DAY 1 - Welcome, TSM

Wu RLE, Chapter 1
NOTE: This document will contain my working notes, but much actual lesson content will appear on the OpenLab in various "Lesson" posts.

NOTE 2023: For a complete (ish) outline of the course as taught in Fall 2023 day-to-day, see [my Planning document].

2023: JONAS: Next time consider moving the Geometry portions closer to the beginning (folks need help on the Geo Regents!). Maybe go from "Number Systems" (3 lectures) to Geometry (6-8 lectures?), then on to Divisibility/Euclidean Algorithm…? NOT 100% CONVINCED THIS IS THE RIGHT MOVE.
COULD ALSO CONSIDER rearranging the "Homeworks" order so that Geo comes first (or maybe after Alg 1 hmwks?)

Welcome, course policies, introductions

FOUR BIG IDEAS FOR THIS COURSE:

1. We all have stuff to learn about mathematics - and it's not the same stuff!
    - Importance of asking questions. ESPECIALLY simple questions, "dumb"-sounding questions, questions that might give away the fact we don't know everything (or even some basic things!)
2. We all *know* stuff about mathematics - usually a *lot*.
    - Often it's a mixture of correct/effective ideas and incorrect/ineffective ideas.
    - Very often, the most effective way to help someone improve their knowledge is **by building on the correct/effective parts**, and helping them to see why other parts may be wrong (or not useful).

3-minute writing assignment: Give a specific example of
- something you know about mathematics
- something you have to learn about mathematics

(Share back answers & discuss)

3. Importance of different perspectives/models/conceptions of mathematical objects

4. Importance of a single **definition.**
    - Having a single, agreed-upon definition of a mathematical term means that, when we are confused (or when we are having a disagreement), we have something we can fall back on -- when in doubt, go to the definition!
    - How does this compare with #3 above?

3 min writing assignment. Answer this question both with your "teacher" hat on, and with your "student" hat on.

- Explain the idea of a fraction

Now form groups of 2 and share what you wrote. Then, in your groups:

3 min writing assignment:

- What is a useful way to think about fractions?

Share back with class

----------------- END OF DAY 1 -------------------

Homework 1: Algebra 1 content

As a New York teacher, you will be responsible for knowing (among other things) the material that is included in the standard New York secondary school math curriculum. Here, "knowing" means multiple things – in particular, it includes both:

the "how" (how to solve various types of problems), and

the "why" (why do we use the method that we do to solve a problem? where does the method come from? how does it fit together with our existing mathematical knowledge?)

Your first exam, taking place in about 3 weeks, will focus on the "how," by asking you to complete problems from the three NY Regents Math examinations (Algebra I, Geometry, and Algebra II). We'll be doing practice problems along the way.

Assignment, due Wednesday, 2/2/22. Complete the 12 multiple choice problems in this document (these problems are from a past version of the Algebra I Regents exam):

Algebra I – Problems 1-12Download

Once you have completed all 12 problems, record your answers in the form below. Also indicate for each problem how confident you feel about your answer. Finally, use the boxes at the bottom of the form to leave a comment or question about three different problems.

Your grade for this assignment will be based on your participation, NOT on the number you get correct – so don't panic if you don't know how to do something!

Record your answers here:

<GRAVITY FORM HERE>

OpenLab: Wu on Textbook School Mathematics (TSM)

H. Wu is a professor at UC Berkeley and is the author of the books on which this course is based. He coined the phrase "Textbook School Mathematics," or TSM, to describe the mathematics curriculum commonly taught in grades K-12 in this country.

[T]here has been a de facto national mathematics curriculum for decades: the curriculum defined by the school mathematics textbooks. There are several widely used textbooks, but mathematically they are very much alike. Let's call this de facto mathematics curriculum Textbook School Mathematics (TSM). In TSM, precise definitions usually are not given and logical reasoning is hardly ever provided (except in high school geometry texts) because the publishers mistakenly believe that intuitive arguments and analogies suffice. Thus, fractions are simultaneously (and incomprehensibly) parts of a whole, a division, and a ratio; decimals are taught independently from fractions by appealing to the analogy with whole numbers; negative numbers are taught by using patterns and metaphors; the central idea of beginning algebra is the introduction of the concept of a variable (which implies, wrongly, that something is going to vary), when it ought to be becoming fluent in using symbols so as to do generalized arithmetic; solving equations is explained by the use of a balance to weigh variables on the weighing platforms; etc.

