Discrete Maths - MID-2

Solve any 5 questions. The 6th one is Bonus.

Problem 1

A. How many ways are there to arrange the letters *a, b, c, d* such that *a* is not followed immediately by *b*?
Answer: There exists 18 ways to arrange the letters a, b, c, d, i.e. a is not followed immediately by b.
For instance. First letter is 'a'; Second place has only two options (c, d) ; Third place has three options (b, c,d) ; and Fourth place has three options (b, c, d).

B. In how many ways can a photographer at a wedding arrange six people in a row, including the bride and groom, if

i. The bride must be next to the groom.

Answer:
The total number of possible arrangements: P(6, 6).
The total number of arrangements that the bride is at the left and
The bride is next to the groom: P(5, 5).
The total number of arrangements that the bride is at the right and
The bride is next to the groom: P(5, 5).
So the answer is: 2 × P(5, 5)
In other words, Think of the bride and groom as one item, which means there are five items to be arranged, giving 5! arrangements. However, the bride and groom can be arranged in two different ways, so for each of the 5! arrangements, there are two possibilities for the arrangement of the bride and groom. The total number of permutations in which the bride is next to the groom is **2 * 5! = 240 ways.**

ii. The bride is not next to the groom.

Answer:
The total number of possible arrangements: P(6, 6).
The total number of arrangements that the bride is at the left and
The bride is next to the groom: P(5, 5).
The total number of arrangements that the bride is at the right and
The bride is next to the groom: P(5, 5).
So the answer is: P(6, 6) – 2 × P(5, 5)
As in part a), there are 240 permutations in which bride is next to the groom. There is a total of 6! = 720 arrangements of the six people. Hence the total number of permutations in which the bride is not next to the groom is **720 - 240 = 480 ways**.

iii. The bride is positioned somewhere to the left of the groom.

Answer:
Choose 2 positions for bride and groom, C(6, 2) ways. Arrange bride and groom in these 2 positions, 1 way (bride on the left side of the groom). Arrange other 4 people,P(4, 4) ways. So, there are C(6, 2) × 1 × P(4, 4) = P(6, 6) · 1/ 2 ways = 720 / 2 = **360 ways**.

Problem 2

A. Find the number of ways of arranging 6 women and 3 men to stand in a row so that all 6 women are standing together?

> **The possible arrangements can be made by considering all the 6 women as a single unit and all the men as different units, which make a total of 4 units. The number of permutations if these 4 units is 4!. However, the 6 women within a single unit can be arranged in 6! Ways.**
> **Therefore total number of arrangements = 4!* 6!= 17280**

B. Four male and five female members of a commission are to sit in a meeting on a round table. How many ways they can sit in the meeting if:
   1. All male members are to sit together?
   2. All female are to sit together and there is one seat for the chair of the commission. Assume each member is qualified to chair the commission?
   3. If no two male members sit together.

> 1. **4! x 5!**
> 2. **4! x 5!**
> 3. **5! x $^5P_4$: First fix any permutation of the females and then place male in the possible 5 locations (in between females) hence $^5P_4$**

Problem 3 (PHP)

A. Imagine you have a birthday. You have bought **The Special Chocolate Cake** with n number of slices (where n is even lets say 10 number of slices). You have only invited one special friend of yours. You would like to eat it all together but eating has some particular rules:
   - ✓ You cannot eat randomly. The First slice has to be eaten by you.
   - ✓ After you have started eating second person has got two choice i.e. eat it one from any two of the neighbours which you selected.
   - ✓ Similarly after your friend took one you also have exactly two choices.

   You absolutely love **The Special Chocolate Cake. Device a strategy (algorithm)** such that you can eat at-least half of the Cake no matter what your special friend choose against your selection of the slice**.**

> **Solution**
>
> Imagine if you have numbered all the slices, because you have even numbers of slices hence you half of them even numbered slices and half odd numbers. Simply by Pigeonhole Principle either ***Even numbered Slices volume will be greater or Odd Numbers Slices volume will be greater***. *Now w.l.o.g. Let me take all Evens volume sum is greater.*
>
> Now here is the strategy:
> - You choose any even numbered slice lets say 6. It is very obvious to see that 6 has neighbourhood 5 and 7, with this selection you have kind of made mandatory to choose from ODD options. Now again without loss of generality if your friend takes 5 then you again have the option to choose even imposing your friend to choose again from odds. Hence with this strategy you will eat all the evens and your friend will eat all the odds. Hence you will eat at-least half of the Cake.

B. We have a class of Cryptography in which 21 students appeared prove the following two claims:
   A. Cryptography class must have at least 11 male students or at least 11 female students.

B. Now prove also that class must have at least 5 male students or at least 16 female students. (**HINT: Use Prove by Contradiction to prove this claim**)

It has two parts.

- There are two choices of being female and male and there are 21 students hence one of the choice has Atleast 11 students by pigeon hole principle: Formally speaking lets assume if both arent the case the sum will be less than 21 hence a contradicton.
- Simple using prove by contradiction we can have the answer: If neither 16 females are there and nor 5 male students then the summation is less than 21 hence a contradiction.

Problem 4

A. Each of the digits 1,1,2,3,3,4,6 is written on a seperate card. The seven cards are then laid out in a row to form a 7-digit number.

1. How many distinct 7-digit numbers are there?
2. How many of these 7-digit numbers are even?
3. How many of these 7-digit numbers are odd?
4. How many of these 7-digit numbers are divisible by 4?
5. How many of these 7-digit start and end with the same digit?

