Fractal Geometry with Applications to Analysis, Number Theory, Mathematical Physics, PDE′s and Dynamical Systems

Scientific Program: Titles and Abstracts

Conference Information

Title: “Fractal Geometry with Applications to Analysis, Number Theory, Mathematical Physics, PDEs, and Dynamical Systems” in honor of Dr. Michel L. Lapidus’ upcoming 70th birthday**.**

Host: The University of California, Riverside

Organizers: Hafedh Herichi (*Santa Monica College*), Claire David (Sorbonne University, France), Machiel Van Frankenhuijsen (*Utah Valley University*), William Hoffer (*University of California, Riverside*)

Tuesday′s Talks

Morning

Marat Markin (California State University, Fresno)

Title: On Spectral Inclusion and Mapping Theorems and Asymptotics for C_0-Semigroups

Abstract: We establish spectral inclusion and mapping theorems for scalar type spectral operators, generalizing their counterparts for normal operators. Thereby, we extend a precise weak spectral mapping theorem along with the spectral bound equal growth bound condition and a generalized Lyapunov stability theorem, known to hold for C_0-semigroups of normal operators on complex Hilbert spaces, to the more general case of C_0-semigroups of scalar type spectral operators on complex Banach spaces. For such semigroups, we obtain exponential estimates with the best stability constants.

We also extend to a Banach space setting a celebrated characterization of uniform exponential stability for C_0-semigroups on complex Hilbert spaces and thereby acquire a characterization of uniform exponential stability for scalar type spectral and eventually norm-continuous C_0-semigroups.

The finer spectrum structure is given itemized consideration.

Nizar Riane (Rabat University, Morocco):

Title: “Membrane Deformation – Finite Elements – Parametric Shape Optimization on the Koch Snowflake”

Abstract: We introduce the finite element method to analyze a membrane with a Koch snowflake-shaped boundary. The fractal nature of this domain presents unique challenges due to its intricate boundary structure. Our approach involves discretizing the domain, estimating the error, and proving convergence. With these aspects addressed, we then solve a shape optimization problem to determine the optimal thickness of the membrane. These findings provide valuable insights into how fractal boundaries affect structural performance and optimization.

Marco Munoz (University of California, Riverside)

Title: The Sierpinski Gasket as a Fractal Variety.

Abstract: In the past decades my thesis advisor Dr. Michel Lapidus had conjectured in works such as In Search of the Riemann Zeros (2008) and Fractal Geometry, Complex Dimensions and Zeta Functions: Geometry and Spectra of Fractal Strings (2013), that fractal geometries characterized by their geometric and spectral zeta functions along with their poles which he called the complex dimensions of the fractal, could be realized as analogs of arithmetic varieties over finite fields. In this talk I wish to outline such a realization via the Sierpinski gasket and a synthesis of V. Nekrashevych's algebraic notion of self similar groups with J. Kigami's harmonic analysis on self similar fractals. In particular, I'll briefly comment on how certain algebraic invariants coming from such a fractal variety interpretation of the Sierpinski gasket naturally corresponds to invariants of the fractal cohomology of the Sierpinski gasket, the latter theory recently developed by Dr. Claire David and Dr. Lapidus and applied to fractals such as the Weierstrauss curve.

Steffen Winter (Karlsruhe University, Germany)

Title: Support Measures in Fractal Geometry

Abstract: The general Steiner formula of Hug, Last and Weil describes the tube volume of any closed set in R^d, and the support measures arising from this formula encode its geometric properties. However, it is not easy to extract fractal properties of a set directly from these measures. We introduce some new geometric functionals for compact subsets of R^d, called basic contents and support contents, which are tightly related to the support measures. We connect appropriate scaling exponents of these contents to the outer Minkowski dimension of the set and explain how its Minkowski content may be represented in terms of these contents providing a geometrically meaningful decomposition. We also comment on the connections to fractal curvatures. Based on joint work with Goran Radunovic.

Afternoon

Frederic Latremoliere (College of Natural Sciences and Mathematics, University of Denver, USA)

Title: The Space of Metric Spectral Triples

Abstract: Spectral triples have emerged as the preferred approach to discuss spectral noncommutative geometry. When the associated Connes' distance induces the weak* topology on the state space, such spectral triples are called metric. We will discuss some aspects of the geometry of the space of metric spectral triples when endowed with the spectral propinquity, which is an analogue of Gromov-Hausdorff distance in this noncommutative context. Applications include approximations of spectral triples on fractals, and the continuity of the spectra of Dirac operators for this metric.

