YEAR 12 – MATHEMATICS

HSC Topic 9 - The first and second derivatives C3.1
Applications of the derivative C3.2

MATHEMATICS ADVANCED

LEARNING PLAN			
Learning Intentions Student is able to:	**Learning Experiences** Implications, considerations and implementations:	**Success Criteria** I can:	**Resources**
1. use the **first derivative** to investigate the shape of the graph of a function	– deduce from the sign of the first derivative whether a function is increasing, decreasing or stationary at a given point or in a given interval – use the first derivative to find intervals over which a function is increasing or decreasing, and where its stationary points are located. – use the first derivative to investigate a stationary point of a function over a given domain, classifying it as a local maximum, local minimum or neither	▪ apply the geometrical significance of the sign of $f'(x)$, including the determination of whether or not $f(x)$ is increasing or decreasing. ▪ A stationary point of $f(x)$ is defined to be a point on the curve $y = f(x)$ where the tangent is parallel to the x-axis. At such a point, $\frac{dy}{dx} = 0$. A turning point of $f(x)$ is a point where the curve $y = f(x)$ is locally a maximum or a minimum.	▪ By considering the sign of the first derivative, show that the function $f(x) = \frac{1}{3x-2}$ is decreasing throughout its domain.

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<td>For a max/min turning point, finding values of x for which $\frac{dy}{dx} = 0$ is <u>sufficient</u> for sketching <u>most</u> curves. However $\frac{dy}{dx} = 0$ does not always imply that there is a turning point; but in all cases of a turning point $\frac{dy}{dx}$ must change sign for points before and after (x_0, y_0), while for some curves $\frac{dy}{dx}$ may not exist at : (x_0, y_0) yet the curve changes direction.

– determine the greatest or least value of a function over a given domain (if the domain is not given, the natural domain of the function is assumed) and distinguish between local and global minima and maxima</td>
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- A useful way to represent and set out the first-derivative test is using a table.

For example, determine the nature and position of the local maxima/minima of
$f(x) = x^2 - 6x + 11.$

Solution: Given $f(x) = x^2 - 6x + 11$, $f'(x) = 2x - 6$

$f'(x) = 0$ at $x = 3$

Construct a gradient table:

<table>
<tr><td>x-value</td><td>2.9</td><td>3</td><td>3.1</td></tr>
<tr><td>$\frac{dy}{dx}$</td><td>–0.2</td><td>0</td><td>0.2</td></tr>
<tr><td>Direction of curve</td><td>\</td><td>—</td><td>/</td></tr>
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<td>2. Understand the significance of the second derivative.</td>
<td>Define and interpret the concept of the second derivative as the rate of change of the first derivative function in a variety of contexts, for example recognise acceleration as the second derivative of displacement with respect to time

– understand the concepts of concavity and points of inflection and their relationship with the second derivative</td>
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- If $\frac{d^2y}{dx^2} > 0$ over an interval, the curve is concave upwards over that interval, and if $\frac{d^2y}{dx^2} < 0$ over an interval, the curve is concave downwards over that interval.
- At a point of inflection, the sign of $\frac{d^2y}{dx^2}$ changes when passing through the point.

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- Sketch the graph of the function $f(x) = x^3 + 3x^2$ by identifying stationary points and determining their nature.