Solution

1. 7!/(2!.2!)= 1260

2. All the numbers ending with an even number i.e. 2,4 or 6. Possible 7-digit number are 3(6!/2!.2!)=540

3. Total possible 7-digits numbers - even number= 1260-540=720
Or the numbers ending with an odd number= 2(6!/2!)

4. Rule for divisibility by 4 states that a number is divisible by 4 if the last two digits are divisible by 4.
All such 7-digit numbers which are divisible by 4 can end at 24,64,16,36,12 or 32.
7-digit numbers ending with 24 = 5!/(2!.2!)=30
7-digit number ending with 64= 5!/(2!.2!)=30
7-digit numbers ending with 16= 5!/(2!)=60
7-digit number ending with 36=5!/(2!)=60
7-digit number ending with 12=5!/(2!)=60
7-digit number ending with 32=5!/(2!)=60
All possible 7-digit numbers divisible by 4= 2(30)+4(60)=300

5. 7-digit number starting and ending with 1= 5!/2!
7-digit number starting and ending with 3= 5!/2!
All possible 7-digit numbers starting and ending with same digit= 120

B. A committee of three people is to be selected from four women and five men. The rules state that there must be at least one man and one woman on the committee. In how many ways can the committee be chosen? Subsequently one of the men and one of the women marry each other. The rules also state that a married couple may not both serve on the committee. In how many ways can the committee be chosen now?

B. Possibility of choosing a committee of 3 members with at least women and at least one men= choosing 2 women and 1 man + choosing 1 women and 2 men= 4C2*5C1 + 4C1*5C2= 70

Possible number of ways of choosing committee of 3 members with at least one man and one women excluding the married woman= 3C1*5C2+ 3C2*5C1= 45
Possible number of ways of choosing committee of 3 members with at least one man and one women excluding the married man= 4C1*4C2+ 4C2*4C1= 48

Possible number of ways of choosing committee of 3 members with at least one man and one women excluding the couple(man and woman both) = 3C2*4C1+ 3C1*4C2= 30 By PIE, Ans= P(excluding woman)+ P(excluding man) - P(excluding couple)= 45+48-30=63

Problem 5. Combinatorial Argument

Provide combinatorial (counting) arguments to prove the following statements. No algebraic manipulation is allowed or required.

A. $C(2n,\ n)\ =\ \sum_{i=0}^{n} C(n,\ i)^2$

B. $C(C(n,2),\ 2)\ =\ 3\ C(n,3)\ +\ 3\ C(n,4)$

A. L.H.S. directly counts the number of ways to select n players from a group of 2n players which include n players from team A and n players from team B. R.H.S. computes the same quantity by picking i players from team A and n-i players from team B. There are C(n, i) ways to pick i players and C(n, n-i) = C(n, i) ways to pick the remaining players. These are multiplied according to the product rule, and the product is summed over all possible values of i.
B. L.H.S. directly counts the number of ways of picking two pairs of elements from a set of all possible pairs of a set of n distinct elements. i.e. number of ways of picking pair sets {{ai, aj},{ak, al}} from the set {a1,a2,...,an}. R.H.S. counts the same quantity by dividing it into two cases. Notice that there are two types of pair sets, {{ai,aj}{ak,al}}. Type1, in which all four of ai, aj, ak and al are distinct, and Type2, in which only three of them are unique and the fourth one repeats. There are C(n,4) ways to pick ai, aj, ak, and al to make pair sets of Type1 and there are three ways to distribute them into two pairs. This makes 3.C(n, 4) pair sets of Type1. There are C(n, 3) ways to pick ai, aj, and ak to make pair sets Type 2, and then there are 3 options for which of these three will repeat as the fourth element al. This makes 3C(n, 3) pair sets of Type2. Following table demonstrates Type1 and Type2 pair sets:

{{a,b},{c,d}} {{a,c},{b,d}} {{a,d},{c,b}}	{{a,b},{c,a}} {{a,b},{b,c}} {{a,c},{c,b}}
3 possibilities based on the distribution of the four elements in a Type1 pair set.	3 possibilities based on which of the three picks is repeating in a Type2 pair set.

Problem 6

6. (Aamir) Generalized Permutation and Combination.

A. How many possible passwords can be formed from the letters: **ASSISTANT_TITAN**

Solution

ASSISTANT_TITAN

No of A: 3
No of S: 3
No of I: 2
No of T: 4
No of N: 2
No of _: 1

No of possible passwords: $\frac{15!}{3!3!2!4!2!1!}$

B. How many possible solutions could be for the equation:

$$x_1 + x_2 + x_3 + x_4 = 20,$$

with the constraints that $x_1 \geq 1,\ x_2 \geq 10,\ x_3 \geq 5,\ x_4 \geq 0$.

Solution

$$x_1 + x_2 + x_3 + x_4 = 20,$$

with the constraints that $x_1 \geq 1,\ x_2 \geq 10,\ x_3 \geq 5,\ x_4 \geq 0$.

With these constraints, we MUST have at least $x_1 = 1,\ x_2 = 10,\ x_3 = 5,\ x_4 = 0,$ and the four quantities should add up to 20.

With the minimum chosen values of four quantities (1+10+5+0 = 16), we are left with 20 - 16 = 4, free choice, that we may choose from any of the 4 quantities (x_1, x_2, x_3, x_4).

So, n = 4, r = 4. We have

$$C(n - 1 + r, r) = C(4 - 1 + 4, 4) = C(7, 4) = \frac{7!}{4!3!} = 35$$