Yat Tin Chow (University of California, Riverside)

Title: Fractality for a class of states of a C* algebra and noncommutative relative distance zeta functional

Abstract: In this work, we first recall the definition of the relative distance zeta function for arbitrary set in the Euclidean space and slightly generalize this notion from sets to probability measures, and then move on to propose a novel definition a relative distance (and tube) zeta functional for a class of states over a C* algebra. With such an extension, we look into the chance to define relative Minkowski dimensions in this context, and explore the notion of fractality for this class of states. Relative complex dimensions as poles of this newly proposed relative distance zeta functional, as well as its geometric and transformation properties, decomposition rules and properties that respect tensor products are discussed. We then explore some examples that possess fractal properties with this new zeta functional and functional equations in our context.

Shanna Dobson (University of California, Riverside, USA)

Title: Fractal Cohomology of the Cantor Set: Complex Dimensions and Cohomological Integers

Abstract: Motivated by the work of Claire David and Michel L. Lapidus on the extended theory of fractality and the fractal cohomology of the Weierstrass curve, we present joint work with Michel L. Lapidus on the development of the fractal cohomology of the ternary Cantor set. We prove that the Cantor function restricted to the Cantor set is discrete H\"older continuous and discrete reverse H\"older continuous with the same H\"older exponent D, where D = log2/log3 is the Minkowski dimension of the Cantor set. Furthermore, the H\"older exponent D is optimal for the H\"older continuity of the Cantor function restricted to the Cantor set (in the discrete sense).

Additionally, we obtain a complete characterization of the corresponding fractal cohomology groups, expressed in terms of fractal Taylor-like expansions involving the cohomological complex dimensions and cohomological integers. We also introduce the local and global cohomological zeta functions, whose poles coincide with the set of cohomological complex dimensions of the ternary Cantor set. Furthermore, we show that the ternary Cantor function belongs to each fractal cohomology group, thereby establishing the non-triviality of the local (or prefractal) and global (or total) fractal cohomology groups.

Adam Richardson (Santa Monica College, USA)

Title: Minkowski Measurability of a Generalized RFD Associated with a Class of N-Dimensional Space-Filling Curves

Abstract: In this talk, we calculate directly the volume formula for the tubular neighborhood of a generalized relative fractal drum (RFD) associated with the class of space-filling curves generated by regular rectilinear dissections of the unit N-cube. With this result, we then compute the N-dimensional relative Minkowski content of the RFD, establishing that this value is indeed equal to 1, which coincides with the N-dimensional Lebesgue measure of the unit N-cube.

Yimin Xiao (Department of Statistics and Probability, Michigan State University, USA)

Title: TBA

Abstract: TBA

Wednesday's Talks

Morning

David Carfi (University of Messina, Italy)

Title: Square-integrable Hausdorff-based measures and their products: examples on some fractals and Quantum applications

Abstract: In this work, we significantly enlarge the usual space of normalizable states of Quantum systems with m degrees of freedom. Specifically, we move in the (relativistic) Schrödinger setting of wave mechanics. In the chosen standard setting, usual quantum states belong to the Hilbert space $L^2(E;C)$, of complex Lebesgue square-integrable functions (equivalence classes) defined upon a convenient m-dimensional Euclidean space E. By our approach, we will extend the collection of possible E-based quantum states to a new class of measures, upon the Euclidean space E, that shall contain the classic Lebesgue-based function space L2, as well as the space of square integrable discrete measures and, more generally, all spaces $L^2_C(E; h_d)$ of complex functions that are square integrable with respect to the normalized Hausdorff measures h_d on E. We, then, construct some examples on classic fractals, we define a natural product between two square integrable measures and we show some applications to Quantum Mechanics.

Therese-Basa Landry (University of California, Santa Barbara, USA)

Title: Towards a Mathematical Theory of the Optimal Transport of Sediment

Abstract: We prove the existence of weak global weak solutions to PDEs describing the sediment flow in the evolution of fluvial land-surfaces, with constant water depth. A solution to this PDE determines a land surface. These PDEs describe the so-called transport-limited situation, where all the sediment can be transported away given enough water. This is in distinction to the detachment-limited situation where we must wait for rock to weather (to sediment) before it can be transported away. Earlier work shows that these PDEs describe the optimal transport of sediment and the evolution of the surfaces in optimal transport theory. Time permitting, we may discuss existing numerical techniques involving fractal interpolation due to Birnir and Cattan, as well as numerical schemes which efficiently approximate gradient flows for Wasserstein metrics. Some synthesis of the two methods could produce substantial improvements in computing time for solutions to our sediment flow PDE. We may also discuss ongoing work in the development of data science techniques to overcome instabilities in simulation and add smoothness to the surfaces after fractal interpolation.

