AP Calc BC Unit 9: Parametric Equations, Polar Coordinates, & Vector-Valued Functions

In math, there are many different kinds of functions, because not everything in the world exists on a plane with two variables. So far, everything we have been doing has been on the **Cartesian** plane: $\mathbb{R}^2$. This unit introduces you to different kinds of functions that model the world around us.

- Parametric Equations, Polar Coordinates, and Vector-Valued Functions
  - Parametric Functions
    - Derivatives of Parametric Functions
      - First Derivative of a Parametric Function
      - Second Derivative of a Parametric Function
    - Parametrics and Motion
      - Parametric Motion Expressed through Vector-Valued Functions
    - Position Vector:
      - Displacement of Parametric Functions
      - Arc Length/Distance of Parametric Functions
  - Polar Functions
      - Polar Conversions
    - Derivatives of Polar Functions
      - Slope of Tangent Line for Polar Functions
    - Area Under the Curve
      - Simple Area Under Polar Curves
    - Area Between Two Curves
    - If the curves intersect, then you may have to find the area inside the curves by splitting the region.
    - Polar Arc Length
      - Arc Length for Polar Functions

9.0 Unit 9 Overview: Parametric Equations, Polar Coordinates, and Vector-Valued Functions

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What To Expect 🤓

Parametric Functions

- Derivatives of Parametric Functions - First Derivative & Second Derivative of a Parametric Function
- Parametrics and Motion - Parametric Motion Expressed through Vector-Valued Functions
- Position Vector - Displacement of Parametric Functions & Arc Length/Distance of Parametric Functions

Polar Functions

- Polar Conversions
- Derivatives of Polar Functions - Slope of Tangent Line for Polar Functions
- Area Under the Curve - Simple Area Under Polar Curves
- Area Between Two Curves If the curves intersect, then you may have to find the area inside the curves by splitting the region.
- Polar Arc Length - Arc Length for Polar Functions

9.1 Defining and Differentiating Parametric Equations

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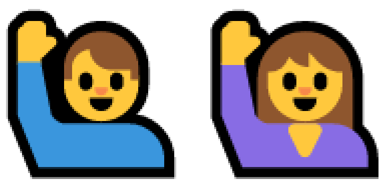

What is a Parametric Function?

Parametric functions are a set of related functions where x and y are independent from each other, but they are connected using the dummy variable t, which represents time. When we use the cartesian graph, we assume that we are moving along the x-axis in only one direction at a constant rate. However, parametric equations give us more freedom to manipulate horizontal motion.

A parametric equation would look something like this:
x(t) = t^2 - 1, y(t) = 3t

In this equation, your x-coordinate would be determined by $t^2 - 1$ and your y-coordinate would be determined by $3t$. So, when $t = 1$, you would plot the point $(0,\ 3)$. In a parametric function, t isn't actually on the graph; we just use t as our constant so that our points are independent from one another.

Derivatives of Parametric Functions

Like we discussed earlier, a parametric function is still graphed in 2D on an xy-plane, so if we wanted to find the slope of the tangent line, we would still need to find dy/dx

If we divide dx/dt and dy/dt, then dt will cancel out.

(dy/dt)/(dx/dt) = dy/dx

First Derivative of a Parametric Function
Given any parametric function defined by $x(t)$ and $y(t)$, $\frac{dy}{dx}$ is valued at $\frac{dy/dt}{dx/dt}$

Example: Find the slope of the tangent line of $x(t) = t^2 - 2t,\ y(t) = t^2 + 1$ at $t = 3$

$$\frac{dx}{dt} = 2t - 2$$

$$\frac{dy}{dt} = 2t$$

$$\frac{dy}{dx} = \frac{2t-2}{2t} = \frac{t-1}{t}$$

Plugging in $x = 3$,

$$\frac{dy}{dx} = \frac{2}{3}$$

The slope of the tangent line is ⅔

9.2 Second Derivatives of Parametric Equations

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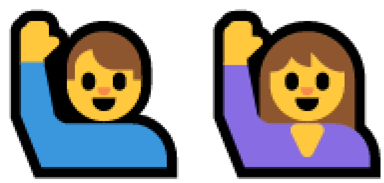