H. WU
Assignment, due Monday, 2/7/22: Watch the following video of Wu speaking about TSM and the way we prepare teachers (16 min). Respond to the questions below by leaving a comment on this post.

What do you think is Wu's main objective in this talk?
What is something Wu said that you agree with? Explain.
What is something Wu said that you disagree with? Explain.
What is a question you have after watching this talk? Your question should have the form:

“I’d like to ask _____(person) the following question: ______“
It can be a question for Prof. Wu, or a question for your classmates, or your professor, or yourself.
Extra Credit. Respond in some way to one of your fellow students’ comments. Do you agree? Disagree? Do you have a response to their question? Did their comments make you think or provoke additional questions? Reminder: Be respectful, be kind.
Video
Wu on K-12 math teacher preparation:
“Stop selling what you have; start selling what they need.” (16min)

DAY 2 - what is a number?

Wu RLE, Chapter 1
OpenLab post: What is a Number?

OUTLINE:

- Hello and welcome
- Discuss algebra I homework (see RESULTS comment on post)
-
- Material from lesson What is a Number?
- Different types of numbers
- Real numbers:
    - Mathematicians definition
    - Our approach: starting points
    - What is a number line?
    - What is a number?
- If a number is a point, how do we talk about size/length?
- How do we talk about addition, subtraction, multiplication, division? LETS LOOK AT ADDITION
- How do we talk about fractions? *These are certain points on the line*
- What is this good for?
    - Thm: kn/k = n
- TIME PERMITTING:
    - Why fractions?
    - Some different concepts of a fraction

NOTES Fa23: This was a very lecture-y class. Would be nice next time if they could actually USE this for something!! Next time?

NOTE: Didn't use this -
FA 23 IDEA: In the video for this week's OpenLab assignment, Wu mentions 4 different examples of "bad mathematics". GO OVER THESE - What's wrong with each of them? Is there anything positive we can take away from them?

What are different kinds of numbers?
Which type is most inclusive (biggest)?
We will focus on real numbers - why?
A. Essentially all of secondary school math happens here (except complex)
B. Based on a very natural picture.

NOTE Fa23: This question was not entirely straightforward to answer, and provoked good discussion (longer than I expected). The distinctions between natural #, integer, rational number/fraction were clear, but the need to move from rationals to reals was not so much. Would be nice for them to have a clear picture in their heads to motivate each successive step. Would this be a good activity/project??

WHAT IS A NUMBER?
WHAT IS A FRACTION?

We start with the following **three building blocks:**

- Counting 1, 2, 3, 4, … This is one of the very first mathematical activities we learn – we learn it at an early age, and it sticks.
- Basic ideas about space. We live in physical world, and our intuitions about physical object and how they behave are hardwired into our brains. For example:
    - Two objects can't occupy the same place at the same time.
    - If I pick an object up and move it, the size doesn't change.
- Points and Lines. Even though we can't find actual physical examples of these in the world (they come from the more abstract world of geometry), we can still use our basic ideas about space to reason about them.
    - If two points occupy the sample place, they are the same point.
    - I can make a line segment by identifying two points on a line.
    - I can slide a line segment along the line without changing its length.
    - I can divide a line segment up into equal pieces.
    - A line doesn't have any holes in it.

From these beginnings, we are going to define the concept "number" (in particular, real number). Once we have done that, we'll be able to talk about different types of numbers – natural numbers, integers, fractions, etc.

Before we get to numbers, we start with a simple, essential picture – a line, and two points. (Even though we haven't defined "number" yet, we still use the word number in the name of this picture – it's called a Number Line).

Number Line
A **number line**, or real line, or x-axis, is a line with two distinct labeled points,
0 and 1.

Since we usually choose number lines to be horizontal, we'll freely talk about the left direction and right direction. The point 1 is to the right of the point 0.

Number
Definition. A **number** (more precisely **a real number**) is a point on the (a) number line.

NOTE: we have just defined explicitly what a number is, namely, a point on the number line. This may not seem like much until you recall that, in TSM, the word number is bandied about repeatedly and yet nobody can say precisely what a number is.