Kasso A. Okoudjou (Tufts University, Department of Mathematics, USA)

Title: Short-time Fourier Transform and Gabor Frames on fractals

Abstract: We introduce a *short-time Fourier transform* on PCF fractals using the heat kernel corresponding to the fractal Laplacian as a window parametrized by a nonnegative time parameter t. We show that the corresponding continuous Gabor-like transform on $L^2(K)$, where K is a PCF fractal. Subsequently, we investigate the invertibility of the Gabor-like transform and present preliminary results on the associated Gabor-like frames on $L^2(K)$. This is a joint work with M. Buck, Luke Rogers, and A Teplyaev.

Stephane Jaffard (Universite Paris Est Creteil,Paris, France)

Title: Some interplays between Diophantine approximation and multifractal analysis.

Abstract: Multifractal analysis deals with the determination of the fractal dimensions of the singularity points of functions and stochastic processes. We will present several types of functions the pointwise Hölder exponent of which is governed by Diophantine approximation properties; these examples will include Haar or Davenport series, and trigonometric series related with the ``nondifferentiable Riemann function'' and which were recently shown to be present in turbulence modeling; we will also mention several open problems raised by generalizations of these functions.

Afternoon

Franklin Mendivil (Acadia University, Canada)

Title: The size of linear cut-out sets

Abstract: Michel Lapidus introduced the powerful tool of zeta functions to the study of the geometry of fractal sets, starting with fractal subsets of ${\mathbb R}$. The critical exponent of his \emph{geometric zeta function} turned out to be the (upper) Minkowski (or box) dimension of the set and the specific properties of the poles of this zeta function giving finer geometric information. All of these are ``global'' properties of the set.

In this talk we will survey some of what is known about the dimensional properties of compact cut-out sets in $\R$. This will include the Hausdorff, packing and box dimensions (which are all ``global'' dimensions) but will also include the Assouad and many Assouad-like ``local'' dimensions.

Adam Yassine (Pomona College, USA)

Title: Compactness of the density space in the Bures metric topology and the non-unital case

Abstract: Building off work of Farenick and Rahaman, we extend the definition of the density space and the Bures metric to the setting of non-unital C*-algebras equipped with a faithful tracial state and prove that the Bures metric is also a metric in this case. Furthermore, we prove a Heine-Borel type theorem for C*-algebras and the density space. In particular, we prove that for any C*-algebra (unital or non-unital) equipped with a faithful tracial state, the density space equipped with the Bures metric topology is not compact if and only if the C*-algebra is infinite dimensional. We also exhibit several examples of sequences that have no converging sequence in the unital and non-unital case including both commutative and noncommutative C*-algebras. This is joint work with Konrad Aguilar, Karina Behera, Katrine von Bornemann Hjelmborg, Tron Omland, Gregory Wickham, and Nicole Wu.

Michael Maroun (TeXDyn Industries, Austin, TX)

Title: Exact Solution of the Dirac Equation with Dirac Potential

Abstract: Exact solutions of the Dirac equation with singular Dirac delta potential in 1, 2, and 3 dimensions are obtained for the first time, since the 2 mathematical objects were first described almost 100 years ago. While point interaction results have been obtained for the Dirac equation in one dimension via the method of self-adjoint extensions (SAE) [Benvegnu and Dabrowski, 1994], there has not been a unifying theory with consistent results in all three dimensions, until this work.

Edward Voskanian (AGBU Manoogian-Demirdjian School)

Title: Diffraction Patterns from the Complex Dimensions of Nonlattice Self-Similar Fractal Strings.

Abstract: In recent joint work with Michel Lapidus and Machiel van Frankenhuijsen, we introduced a variation of the well-known diffraction framework by A. Hof to derive a formula for the diffraction measure of the complex dimensions of any lattice self-similar fractal string. We are now leveraging this diffraction formula for the lattice case to derive one for the nonlattice case, where the complex dimensions are quasiperiodically distributed and are approximated by those of the lattice case. Ultimately, we aim to better understand how the complex dimensions in the nonlattice case fit into the broader landscape of mathematical structures considered to be mathematical quasicrystals.