If we wanted to find the **second derivative** of a parametric function **d^2y/dx^2**, we would simply use the chain rule:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy/dt}{dx/dt}\right) = \frac{d}{dt}\frac{dt}{dx}\left(\frac{dy/dt}{dx/dt}\right) = \frac{\frac{d}{dt}\left(\frac{dy/dt}{dx/dt}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Second Derivative of a Parametric Function

Given any parametric function defined by $x(t)$ and $y(t)$, $\frac{d^2y}{dx^2}$ is valued at

$$\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Example: Show that the cycloid defined by $x(t) = 2(t - \sin t)$ and $y(t) = 2(1 - \cos t)$ is concave down on $t \in (0, 2\pi)$

💡 **Remember:** Concavity is determined by the second derivative

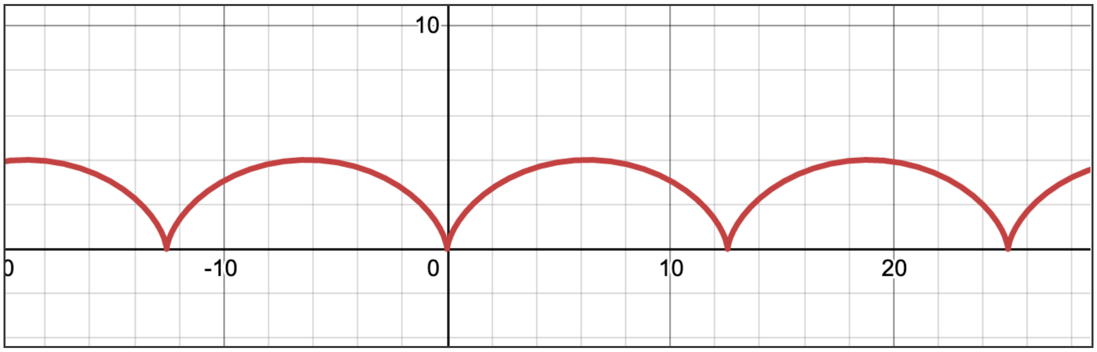
Finding $\frac{d^2y}{dx^2}$:

$$\frac{dy}{dx} = \frac{2(sin(t))}{2(1-cos(t))} = \frac{sin(t)}{1-cos(t)}$$

$$\frac{d^2y}{dx^2} = \frac{cos(t)(1-cos(t))-(sin(t))(sin(t))/(1-cos(t))^2}{2(1-cos(t))} =$$

$$\frac{cos(t)-cos^2(t)-sin^2(t)}{2(1-cos(t))^3} = \frac{cos(t)-1}{2(1-cos(t))^3} = \frac{-1}{2(1-cos(t))^2}$$

Since $\frac{d^2y}{dx^2}$ is always negative, the cycloid is always concave down



9.3 Finding Arc Lengths of Curves Given by Parametric Equations

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Arc Length/Distance of Parametric Functions

Given any parametric function defined by $x(t)$ and $y(t)$,

$$L = \int_{\alpha}^{\beta} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

Example: Find the distance traveled for $x(t) = \cos t, y(t) = \sin t$ for $t \in [0, 4\pi]$

$$D = \int_0^{4\pi} \sqrt{cos^2(t) + sin^2(t)}\, dt = \int_0^{4\pi} 1dt = 4\pi$$

Note that the unit circle is one of the rare cases in which the arc length does not equal the distance. Since this equation traces over the same line twice, the arc length would be 2π while the distance would be 4π. However, for the purposes of the AP Calc exam, you should always use the above formula to calculate arc length and distance.

