Aleph-4 Problem Solving Test
Do not open the test until instructed to do so.

Information:
- The Problem-Solving Round contains 10 questions that you will be given 40 minutes to answer.
- There will be an 11th estimation question that will be used as a tiebreaker. There is no penalty for making an educated guess.
- Do not expect to be able to solve every question. Likewise do not be afraid to skip questions. The questions are only roughly but not strictly in order of difficulty.
- Please write legibly and in the specified form; all answers that cannot be read or are in the incorrect form will be considered wrong.

Instructions:
- You will receive one answer sheet.
- On the answer sheet, make sure to LEGIBLY write your name, Competition ID, and the school that you attend.
- Only answers on the answer sheet will be graded, not answers in the test booklet.
- Only basic writing implements (i.e. pencils, pens, erasers) are allowed. All other tools (e.g. calculators, compasses, rulers, etc…) or external help is prohibited.

Good luck and have fun! If you have any questions, raise your hand.

1. $\frac{cosx - sinx}{cos(2x)} = 1$. Let the sum of all real solutions for $0 \le x \le 2\pi$ be $n\pi$. What is n?

2. James is playing a game with nine cards labeled 1 to 9. He wants to arrange eight of them in a line such that adjacent cards differ by at least 5. What is the largest possible card that can be in the 3rd position?

3. Regular pentagon ABCDE and regular hexagon AFGHIJ, both labeled clockwise, are inscribed in a unit circle. Given that A is a vertex of both shapes, what is the fraction of the minor arc lengths of DH over BF?

4. Bah-Lah-Kay writes all of the 4-digit palindromes in base 11. His twin Kai-Den wants to circle all of the numbers divisible by 10. For example, Bah-Lah-Kay writes $2332_{11}=$ 3060, which is divisible by 10, so Kai-Den circles this number. The fraction of the numbers that Kai-Den circles is $\frac{a}{b}$ such that a and b are coprime, positive integers. What is $a + b$?

5. Friat is waiting for a 24-hour digital clock to display a palindromic time like 01:10 or 14:41 where the FOUR digits of the time are the same backwards and forwards. Assuming the time is currently palindromic, let $\frac{m}{n}$ be the number of minutes should Friat expect to wait for the next palindromic time, given that m and n are coprime. *Note that the leading digit must be included even if it is a zero, so 01:01 would not be palindromic.*

6. There is a circular right cylinder of height 2 and radius 1. A sphere is placed at the center of the cylinder such that half of the cylinder's surface area is inside the sphere. The sphere's radius can be written as $\sqrt{\frac{a}{b}} - \sqrt{c}$, where a, b, and c are positive integers. What is the value of $a + b + c$?

7. Quazil has five candles that each take 1 hour to burn completely. He starts a stopwatch at the same time as lighting the first candle. While the first candle is burning, he randomly chooses to light the second candle. This process repeats for each of the subsequent candles. After the fifth candle finishes burning, he stops the stopwatch. Let the probability that the stopwatch reads more than 4 hours be a fraction $\frac{a}{b}$ such that a and b are coprime, positive integers. What is $a + b$?

8. There is an infinite series of rooms numbered with the nonnegative integers, each equipped with a light. For a given room k, the light has a probability 3^{-pnk} of staying on for n minutes and then turning off (such that n is a positive integer). Lisa picks a random positive integer p such that the probability of picking a given value of p is 2^{-p}. Rihbi goes into a random room and turns on the light. The light stays on for four minutes, but Rihbi gets tired of waiting and leaves the room. Let $\frac{a}{b}$ be the probability that Rihbi was in the room numbered "0" such that a and b are coprime, positive integers. What is $a + b$?

9. A blind ant is on the surface of a right circular cone of radius 5 and height $5\sqrt{3}$. Point A is on the edge of the circular base, and point X is located opposite point A such that AX is a diameter of the base. Point O is the vertex of the cone, and point B is located on XO such that BX = 2. Starting from point A, the ant chooses a random direction (in a straight line on the cone's 2D net) to walk 15 units along the surface of the cone except that the ant cannot walk on the base of the cone. Let the probability that the ant travels within 1 unit of B (along the cone's surface) equal $\frac{2}{\pi} arcsin(\frac{\sqrt{a}}{b})$ such that a is square-free and that a and b are positive integers. What is $a + b$?

10. The Pell numbers are defined: $P_{n+1} = 2P_n + P_{n-1}$ for $n \in \mathbb{Z}$ such that $P_0 = 0$, $P_1 = 1$. Silyna begins writing the Pell numbers with leading zeros to make each number three digits long (e.g. 000, 001, 002, 005, etc.). She concatenates these values after a decimal point to form: $0.000001002005012029070169408...$ Let (at least) the first 27 digits of the decimal expansion of $\frac{a}{b}$ be equal to this decimal expansion such that a and b are positive integers. Find the smallest possible value of $a + b$.

11. **TIEBREAKER (Estimation):** Your answer will only be used in the event of a tie. Note that an exact answer may not be practical without computer aid, so you are expected to provide an <u>estimation</u>. Mark draws a circle of radius 1 with center A_1. He then draws another circle that passes through A_1 and is tangent to the original circle. He repeats this process: drawing a circle with center A_{k+1} that passes through center A_k and is tangent to the circle with center A_k. What is the expected distance between A_1 and A_n as n approaches infinity?