Jun Kigami (Kyoto University, Japan)

Title: Heat kernel lower bound estimates for pure jump processes via averaged jump kernels

Abstract: We derive a heat kernel lower bound estimates for symmetric pure jump processes on general volume doubling metric measure spaces with possible degenerate and/or singular jump kernels, where no off-diagonal estimate has been available, using averaged jump kernels. The main result of this paper is applied to derive a lower bound estimate for the transition density function of the trace of Brownian motions on Sierpinski gaskets on the bottom of the Sierpinski gasket.

Thursday’s Talks

Morning

Christophe Godin (ENS Lyon, INRIA, France)

Title: The Genetics of Fractals: how do plants make cauliflowers ?

Abstract: Fractals forms are ubiquitous in nature, clouds, lightenings, coastlines, snowflakes in the inorganic world, but also trees, ferns, lungs, and corals in the living world. One of the most iconic example of a biological fractal is certainly the cauliflower curd, whose fractal appearance culminates in the Romanesco cultivar. Thirty years ago, it was discovered that the mutation of a couple of genes in the model plant *Arabidopsis thaliana* (of the same plant family as the edible cauliflower), would result in the growth of small cauliflower-like curds instead of regular inflorescences. However, why and how such a mutation would produce a fractal-like structures has long remained a mystery.

Using a combination of biological, mathematical, computational approaches, we started to address this question with the group of F. Parcy, a geneticist of flower development living in Grenoble, France, 15 years ago. In this talk, I will present our findings and show that cauliflowers result from plant apices repeatedly trying, but failing, to produce lateral flowers. However, the transient passage in a floral state irreversibly modifies the fate of the primordia, and triggers a chain reaction that results in the curd fractal structure. Additional alterations of growth regulation may induce the production of conical structures reminiscent of the conspicuous fractal Romanesco shape. This study reveals how fractal-like forms may emerge from the combination of specific perturbations of floral developmental programs and growth dynamics.

Megha Pandey (Northwest University Xian, China)

Title of The Talk: Multifractal Analysis for Unstable Entropies

Abstract: We introduce a multifractal unstable entropy framework for partially hyperbolic dynamical systems. Specifically, we define the (μ,q,α)-Bowen unstable topological entropy and the (μ,q,α)-packing unstable topological entropy for subsets of such systems, extending the classical Bowen and packing unstable entropies by incorporating multifractal parameters. Motivated by the measurability theory of multifractal measures due to Olsen and the general multifractal formalism of Takens, we establish the basic structural properties of these entropic quantities and prove their measurability with respect to the σ-algebra generated by analytic sets.

We analyze their behavior on multifractal level sets and show that they coincide with the classical unstable entropies on appropriate subsets. Moreover, we prove that the domain of the (μ,q,α)-Bowen unstable topological entropy contains the domain of the multifractal spectrum associated with unstable local entropies. These results provide a natural multifractal extension of unstable entropy theory and yield new tools for the quantitative study of partially hyperbolic dynamics.

William Hoffer (University of California, Riverside)

Title: On Complex Dimensions and Heat Content of Self-Similar Fractals

Abstract: Complex fractal dimensions, defined as poles of appropriate fractal zeta functions, describe the geometric oscillations in fractal sets. In this work, we show that the same possible complex dimensions in the geometric setting also govern the asymptotics of the heat content on self-similar fractals. We illustrate our method in the case of generalized von Koch snowflakes and, in particular, extend known results for these fractals with arithmetic scaling ratios to the generic (in the topological sense), non-arithmetic setting.

Goran Radunovic (University of Zagreb, Croatia)

Title: Support measures, zeta functions, and complex dimensions

Abstract: The general Steiner formula of Hug, Last and Weil describes the tube volume of any closed set in R^d, and the support measures arising from this formula encode its geometric properties. Recently, basic contents and support contents have been introduced as tools to extract fractal properties of a set from these measures. The original motivation was to elucidate the geometric meaning of the coefficients in fractal tube formulas that arise in the theory of complex dimensions by Lapidus, Radunovic, and Zubrinic. This is achieved by introducing appropriate zeta functions associated to each support measure, which turn out to be useful tools for computing basic contents and support contents. Moreover, they allow to decompose fractal tube formulas into the contributions by different support measures, allowing a geometric interpretation of their terms.

