultant matrix do not change
- When there is no noticeable change from one state matrix to the next the system is said to have reached its **steady state** or **long term state.**

To calculate the steady state....

$$\boxed{S_\infty = T^{100} S_0}$$

- To prove a system has reached the long term state you need to do 2 calculations

$S_{100} = T^{100} S_0$

$S_{101} = T^{101} S_0$

Example

For the car rental problem:

$$S_0 = \begin{bmatrix} 50 \\ 40 \end{bmatrix} \text{ and } T = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$$

Estimate the steady-state solution by calculating S_n for $n = 10, 15, 17$ and 18.

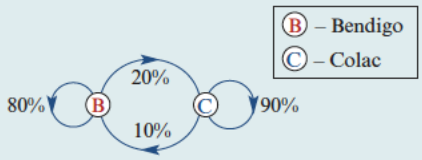
Ex 11c
Ex 11d - Transition Matrices – Using the Rule (Culling and Restocking)

- Must be done recursively

Example

A rental starts with 90 cars, 50 located at Bendigo and 40 located at Colac.

Cars are usually rented and returned in the same town. However, a small percentage of cars rented in Bendigo are returned in Colac and vice versa. The transition diagram opposite gives these percentages.



To increase the number of cars, two extra cars are added to the rental fleet at each location each week. The recurrence relation that can be used to model this situation is:

$$S_0 = \begin{bmatrix} 50 \\ 40 \end{bmatrix}, \; S_{n+1} = TS_n + B \quad \text{where} \quad T = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Determine the number of cars at Bendigo and Colac after:

a 1 week **b** 2 weeks.

Ex 11d

Ex 11e – Leslie Matrices

- Leslie matrices are a special kind of transition matrix that can be used to describe the way a population changes over time.
- They are used to model changes in the sizes of different age groups within a population

Special parts of a Leslie Matrix

Birth rates

- We ignore migration, and so the population growth is entirely due to new female births
- The birth rate b_i for age group i is the average number of female offspring from a mother in age group i during one time period

Survival Rates

- The survival rate s_i for age group i is the proportion of the populatin in age group i that progress to age group $i + 1$
- Note: $0 \leq s_i \leq 1$

Summary

Leslie matrices

An $m \times m$ **Leslie matrix** has the form

$$L = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_{m-1} & b_m \\ s_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & s_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & s_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & s_{m-1} & 0 \end{bmatrix}$$

where:

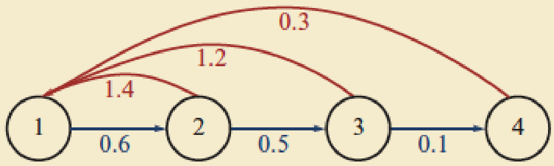
- m is the number of age groups being considered
- s_i, the survival rate, is the proportion of the population in age group i that progress to age group $i + 1$
- b_i, the birth rate, is the average number of female offspring from a mother in age group i during one time period.

Leslie matrix and its interpretation

From age group

$$L = \begin{array}{c} \begin{matrix} 1 & \quad 2 & \quad 3 & \quad 4 \end{matrix} \\ \begin{bmatrix} 0 & 1.4 & 1.2 & 0.3 \\ 0.6 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \quad \text{To age group}$$

This is a Leslie matrix with 4 age groups. The corresponding life-cycle transition diagram is shown here.



Recursive rules

The population matrix S_n is an $m \times 1$ matrix representing the size of each age group after n time periods. This is calculated using a recursive formula

S_0 is the initial state matrix, $S_{n+1} = LS_n$

or the explicit rule

$S_n = L^n S_0$

Example

Use the Leslie matrix and initial state matrix below to answer the following questions.

$$L = \begin{array}{c} \text{From age group} \\ \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1.8 & 2.6 & 0.1 \\ 0.2 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \text{ To age group} \qquad S_0 = \begin{bmatrix} 1000 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

a Write down

i the birth rate for age group 2 **ii** the survival rate for age group 3

b Complete a life cycle diagram for this Leslie matrix.

c Evaluate the following population state matrices.

S_1, S_5 and S_{20}

d Given that $S_{16} = \begin{bmatrix} 9.53 \\ 2.42 \\ 1.22 \\ 0.16 \end{bmatrix}$, determine S_{17}

Example

Information about a population of female goats is given in the following table.

Age group (years)	0 – 1	1 – 2	2 – 3	3 – 4	4 – 5
Initial population	10	25	40	20	15
Birth rates	0	0.2	0.9	0	0
Survival rates	0.6	0.7	0.5	0.2	0

a Write down the initial population state matrix, S_0.

b Using the information in the table above, write down a Leslie matrix to describe the change in population of female goats.

c Construct a life cycle transition diagram for this Leslie matrix.

d Determine the number of 3 – 4 year old female goats in the population after 3 years. Round your answer to the nearest whole number.

Long Term (limiting) behaviour of population numbers

Example

Consider the following Leslie matrix L and initial population matrix S_0:

$$L = \begin{bmatrix} 0 & 4 & 4 \\ 0.25 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \quad \text{and} \quad S_0 = \begin{bmatrix} 1000 \\ 0 \\ 0 \end{bmatrix}$$

a Find

i S_5 **ii** S_{10} **iii** S_{50}

Premultiply each of these state matrices by $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ to calculate the total populations at each of these stages and comment.

b Determine S_{25} and S_{26}. Divide each age group population for S_{26} by the corresponding age group population for S_{25} and show that $S_{26} \approx 1.1915 S_{25}$ and comment.

Limiting behaviour of a Leslie Matrix

- Often we will find that after a long enough time, the proportion of the population in each age group does not change from one time period to the next.
- This happens if we can find a real number k such that $LS_n = kS_n$ fro some sufficiently n
- This does not happen with every matrix

Example

Consider the following Leslie matrix L and initial population matrix S_0:

$$L = \begin{bmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix} \quad \text{and} \quad S_0 = \begin{bmatrix} 1600 \\ 800 \\ 200 \end{bmatrix}.$$

a Determine

i S_{10} **ii** S_{14} **iii** S_{15}

b The rate of increase of the population is a constant and each of the age group populations increase in the same way. Find this rate by comparing S_{14} and S_{15}.

c Confirm the ratio of the age group populations stays constant for S_0, S_1 and S_{10} at $8 : 4 : 1$

Example

Consider the following Leslie matrix L and initial population matrix S_0:

$$L = \begin{bmatrix} 0 & 0 & b_3 \\ 0.25 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \quad \text{and} \quad S_0 = \begin{bmatrix} 1000 \\ 0 \\ 0 \end{bmatrix}$$

Investigate the long-term behaviour of the population if:

a $b_3 = 8$ **b** $b_3 = 4$ **c** $b_3 = 10$