NOTE: If this is our definition of number — a number is a point on a line — then when we want to talk about doing various things to numbers, like adding, subtracting, multiplying, dividing them, we better give definitions of those concepts in terms of this number line idea. This may take some getting used to!

Let's start by just identifying on the line some common, familiar numbers.
Question: How do we place the number 5 on the number line?
Question: How do we place the number -7 on the number line?

We can use the unit distance (that is, the length of the interval [0,1]) to build the entire sequence of whole numbers on the number line.

At its heart, this process relies on:
- sliding line segments (so that one endpoint of the new segment occupies the same point as the the other endpoint of the original segment)
- repeating this process, and counting how many times we repeat it.
- If we repeat this process to the left as well, we get the integers.

We call the resulting collection of points an infinite sequence of equidistant points, which gives us the whole numbers (to the right of 0) and the integers (along the whole line).

If a number is a point on the line, how do we talk about size, or distance? These are properties that apply to intervals

How do we place the fractions on the number line?

NOTE: We will assume that we can divide a given segment into any number of segments of equal length.

When we think of "the whole" as in "parts of the whole", we always mean the length of the unit segment
. NOTE: The whole is not the unit segment [0,1], but the LENGTH of the unit segment [0,1].

How do we place the fractions with denominator 3? e.g.

etc.

- Divide the unit segment
- into three equal segments.
- Divide also each of
- ,
- ,
- also into three equal segments.
- These division points, together with the whole number themselves, form an infinite sequence of equidistant points, called the sequence of thirds.
- We call the first of these short segments, the segment with left endpoint
- , the standard representation of
-
- .
- The number
-
- is what we call the right endpoint of this segment.
- What is the number
-
- ?
- In general, what is the fraction
-
- , where
- is a whole number?

QUESTION: How do we locate the fraction

on the number line?

Fractions
Definition. The collection of all the points in all sequences of n-ths, as n runs through the nonzero whole numbers
, is called the fractions.

For a nonzero whole number m, the m-th point to the right of 0 in the sequence of n-ths is denoted by

. The number m is called the numerator and n is called the denominator of the symbol

. By the traditional abuse of language, it is common to say that m and n are the numerator and denominator, respectively, of the fraction

. By convention, 0 is denoted by

for any n.

What can we do with this definition?

Homework 2: Algebra 1 part 2
Answers to the previous homework, Homework 1: Algebra 1 content, are here:

Assignment, due Monday, 2/7/22. Complete the 12 multiple choice problems and 8 short answer problems in this document (these problems are from a past version of the Algebra I Regents exam):

AlgI-Problems13-32 - Download

Record your answers to the 12 multiple choice problems below.
Complete the 8 short answer problems on paper, bring them with you to class on Monday.
Record your confidence level for each problem (including the short answer problems) below.
Finally, use the boxes at the bottom of the form to leave a comment or question about three different problems (multiple choice or short answer).
Your grade for this assignment will be based on your participation, NOT on the number you get correct – so don't panic if you don't know how to do something!

Record your answers here:

Day 3, Numbers and Fractions

Spend some time on Regents review in class (questions from homework 2)

NOTE: Should be capturing OVERALL regents problems that challenge/confuse - look for patterns etc!! Track these from various homeworks? Maybe assign something to do with them?

- Use the analysis that I post as a comment on each homework (listing challenging problems) as a basis

Start with pop ~~quiz~~ worksheet (5 min)
You may use all class notes and materials to answer these questions.

1. What are the four main types of numbers that we work with in secondary school, starting with the numbers we count with and working up to the numbers on the number line?
2. What basic knowledge and skills do we rely on in our students, in order for them to understand the definition of number?
3. State and explain the definition of number.

Now get into groups of 3. Share & discuss your answers. Feel free to make any changes to your own answers that you wish.
Hand in quiz.

NOTE: Group Work question 1 took a lot more time than I expected Might be good to just focus on this one today? Might be good to think about re-phrasing/restating? Might be worth taking a little more time on this, maybe having groups present results to each other? Also talking a bit about "what is the point of this" -- thinking of numbers in terms of ***systems***?

Group work:

1. In the following three questions, your equations should use ONLY NATURAL NUMBERS (you can also use an equals sign, a variable, and arithmetic operations (addition, subtraction, multiplication, division)).
    a. Give an example of an equation that has a solution in the integers but does not have a solution in the natural numbers.
    b. Give an example of an equation that has a solution in the rational numbers but not in the integers.
    c. Give an example of an equation that has a solution in the real numbers but not in the rational numbers.