Afternoon

Claire David (Sorbonne University, France)

Title: The Weierstrass Drift

Abstract: We revisit the use of the Weierstrass function as a deterministic driving term for the Schramm--Loewner evolution process (SLE), a setting first considered in earlier work by Joan Lind and Jessica Robbins in 2017.

Building on this idea, we use a refined deterministic toy model for SLE driven by a Weierstrass drift, based on explicit and sharp estimates for the Lip(delta) norm of the Weierstrass function.

Our approach yields improved quantitative bounds -- valid for all delta in]0,1/2] -- which significantly sharpen the classical Lip(1/2) estimates obtained by truncation methods by Lind and Robbins. In particular, we determine precise numerical bounds for the critical scaling parameter multiplying the driving term at which the Weierstrass-driven Loewner trace transitions from simple to non-simple curves.

We further compare the dynamics generated by prefractal and truncated Weierstrass drifts, showing that the former better captures the roughness, multifractal behaviour, and geometric complexity typically associated with Brownian-driven SLE. These results support the idea that carefully constructed Weierstrass drifts provide a robust and tractable deterministic analogue for studying multifractality and phase transitions in SLE.

Jade Leathrum (University of California, Riverside)

Title: Another One Bites the Dust

Abstract: We investigate modified Sierpiński Carpet fractals, constructed by dividing a square into a uniform nxn grid and removing a subset of the squares at each step. If enough squares are removed and in the proper places, we get “Dust Type” carpets, which have a path-connected complement and are themselves not path-connected. We present an analytical and combinatorial algorithm to compute the complex dimensions of every Sierpiński Carpet modification of Dust Type.

Matthew Overduin (Arkansas Tech University)

Title: Fractional Taylor Series Intepretation of the Fractal Explicit Formulas

Abstract: Dr. Lapidus and his collaborator M. van Frankenhuijsen, found a way of recovering a generalized fractal string from its complex dimensions. This became known as the explicit formula, and is found in their book titled, Fractal Geometry and Complex Dimensions. In this talk, I will present my joint work with Dr. Michel Lapidus, which involves re-expressing the (distributional) explicit formula as a fractional distributional Taylor series. The idea of taking a fractional derivative stem from Schwartz, in his book titled, Les Distributions. This allows us to generalize well-known formulas in the subject of calculus to much more general settings (involving distributions).

References:

[1] M. L. Lapidus, M. van Frankenhuijsen. *Fractal Geometry, Complex Dimensions and Zeta Functions: Geometry and Spectra of Fractal Strings*, Springer Monographs in Mathematics. Springer, New York, second revised and enlarged edition (of the 2006 edition), 2013.

[2] Laurent Schwartz. *Th’eorie Des Distributions*. Hermann, Paris, 1978.

Michel L. Lapidus (University of California, Riverside, USA)

Title: A Journey Through Complex Fractal Dimensions

Abstract: See the next page.

A Journey Through Complex Fractal Dimensions

Michel L. Lapidus

Distinguished Professor of the Graduate School and Research Professor.

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We will give a brief and impressionistic overview of the author's journey through the mathematical theory of complex fractal dimensions, which he has developed since the 1990s, jointly with a number of collaborators---including, especially, Carl Pomerance and Helmut Maier ([LapPom1-2] and [LapMai], respectively) (Phase 0: the prehistory), Machiel van Frankenhuijsen [Lap-vFr1-3] (phase 1), Goran Radunovic and Darko Zubrinic [LapRadZub1] (phase 2), and Claire David [DavLap1-9] (phase 3).

As was just alluded to just above, in addition to the prehistoric phase (to be further discussed later and focusing on the vibrations of fractal drums and the Weyl-Berry conjecture ([Lap1-2] and [KigLap1-2]), both for Laplacians and other elliptic operators on domains with fractal boundaries (fractal drums) [Lap1-2] and for Laplacians on fractals themselves ([KigLap1-2], [Lap3-4]), as well as on direct and inverse spectral problems for fractal strings along with their intimate connections with the Riemann zeta function [LapPom1-2] and the Riemann Hypothesis [LapMai]), the development of the theory of complex fractal dimensions can be naturally divided into three periods (or 'phases'):