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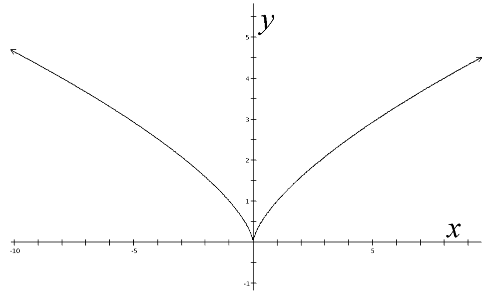
	– use the second derivative to determine concavity and the nature of stationary points – understand that when the second derivative is equal to 0 this does not necessarily represent a point of inflection $\frac{d^2y}{dx^2}=0$ **is not a sufficient** test for inflexion points, since $\frac{d^2y}{dx^2}=0$ when $x=0$ for both $y=x^2$ and $y=x^3$ but $y=x^2$ has a minimum turning point at $(0,0)$ while $y=x^3$ has a horizontal inflexion point at $(0,0)$. Students must test for a change on concavity about a point of inflection.	▪ Consider the curve $y=\frac{1}{4}x^4-x^3$ (a) Find any stationary points and determine their nature. (b) Find any points of inflection. (c) Sketch the curve for $-1.5\le x\le 4.5$, indicating where the curve crosses the x-axis. (d) For what values of x is the curve concave down?	
3. Sketch a function: use calculus to determine and verify the nature of stationary points, find local and global maxima and minima and points of inflection (horizontal or vertical), examine behaviour of a function as $x\to\infty$ and $x\to-\infty$ and hence sketch the graph of the function. This can be applied to exponential, trigonometric and logarithmic functions.	Eg. sketch $y=x^3-3x^2-9x+2$, $y=x^2+\frac{2}{x}$. Curves may also have inflexion points where the tangent is vertical – consider $y=\sqrt[3]{x}$. There is still a change in y" before and after $(0,0)$. Consider curves such as $y=\sqrt[3]{x^2}$ and $y=\|x\|$ for	Further Exam Style Question Consider $y=xe^x$ (i) Show $y'=(1+x)e^x$ and $y''=(2+x)e^x$. (ii) Show that there is one stationary point and determine its nature. (iii) Find the coordinates of the point of inflexion	

	which the gradient functions do not exist at their max/min turning points.  	(iv) Examine the behaviour as $x \to -\infty$ (v) Sketch the curve **Note: always check for asymptotes when sketching Exponential functions**	
5.Solve optimisation problems: Include topics of displacement, velocity, acceleration, area, volume, business, finance and growth and decay	– define variables and construct functions to represent the relationships between variables related to contexts involving optimisation, sketching diagrams or completing diagrams if necessary – use calculus to establish the location of local and global maxima and minima, including checking endpoints of an interval if required – evaluate solutions and their reasonableness given the constraints of the domain and formulate appropriate conclusions to optimisation problems. Eg.1: Constructing various containers or enclosures to maximise/minimise areas, volumes, costs etc. given fixed perimeters, surface areas etc. Prove that a closed cylinder of fixed surface	▪ A box without a lid is made by cutting out four equal squares from the corners of a sheet of heavy card, then folding up the sides. If the card has dimensions 20 cm by 10 cm, what are the dimensions of the box with largest volume that can be constructed in this way? ▪ A right circular cone is inscribed in a sphere of radius a, centred at O. The distance from the base of the cone to the top of the sphere is x and the radius of the base is r, as shown in the diagram. (a) Show that the volume, V, of the cone is given by $V = \frac{1}{3}\pi\left(2ax^2 - x^3\right)$. (b) Find the value of x for which the volume of the cone is a maximum. Give reasons why this value of x gives the maximum volume.	

	area has maximum volume when its diameter equals its height. Eg.2: Given the hourly running cost of a ship as a function of its velocity, find the most economical running speed. Students need to pay particular attention to **restrictions** on variables and their explanation of why there is a **local** as well as an **absolute** max./min. for the values under consideration. * Consider problems in which more than one case needs to be analysed – HSC 1988 Q7b	▪ Investigate functions of the form $f(t) = Ae^{-at} - Be^{-bt}$, $t \geq 0$, and $f(t) = Ate^{-kt}$, $t \geq 0$ which may both be used to model the amount of a drug in the bloodstream after a dose of the drug has been taken. Where are the maximum turning points in each case? Where are any inflection points?	

Established Goals(Syllabus Outcomes): MA12-3, MA12-6, MA12-7, MA12-9, MA12-10