9.4 Defining and Differentiating Vector-Valued Functions

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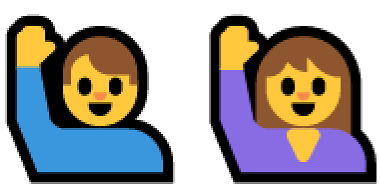

Parametrics and Motion

Because parametric functions are associated with time, they are also generally used to calculate motion and velocity, and the College Board usually uses parametrics in this context.

When we deal with parametrics in the context of motion, we express them as **vector-valued functions**. Vector-valued functions aren't graphed with the points x and y like we are used to seeing. Instead, each "point" on a vector-valued function is determined by a **position vector** (a vector that starts at the origin) that exists in the direction of the point.

For example, if the parametric function $x(t) = t^2 - 1,\ y(t) = 3t$ was written as a vector-valued function, it would be $\langle t^2 - 1,\ 3t\rangle$. This means that for any value t on the function, there is a vector that starts at the origin and goes $t^2 - 1$ units in the direction of the x-axis and $3t$ units in the direction of the y-axis. If we connect the ends of all of those vectors, we will get the same curve as we would get if we graphed the parametric equation $x(t) = t^2 - 1,\ y(t) = 3t$.

Just like Cartesian functions, if we take the derivative of the position vector, we would get the velocity vector, and if we take the derivative of the velocity vector, we would get the acceleration vector. When we were taking the derivative of a parametric function to find dy/dx, we were trying to find the slope of the tangent line that was determined by both the x and y functions of the curve. However, when we are looking at vector-valued functions, we aren't looking at the curve itself; we are looking at **how much our particle is moving in the direction of x and how much it is moving in the direction of y.** This means that when we are taking derivatives of vector-valued functions, we take the derivative of the components separately.

Parametric Motion Expressed through Vector-Valued Functions		
Position Vector: $\langle x(t),\ y(t)\rangle$	Velocity Vector: $\langle x'(t),\ y'(t)\rangle$	Acceleration Vector: $\langle x''(t),\ y''(t)\rangle$

9.5 Integrating Vector-Valued Functions

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Similarly, we can take integrals of vector-valued functions simply by taking the integrals of the individual x- and y-components. We use integrals either to backwards along the position-velocity-acceleration chain or to calculate **displacement**.

Example problem in 5.6 Solving Motion Problems Using Parametric and Vector-Valued Functions!

Displacement of Parametric Functions
Given any parametric function defined by $x(t)$ and $y(t)$, Displacement = $\int_{\alpha}^{\beta} v(t)$

9.6 Solving Motion Problems Using Parametric and Vector-Valued Functions

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Example: Given that an object in motion has $v(t) = \langle t^2 + 2, t^3 - 1\rangle$ and the initial position $\langle -1, 3\rangle$, find the position at $t = 2$

$$\int \langle t^2 + 2, t^3 - 1\rangle dt = \langle \frac{t^3}{3} + 2t + C, \frac{t^4}{4} - t + C\rangle$$

Since the position is $\langle -1, 3\rangle$ when $t = 0$,

The position vector is $\langle \frac{t^3}{3} + 2t - 1, \frac{t^4}{4} - t + 3\rangle$

Substituting $t = 2$,

$\langle x(2), y(2)\rangle = \langle \frac{17}{3}, 5\rangle$

Remember from previous units that if we take the integral of the speed (the absolute value of velocity), we can find the distance traveled (imagine adding up all of the tiny instantaneous distances to find a total distance).

This same concept applies in parametric equations, but since velocity is expressed as a vector, we need to take the integral of the **magnitude** of velocity. (In vector-valued functions, the magnitude is equivalent to the distance formula, which is essentially taking the absolute value of the vector.)