2023 Fall: Most groups did not get to the following questions this time around:

2. *Explain how to view a fraction as each of the following (feel free to explain in your own words, use example(s), and so on):*
    a. *a multiplication*
    b. *a division*
    c. *a ratio*
3. *Using the definition of number as "a point on a number line", describe how to locate the number 23/5 on a number line.*
   Your description should rely only the basic knowledge and skills necessary for the definition of a number (see below).
4. *Create a definition for addition on the number line. That is, given two numbers a and b (points on the number line), describe how to locate the point a+b on the number line.*
   Your description should rely only the basic knowledge and skills necessary for the definition of a number (see below).

Fraction as ratio - think about multiple number lines here?(look ahead at wu definition of multiplication & division??)

Homework 3: Algebra 2 part 1

Answers to the Homework 2: Algebra 1 part 2 are here – you'll find the rubric for scoring the short answer questions (25-32) starting on page 5:

Aug 2017 Algebra 1 Regents – ANSWER KEY
Download
For your convenience, the answer key to the multiple choice questions (13-24) is provided here:

Assignment, due Wednesday, 2/9/22. Complete the 12 multiple choice problems in this document (these problems are from a past version of the Algebra II Regents exam):

AlgII-Problems1-12
Download
Record your answers to the 12 multiple choice problems below.
Record your confidence level for each problem.
Use the boxes at the bottom of the form to leave a comment or question about three different problems (multiple choice or short answer).
Your grade for this assignment will be based on your participation, NOT on the number you get correct – so don't panic if you don't know how to do something!

Record your answers here:

Day 4, Divisibility

Wu RLE, Chapter 3
OpenLab post: Divisibility: Addition, Multiplication, and the Integers

Spend some time on Regents review in class (questions from homework)

2023 NOTE: question from hmwk 3 about normal distribution prompted mini-lesson on normal dist and 68-95-99.7 rule which took about 45 min. Ended up not finishing lesson, just covered through gcd & connection to reducing fractions

LESSON:

BIG IDEAS TODAY:
- multiple
- divisor
- linear combination
- division-with-remainder
- euclidean algorithm

TODAY we talk about the integers and divisibility.
"I want you to FEEL it!" - JR
This topic is connected to both major arithmetic operations, to addition/subtraction and to multiplication/division. It lies at the heart of so much of our basic understanding of numbers, some people say that it is the source of much of the complexity that we find in numbers. If you've taken number theory, you've already learned a *lot* about this relationship.

QUES: What does it mean to say "a goes into b" or "a goes into b evenly"?
Give example.

FACT: We use the following different kinds of language to describe this situation:
b is a **multiple** of a
a is a **divisor** of b, or "a divides b".

Definition. (Defn of multiple, divisor)

QUES: How many multiples does 17 have? List them.

QUES: How many divisors does 12 have? List them.

DEMO: (Desmos) - Show the multiples of a number, e.g. 6.

QUES: What are the divisors of 6? Are any of them also multiples of 6? NOTE: The only divisor of 6 that is also a multiple of 6 is 6 itself.

GROUP ACTIVITY: Write a definition for gcd, lcm.

PART 1: Divisibility with addition/subtraction.

1. REDUCE the following fractions. What number did you cancel out of the top & bottom to get the reduced form?

a, $\frac{25}{10}$ b. $\frac{14}{22}$, c. $\frac{12}{60}$, d. $\frac{13}{33}$ e. $\frac{1147}{899}$

NOTE: unless the numerator and denominator are relatively small, it's not always easy to tell! If classroom instruction focuses on single-digit numerator/denominator, we get the impression we can always tell "just by looking" if a fraction is reduced.