> (i) Phase 1 (Complex Dimensions of Fractal Strings): The case of fractal strings (or one-dimensional drums with fractal boundaries) and their geometric zeta functions, with applications to number theory, fractal geometry and dynamical systems, via fractal explicit formulas and fractal tube formulas. See, e.g., the research monographs [Lap-vFr1-3], joint with Machiel van Frankenhuijsen.
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> (ii) Phase 2 (Complex Dimensions of Fractal Drums): The higher-dimensional case of the theory of complex dimensions, via (relative) fractal drums and their fractal zeta functions, with applications to mathematical physics, harmonic analysis, spectral geometry, fractal geometry, and dynamical systems). See, e.g., the research monograph [LapRad1], joint with Goran Radunovic and

Darko Zubrinic, along with about ten other papers with the same authors (including, e.g., [LapRadZub2-5]).

(iii) Phase 3 (Extended Theory of Fractal Dimensions): The extended theory of complex dimensions, with applications to highly complicated deterministic and random fractals, such as the deterministic nowhere differentiable but continuous Weierstrass and von Koch snowflake curves (see [DavLap1-9], joint with Claire David) and the random traces of the stochastic Schramm-Loewner evolution equation (SLE) [DavDupLap]. A major new development is a rigorous geometric formulation and computation of fractal cohomology theory---as initially conjectured to exist by the author in the mid-1990s and in his book, [Lap5], as well as in [Lap-vFr1-3]---and building, in particular, on [CobLap], joint with Tim Cobler, and on the author's forthcoming book---along with a fractal analogue of the classic Hodge theory in arithmetic, differential, and algebraic geometry. See, especially, [DavLap2], [DavLap6], and [DavLap7].

A new striking phenomenon is that the complex dimensions are constructed dynamically and that, starting at a certain stage of approximation (corresponding, physically, to a 'Euclidean/fractal phase transition'), new complex dimensions appear with each new prefractal graph approximation and that, therefore, the latter prefractals are themselves fractals, without having to take the usual mathematical limit. Naturally, the limiting geometric object is also fractal, and its set of complex dimensions is the increasing union of the complex dimensions of all the admissible prefractal approximations. In the case of the Weierstrass curve, the complex dimensions are calculated in three different ways: first, in [DavLap1] and [DavLap3], respectively, by using (local and global) Euclidean as well as polyhedral tube zeta functions; and then, in [DavLap2], by using the characterization of the prefractal cohomology groups obtained in that paper in terms of fractal (Taylor-like) expansions. Reassuringly, all three methods in [DavLap1-3] yield the same sets of local and global complex dimensions.

In short, the theory of complex fractal dimensions is a theory of oscillations (or vibrations) that ate intrinsic to the fractal geometry, the arithmetic geometry, the spectrum or the dynamics, of the underlying object. Intuitively, one can think of 'waves' propagating through the fractal (or rather, through the underlying 'space of scales'). Then, the amplitudes of these waves correspond to the real part of the complex dimensions, while their frequencies correspond to the imaginary parts of the complex dimensions. This heuristic interpretation is fully justified by the fractal explicit formulas and the fractal tube formulas obtained in the theory.

From this point of view, fractality can now be rigorously defined as the presence of nonreal complex dimensions—or, equivalently, by the presence of oscillations that are intrinsic to the underlying geometry, number theory, spectral geometry, or the dynamics. Accordingly, all number theories and arithmetic geometries are ‘fractal’, in this very general sense. So are most (and conjecturally, all) classic fractals, including the Devil’s staircase (or graph of the Cantor-Lebesgue function) which everyone would like to call “fractal”, whereas it is not fractal according to Mandelbrot’s definition of fractality.

Complex dimensions are defined as the poles of the meromorphic extension of the geometric zeta function (for a fractal string) or of suitable fractal zeta functions, the so-called distance and tube zeta functions (for an an arbitrary bounded set in m-dimensional Euclidean space or, more generally, for any higher-dimensional relative fractal drum). In the case of a fractal string, both the usual deterministic case [Lap-vFr1-3] and the random case [HamLap] are considered. Furthermore. Still for a fractal string, the distance zeta function of the associated relative fractal drum is very simply expressed in terms of the geometric zeta function, which enables us to recover the earlier theory of complex dimensions of fractal strings.