9.7 Defining Polar Coordinates and Differentiating in Polar Form

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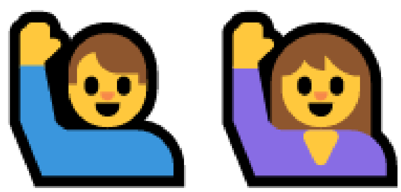


🎥**Watch: AP Calculus BC - Polar Coordinates and Calculus** (for teachers)

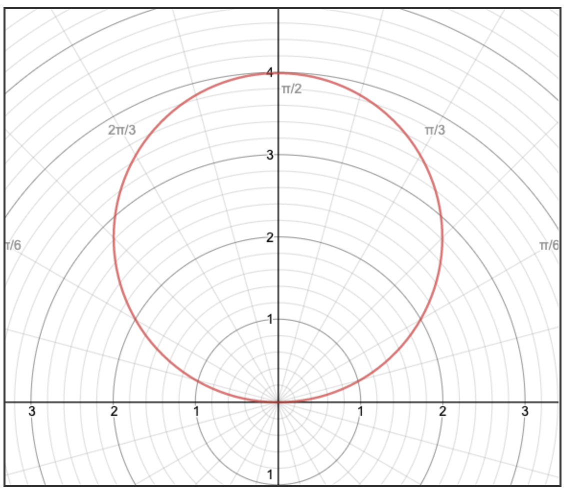
Polar functions are functions that are graphed around a pole in a circular system rather than the Cartesian rectangular system. Polar functions are graphed with the points (r, θ) rather than (x, y).

When we are working with polar graphs, we can't differentiate them right away. We have to convert them to Cartesian graphs. Converting polar equations to Cartesian also helps us visualize them.

	Polar Conversions	
$x = r\cos\theta$	$y = r\sin\theta$	$r = \sqrt{x^2 + y^2}$

Example: Convert $r = 4sin\theta$ to a Cartesian function

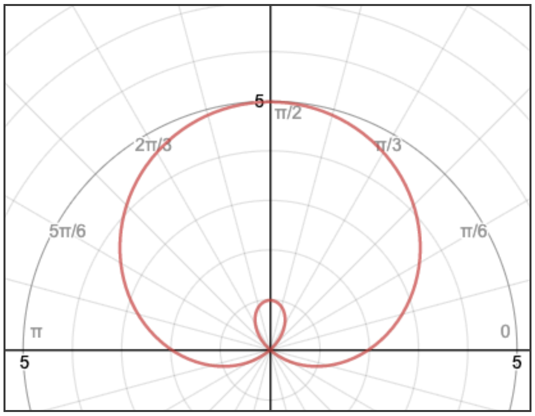
$$sin\theta = y/r$$
$$r = 4(y/r)$$
$$r^2 = 4y$$
$$x^2 + y^2 = 4y$$
$$x^2 + (y - 2)^2 = 4$$



Example: Find the values of θ on $r = 2 + 3sin\theta$ where $x = 2$

$$rcos\theta = 2$$
$$rcos\theta = cos\theta(2 + 3sin\theta) = 2$$

Plugging this into your calculator,
$\theta = 0,\ 1.133$

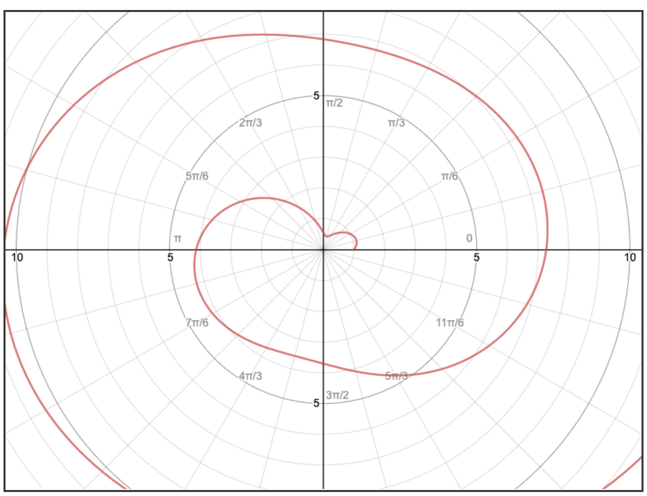


Derivatives of Polar Functions

When we take **derivatives of polar functions**, we can take them as **dr/dθ**, which would give us the points that are furthest away from the origin on the polar coordinate system. We find dr/dθ in the same way we would find any normal derivative: by taking the derivative of the polar function:

Example: Find the points closest and furthest from the origin for

$r = \theta + cos2\theta, \ \in [0, \pi\}$



$$\frac{dr}{d\theta} = 1 - 2sin2\theta = 0$$

$$sin2\theta = \frac{1}{2}$$

$$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}$$

$$r\left(\frac{\pi}{12}\right) = 1.128$$
$$r\left(\frac{5\pi}{12}\right) = 0.443$$

Checking the endpoints:

$r(0) = 1$

$r(\pi) = \pi$

The point closest to the origin is 0.443, and the point furthest from the origin is π.

While dr/dθ can tell us relative maximum and minimum values, it doesn't tell us the slope of the tangent line, since we can't have linear graphs on the polar coordinate system. In order to find the slope of the tangent line, we need to find the derivative on the Cartesian system, which requires us to calculate dy/dx.

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} = \frac{r\cos\theta + \frac{dr}{d\theta}(\sin\theta)}{-r\sin\theta + \frac{dr}{d\theta}(\cos\theta)}$$

Slope of Tangent Line for Polar Functions

$$\frac{dy}{dx} = \frac{r\cos\theta + \frac{dr}{d\theta}\sin\theta}{-r\sin\theta + \cos\theta\frac{dr}{d\theta}}$$

Of course, you can memorize this formula, but most students find it much easier to simply derive it using the chain rule.

Example: Find the equation of the line tangent to the polar curve $r = \theta + \cos 2\theta$at

$$\theta = \frac{\pi}{3}$$

$$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{\frac{d}{d\theta}(\theta sin\theta + sin\theta cos2\theta)}{\frac{d}{d\theta}(\theta cos\theta+cos\theta cos2\theta)}=$$

$$\frac{sin\theta+\theta cos\theta+cos\theta cos2\theta-2sin\theta sin2\theta}{cos\theta-\theta sin\theta-sin\theta cos2\theta+2cos\theta sin2\theta}$$

Substituting $\theta = \frac{\pi}{3}$

$$\frac{sin\frac{\pi}{3}+\frac{\pi}{3}cos\frac{\pi}{3}+cos\frac{\pi}{3}cos\frac{2\pi}{3}-2sin\frac{\pi}{3}sin\frac{2\pi}{3}}{cos\frac{\pi}{3}-\frac{\pi}{3}sin\frac{\pi}{3}-sin\frac{\pi}{3}cos\frac{2\pi}{3}+2cos\frac{\pi}{3}sin\frac{2\pi}{3}}= 0.429$$

Finding the x- and y-coordinates of the tangent line:

$$x = rcos\theta = (\frac{\pi}{3}+ cos\frac{2\pi}{3})(cos\frac{\pi}{3}) =.274$$

$$y = sin\theta = (\frac{\pi}{3}+ cos\frac{2\pi}{3})(sin\frac{\pi}{3}) =.474$$

The equation of the tangent line is:

$$y - 0.474 = 0.429(x - 0.274)$$

9.8 Find the Area of a Polar Region or the Area Bounded by a Single Polar Curve

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When we calculate the **area under the curve for Cartesian graphs**, we would integrate with rectangles, since it is a rectangular plane. However, when we find the area under the curve for polar functions, we need to add up the area of triangles (imagine cutting a pizza into really really thin slices).

Since the area of a triangle is calculated by (1/2)bh, where h = r and b = rdθ (the base would be proportional to the radius, multiplied by the tiny value d to obtain an infinitely tiny base).

Simple Area Under Polar Curves
$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$

The tricky part about calculating the area is finding the interval on which you want to integrate. Sometimes, they will give you the graph of the function, or you will be able to graph it on your calculator. However, for non-calculator sections, you might have to figure out the endpoints just with the function.

Here are a few general guidelines you can follow.