2. FIND the GCD of each pair:
- 25, 10
- 14, 22
- 12, 60
- 13, 33
- 1147, 899

One reason why gcd matters: in a fraction m/n, the gcd(m,n) is the biggest number you can cancel out (and it always leaves the fraction in reduced form)

Homework 4: Algebra II part 2
Answers to the previous homework, Homework 3: Algebra 2 part 1 are here:

Problems 1-8 ANSWERS
Problems 9-12 ANSWERS
Assignment, due Monday, 9/18/23. Complete the 12 multiple choice problems and 8 short answer problems in this document (these problems are from a past version of the Algebra II Regents exam):

AlgII-Problems13-32

Download
Record your answers to the 12 multiple choice problems below.
Complete the 8 short answer problems on paper, bring them with you to class on Monday.
Record your confidence level for each problem (including the short answer problems) below.
Finally, use the box at the bottom of the form to leave a comment or question about a problem (multiple choice or short answer).
Your grade for this assignment will be based on your participation, NOT on the number you get correct – so don't panic if you don't know how to do something!

Record your answers here:

Day 5, Euclidean Algorithm

Wu RLE, Chapter 3
OpenLab post: Divisibility: Addition, Multiplication, and the Integers

Linear combos group work:
Example: Rosa wants to buy a piece of candy from Jamal for ten cents. Rosa only has quarters. Jamal checks his pocket and finds he has only dimes. Explain how Rosa can buy the candy from Jamal and pay the correct amount.

Definition. A linear combination of two numbers is a sum of multiples of those numbers.

You can create all linear combinations of, for example, 25 and 10, by starting at 0 and repeatedly adding or subtracting either 25 or 10 at each step.

Let's play a linear combination game with the numbers 8 and 22. Start at 0 and, at each step, either add or subtract either 8 or 22. Repeat as often as you like.
EXERCISE: Starting with the numbers 8 and 22, can you write a linear combination that equals:
A. 14
B. 12
C. 10
D. What is the **smallest positive number** you can create as a linear combination of 8 and 22?

3. Create the smallest possible positive number as a linear combination of the two numbers:
   1. 10, 25
   2. 14, 22
   3. 60, 12
   4. 13, 33
   5. 1147, 899

DEMO: Desmos, demo about multiples of a number, linear combos of two numbers(?)

FACT: The linear combinations of two integers a and b are exactly the multiples of the GCD of a and b.

COROLLARY: The GCD of a and b can be written as a linear combination of a and b.

EXAMPLE: How do we do this? Given the numbers 1147 and 899, how do we:
a) find the GCD of 1147 and 899?
b) Write the GCD of 1147 and 899 as a linear combination of 1147 and 899?

LECTURE: Euclidean Algorithm.

This is important because it is an actual set of steps (NOT "guess-and-check") that will give you the GCD of any two numbers.

Example. Find the GCD of 258 and 60. Write the GCD as a linear combination of 258 and 60.

Example (class). Find the GCD of 22 and 14, and write the GCD as a linear combination of 22 and 14.

The Euclidean algorithm works in stages. Each stage consists of 4 steps:

1. Start with two numbers, a larger one and a smaller one.
2. Divide the smaller number into the bigger number (use division with remainder), let r be the remainder.
3. Write down an equation giving r as a linear combination of the two numbers.
4. Now go back to step 1, *replacing the larger number with the number r found in step 2*.

Repeat until you reach a remainder of 0. The previous stage (the last nonzero remainder) is the GCD of the original two numbers.

Now work backwards, starting with the GCD. In each stage, use the equation from step 3 to write the GCD in terms of the two numbers in step 1.

Example: Find the GCD of 2964 and 288. Write the GCD as a linear combination of 2964 and 288.

Exercise: Find the GCD of 710 and 170 using the Euclidean Algorithm.

Exercise: Find the GCD of 1147 and 899.

Exercise. Find GCD(10049,1190)

Day 6, Geometry - Isometries and Congruence

Wu RLE, Chapter 4
OpenLab post: [Geometry - what does it mean for two things to be the same?]

2023: Writing assignment or midterm assignment on congruence - "What is the "mathworld" equivalent of moving one object on top of another to show they are the same size and shape?"

2023: todays lecture was too lecture-heavy - how to make it more engaging???

Geometry – When are two things the same?

Plane geometry
Our geometric intuition is grounded in our experience of the real world. For many basic concepts in geometry, that intuition is a great (indeed, fantastic) tool! However, it can be helpful to recognize some of the underlying assumptions we make about geometry and geometric objects that come from our real-world experience.

Many basic concepts will not be defined here — for example, point, line, plane, angle, and so on (you can take a look at Wu RLE Chapters 4 and 5 for a detailed exposition of basic concepts). It's also important to note at the outset that the geometry curriculum does NOT make use of a coordinate plane (complete with x- and y- axes, etc), but just a "plain plane" – so things like coordinates of points, and the use of equations to describe lines and other figures, will not appear.