The aforementioned fractal explicit formulas [Lap-vFr, Chapter 5]—which significantly extend Riemann’s beautiful explicit formula for the prime number counting function--express a suitable geometric, spectral, or dynamical counting function in terms of the underlying complex dimensions---and, more generally, the geometric counting function of any generalized fractal string, in the sense of [Lap-vFr3, Chapter 4]---in terms of the underlying (visible) complex dimensions. Similarly, the fractal tube formulas (obtained in [Lap-VF1-2] and in [Lap-vFr3, Chapter 8] for fractal strings and in ([LapRadDar1, Chapter 5] and [LapRadDar5]) for higher-dimensional relative fractal drums) express the volume of small tubular neighborhoods as suitable power series (or generalized fractional Taylor series, see [LapOve]) with exponents the underlying complex codimensions.

The techniques used in the above works come from many areas of mathematics, including fractal geometry, number theory, real and complex analysis, partial differential equations, spectral theory, spectral geometry, mathematical physics, probability theory, functional analysis and operator theory, operator algebras and noncommutative geometry, geometric measure theory, p-adic analysis, algebraic topology, as well as arithmetic geometry, algebraic, and differential geometry.

A long-term goal of the theory of complex dimensions is to unify many aspects of mathematics (and physics), including fractal geometry and number theory, arithmetic geometries and other ‘exotic’ geometries, such as noncommutative geometry, ‘quantum geometries’, and tropical geometry. Especially in its phase 3 [DavLap1-9], the theory provides a new and very useful bridge between the discrete and the continuous realms, via the systematic use of prefractal graph approximations and of suitable geometric scales, exhibiting a natural phase transition between 'Euclidean' and 'fractal' geometries.

As was alluded to above, in this talk, we will give an impressionistic overview of the theory of complex fractal dimensions (phases 1-3 above), beginning with the prehistory of the theory ([Lap1-2, LapPom2-3], [LapMai]). For simplicity, the 'prehistoric phase' (phase 0) is viewed as consisting of three parts:

- The author's obtention in [Lap1] of a general Weyls' asymptotic formula with sharp error term for the spectra of fractal drums, thereby leading to a partial resolution of the Weyl-Berry conjecture and the formulation of a corresponding 'modified Weyl-Berry (MWB) conjecture'. Also, an analogue of Weyl's formula (obtained in [KigLap1-2]) for Laplacians on self-similar fractals (rather than on bounded open sets with fractal boundaries), thereby specifying and proving Berry's conjecture in this case and the author's related conjecture for self-similar fractal drums formulated in [Lap3]; see [KigLap1-2] and [Lap2-4], along with [ChrIvaLap], [LapSar] and [LanLapLat].
- The proof in [LapPom1] (jointly with Carl Pomerance) of the modified Weyl-Berry (MWB) conjecture (from [Lap1]) in the case of fractal strings (i.e., for drums with one-dimensional fractal boundaries) and the obtention in the process of a direct connection between direct spectral problems for fractal strings and the Riemann zeta function within the critical strip $0 < \mathrm{Re}(s) < 1$.

  (iii) Lastly, the geometric and spectral reformulation of the Riemann Hypothesis (obtained in [LapMai], jointly with Helmut Maier) via inverse spectral problems for fractal strings and the then mostly intuitive notion of complex dimensions.

  We note that the latter reformulation of the Riemann Hypothesis (RH)—viewed as a symmetric criterion for RH---has later been revisited and significantly extended to most arithmetic L-functions in [Lap-vFr1-3] within the rigorous theory of complex dimensions of fractal strings and fractal explicit formulas. See also the research monograph [HerLap1] (and its sequel in preparation, [HerLap2]), on 'quantized number theory', where a 'quantized' (or operator-valued) version of this criterion is provided, along with a very different 'asymmetric criterion for the Riemann Hypothesis' obtained by the author in [Lap6] in connection with the universality of the Riemann zeta function in the right critical strip $\frac{1}{2} < \mathrm{Re}(s) < 1$. (For a survey of the results in [LapMai], see, e.g., [Lap7].)

If time permits, we will very briefly discuss the author's theory of fractal membranes developed in the author's book (or essay) [Lap5] (entitled "In Search of the Riemann Zeros"), along with the associated moduli space within which is conjectured the existence of a suitable noncommutative flow of generalized quasicrystals, as well as of the associated partition functions (or zeta functions) and complex dimensions (zeros and poles). This 'Riemann flow' is aimed at understanding why the (Generalized or Extended) Riemann Hypothesis (ERH) is true, and at eventually proving it for all number-theoretic L-functions.