- If the equation has a sinθ or cosθ without any squares or coefficients, then the interval will always be from 0 to 2π, since it will go around a full circle once within that interval
- If the equation has a coefficient inside the trig function (sin4θ or cos2θ, etc), then it the function likely has multiple petals. You can make a chart in order to find where the function meets zero, and then just calculate the area of one petal and multiply it by the number of petals.
- If you can't figure out the boundaries, you can usually guess that they are 0 to 2π. However, you should only use this as a last resort.

Example: Find the area of the region bounded by $r = 2(1 - sin\theta)$

First, we figure out the boundaries of the region:

θ	r
0	2
$\frac{\pi}{2}$	0
π	2
$\frac{3\pi}{2}$	4
2π	0

Based on the chart, we can see that the function comes back to the pole once at $\frac{\pi}{2}$ and 2π, which means that it will go around once.

$$A = \int_0^{2\pi} \frac{1}{2}[2(1 - sin\theta)]^2 d\theta$$

$$= 2\int_{0}^{2\pi}(1 - 2sin\theta + sin^{2}\theta)d\theta$$

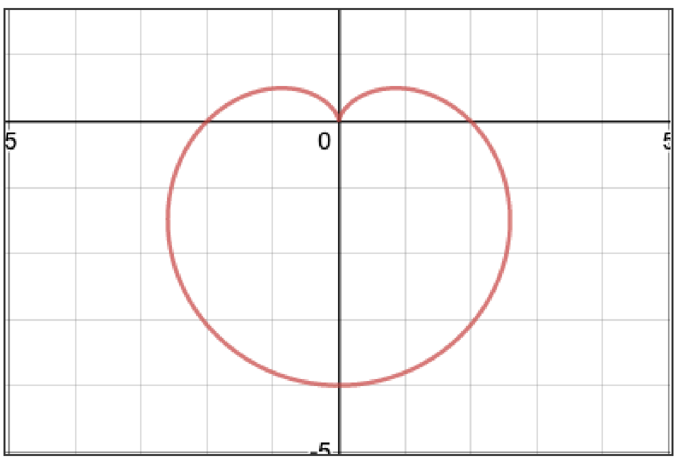
$$= 2\int_{0}^{2\pi}(1 - 2sin\theta + \frac{1-cos2\theta}{2})d\theta$$

$$= 2\int_{0}^{2\pi}(\frac{3}{2} - 2sin\theta - \frac{cos2\theta}{2})d\theta$$

$$= 2[\frac{3\theta}{2} + 2cos\theta - \frac{sin2\theta}{4}]$$

$$= 2[(3\pi + 2) - (2)]$$

$$= 6\pi$$



Example: Find the area of the region enclosed by the polar curve $r = sin4\theta$

First, since there is a coefficient inside of the sine function, we can assume that there will be petals to the function

We can figure out the length of one petal by making a chart:

θ	r	θ	r
0	0	$\frac{5\pi}{8}$	1
$\frac{\pi}{8}$	1	$\frac{3\pi}{4}$	0

$\frac{\pi}{4}$	0	$\frac{7\pi}{8}$	-1
$\frac{3\pi}{8}$	-1	π	0
$\frac{\pi}{2}$	0		

We can see that this pattern will continue; the graph will come back to the origin 8 times over $[0,\ 2\pi)$, so there are 8 petals