More specifically, conjecturally, this flow of generalized quasicrystals (or of fractal membranes) is pushing in antipodal pairs--- as time tends to infinity (or, equivalently, as the underlying absolute temperature tends to zero)--- the nonreal zeros of the spectral partition functions (or zeta functions) of fractal membranes onto the Equator of the Riemann sphere (which represents the critical line), thereby explaining why the Extended Riemann Hypothesis should be true for all arithmetic L-functions; the latter can be viewed as (noncommutative) fixed points of the induced flow of zeta functions. Correspondingly, the core of the moduli space of fractal membranes (the attractor of the Riemann flow) consist of all arithmetic geometries (or 'pure crystals', in this context) and can be viewed as a mathematical incarnation of 'Deninger's arithmetic site'. Similarly, the Equator (or critical line) is the attractor of the zeros, and the space of arithmetic L-functions--which necessarily satisfy a true functional equation and therefore, are 'self-dual', in the sense of T-duality or the AdS/CFT correspondence in string theory, see [Lap6, Chapter 2]) is the attractor of the moduli space of partition functions of the generalized quasicrystals (or fractal membranes).

The noncommutative Riemann flow can also be thought of as a noncommutative analogue of the Ricci flow, a geodesic flow on an appropriate Teichmuller space associated with the KdV equation, or, alternatively, as a flow of KMS states in quantum statistical physics associated with normal (but not necessarily self-adjoint) generalized Polya-Hilbert operators, and with the analytic continuation in mass (or in the diffusion constant) of Feynman integrals [JohLap, Chapters 11 and 13] for a Schrodinger equation with a highly singular potential (such as the attractive inverse-square potential in quantum mechanics); the unitarity of the underlying flow is then only achieved in the limit as the absolute temperature tends to zero and hence, when the entropy is extremized. See [Lap5, Chapter 5].

We will also briefly discuss 'quantized number theory' and the theory of the spectral operator developed in [HerLap1-2] (jointly with Hafedh Herichi) and aimed at providing 'quantum' or operator-theoretic analogues of many aspects of number theory—and, particularly, of many arithmetic zeta functions, their Dirichlet series and Euler products, in regions of the complex plane (such as the critical strip $0 < \mathrm{Re}(s) < 1$) where the latter complex-valued analytic objects do not converge. Several mathematical phase transitions appear naturally in this context, particularly at $s = 1$ (corresponding to the pole of the Riemann zeta function, RZF) and at $s =1/2$ (corresponding to the critical line $\mathrm{Re}(s) = 1/2$ where, according to RH, the nontrivial zeros of the RZF are located).

An overview of the theory of complex fractal dimensions, both for fractal strings and in the higher-dimensional case of relative fractal drums (phases 1 and 2, respectively), can be found in the author's survey article, [Lap8] as well as in Chapters 2 and 3 of the author's forthcoming book, [Lap9]. See also the AMS graduate textbook [LapRad] and its forthcoming sequels [DavLapRad1-2] and [DavLap9].

Needless to say, this body of work relies on numerous other works by many different mathematicians (and physicists) whose contributions are discussed in the enclosed references. Also, in addition to the previously mentioned references, [HerLap1-2], [CobLap], and [LapOve] (joint with the author's former Ph.D. students, Hafedh Herichi, Tim Cobler, and Matthew Overduin, respectively), many results which are not explicitly discussed in this abstract---and which played (or are still currently playing) a very useful role in the development of the theory of complex fractal dimensions---were obtained jointly by the author and a number of his current or former Ph.D. students and postdoctoral fellows, including (in approximate chronological order) Christina Q. He [HeLap], Erin P. J. Pearse [LapPea1-3] and Stephen Winter [LapPeaWin1-2], John A. Rock ([LapRoc], [LapLevRoc], [EllLapMacRoc], and [LapRocZub]), Nishu Lal [LalLap1-2], Hung (Tim) Lu or Lu Hung ([LapLu1-2], [LapHun]) and Machiel van Frankenhuijsen ([LapHun-vFr1-2] and [LapLu-vFr]), Cristina Ivan [ChrIvaLap], Jonathan Sadhad [LapSar], Therese-Marie Landry [LanLapLat], as well as most recently, Edward Voskanian and Machiel van Frankenhuijsen [Lap-vFrVos1-2], Will Hoffer [HofLap], Sean Watson [LapWat], and Shanna Dobson [DobLap]. The author is grateful to all of them, as well as to his other co-authors, for their very significant contributions as well as for their fruitful and friendly collaboration.

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