$$A = 8 \int_0^{\pi/4} \frac{1}{2}(sin 4\theta)^2 d\theta$$

$$= 4 \int_0^{\pi/4} (sin 4\theta) d\theta$$

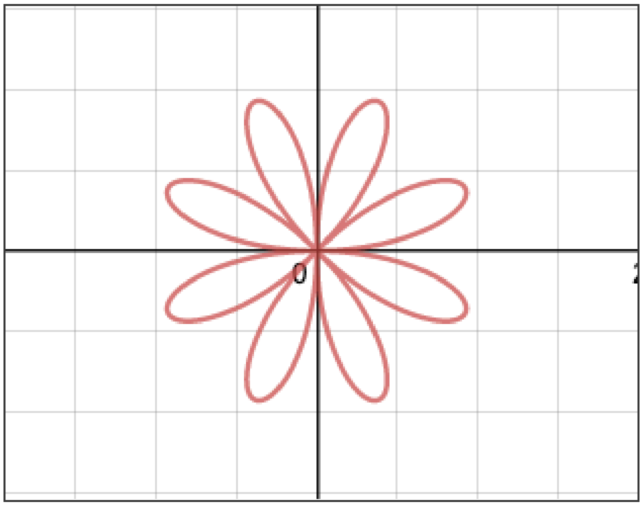
$$= 4 \int_0^{\pi/4} \frac{1}{2}(1 - cos 2(4\theta))\, d\theta$$

$$= 2 \int_0^{\pi/4} (1 - cos 8\theta) d\theta$$

$$= 2\left(\theta - \frac{sin 8\theta}{8}\right)$$

$$= 2\left(\frac{\pi}{4} - 0 - 0 + 0\right)$$

$$= \frac{\pi}{2}$$



9.9 Finding the Area of the Region Bounded by Two Polar Curves

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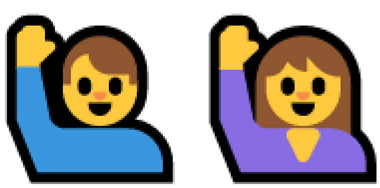


Once you get the hang of finding the area under one curve, **finding the area between two curves** is pretty simple. Remember from previous units that when you find the area between two curves, you subtract the bottom curve from the top curve. This is the same in polar functions, but instead of subtracting “top minus bottom,” you’ll subtract **“outer minus inner.”**

Example: Let R be the region bounded by $r = \theta$ and $r = \frac{1}{2}(\theta + \pi)$ on $\theta \in [0, \pi]$.

Find the area of R.

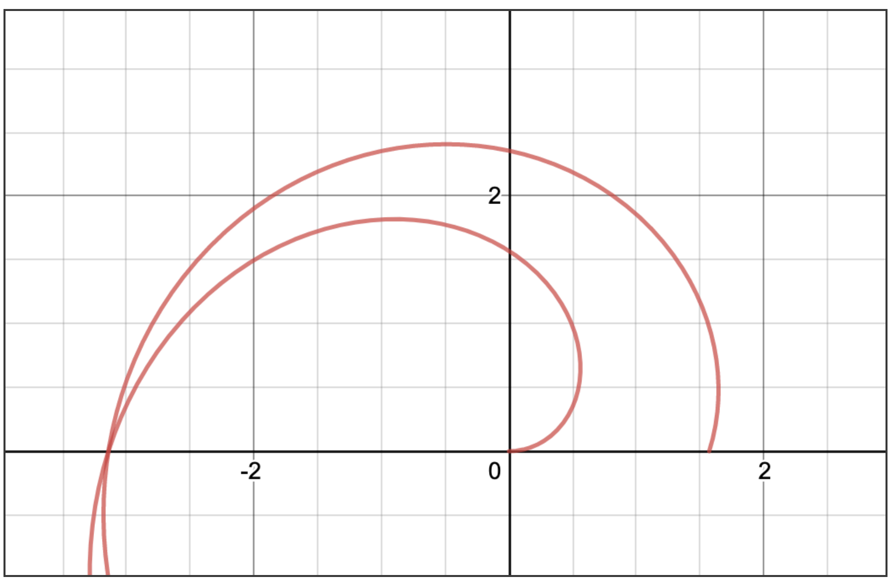
$$A = \frac{1}{2}\int_0^{\pi}[(\frac{1}{2}(\theta + \pi))^2 - (\theta)^2]d\theta$$

$$= \frac{1}{2}\int_0^{\pi}[(\frac{1}{4}(\theta^2 + 2\theta\pi + \pi^2) - \theta^2]d\theta$$

$$= \frac{1}{2}\int_0^{\pi}(-\frac{3\theta^2}{4} + \frac{\pi\theta}{2} + \frac{\pi^2}{4})d\theta$$

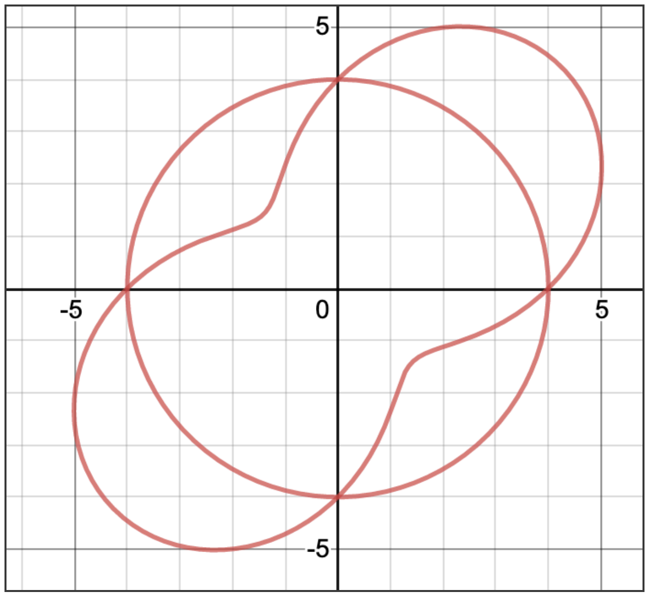
$$= \frac{1}{2}\left(-\frac{3}{4}\left(\frac{\theta^3}{3}\right) + \frac{\pi}{2}\left(\frac{\theta^2}{2}\right) + \frac{\pi^2\theta}{4}\right)$$

$$= \frac{\pi^3}{8}$$



If the curves intersect, then you may have to find the area inside the curves by splitting the region.

Example: Let R be the region inside the graph of the polar curve $r = 4$ and $r = 4 + 2sin2\theta$ on $0 \leq \theta \leq \pi$. Find the area of R.



First, we need to find the points of intersection:

$4 = 4 + 2sin2\theta$

$2sin2\theta = 0$

$sin2\theta = 0$

$2\theta = 0, \pi, 2\pi$

$\theta = 0, \frac{\pi}{2}, \pi$

Since the graphs intersect at θ = 2 we can see that when θ < π/2, r = 4 is on top, and when θ > π/2, r = 4+2sin(2θ) is on top. Based on this information, we can **construct two integrals:**

$$A = \int_{0}^{\pi/2} \frac{1}{2}(4 - 4 - 2sin2\theta)^2 d\theta + \int_{\pi/2}^{\pi} \frac{1}{2}(4 + 2sin2\theta - 4)^2 d\theta$$

$$= \frac{1}{2}\int_{0}^{\pi/2} (4sin^2 2\theta)d\theta + \frac{1}{2}\int_{\pi/2}^{\pi} (4sin^2 2\theta)d\theta$$

$$= \int_{0}^{\pi?2} (2sin^2 2\theta)d\theta + \int_{\pi/2}^{\pi} (2sin^2 2\theta)d\theta$$

$$= 18.7$$

Polar Arc Length

There is one last thing you need to know about polar functions: arc length. Finding arc length is pretty straightforward, but you do need to have the formula memorized for the exam.

Arc Length for Polar Functions
$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + (dr/d\theta)^2} d\theta$$

Example: Find the arc length of the polar curve $r = 5 + cos\theta$on $\theta \in [0, 2\pi]$.

$$L = \int_{0}^{2\pi} \sqrt{(5 + cos\theta)^2 + (-\ sin\theta)^2}\, d\theta$$

$$= \int_{0}^{2\pi} \sqrt{25 + 5cos\theta + cos^2\theta + sin^2\theta}\, d\theta$$

$$= \int_{0}^{2\pi} \sqrt{26 + 5cos\theta}\, d\theta$$

$$= 31.731$$