The point of this form of geometry is that it can be done independently of calculations and numbers. I think this is an important idea to teach: mathematics is not about numbers, but about objects adhering to certain rules (axioms).

MATH.SE USER GREG GRAVITON
When a coordinate plane is introduced, the subject becomes analytic geometry (and is concerned with equations of lines, circles, and so on).

What does it mean for two objects to be "the same"?
Example. Consider two pieces of paper. Are they the same? If I give them to two different students, do the students have the same piece of paper? What is the same about them? How can we tell? (lay one on top of the other).

Example. What does it mean for two points to be "the same"?

Example. What does it mean for two line segments to be “the same”?

In particular, what’s the idea in the plane that corresponds to our real-world notion of “moving one thing on top of the other”? We don’t literally move things like triangles and line segments around! How do we describe what’s going on mathematically?

The basic notion that will describe so much of our work in the plane is called a transformation – a function from the plane to itself.

Transformation

Transformation

Let Π denote the plane. A **transformation F of Π** is a rule that assigns to each point P of Π a unique point $F(P)$ (read: “F of P”) in Π.

We also say F **maps** P **to** $F(P)$ or, sometimes, F **moves** P **to** $F(P)$. Indeed, it is intuitively appealing to think of a transformation as a way of “moving” the points of the plane around.

Discuss the language: instead of “moving a point from one place to another,” we talk about the “image of a point under a transformation.”

Definition. We call two subsets of the plane congruent if there is an isometry taking one to the other.

Isometry

A transformation F is an **isometry** if it preserves distances/lengths (sometimes called a **rigid motion**). That is, if $\operatorname{dist}(P, Q) = \operatorname{dist}(F(P), F(Q))$ for all P, Q in Π.

NOTE: Preserving distance is a very strong property – in fact, any isometry also preserves angles, maps lines to lines and circles to circles, and so on.

The basic isometries

Rotation around a point P by angle t (t is measured in degrees, if t is positive then the rotation is counterclockwise, if t is negative the rotation is clockwise).

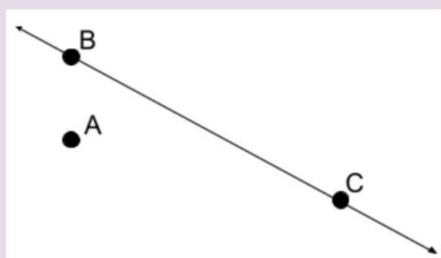
Reflection across a line L.

Translation along a vector $\overrightarrow{AB}$.

Note that we can do multiple transformations in a row, and the result is a new transformation – this exactly corresponds to composition of functions.

Question: If we do multiple isometries in a row, is the result an isometry?

Example

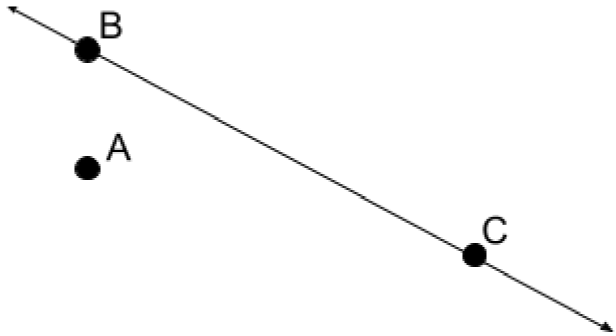


Suppose F is the rotation of 90° about the point A, and G is the reflection across the line $\overline{BC}$.

- Locate $F(B), F(C), F(A)$ (referred to as the image of B, C, and A under the transformation F).
- Locate $G(A), G(B), G(C)$.

Now suppose we *first* do F, followed by G.

- Locate $G(F(B)), G(F(C)), G(F(A))$.
- Locate $F(G(B)), F(B(C)), F(G(A))$.



Congruent

A subset S of the plane is **congruent** to another subset S' of the plane if there is an isometry F so that $F(S) = S'$. In symbols, we say $S \cong S'$.

NOTE: Spend second half of class doing group work
GROUP WORK: short-answer problems from Geometry Regents

Day 7, Geometry - Dilation and Similarity

Wu RLE, Chapter 5
OpenLab post: Geometry - Dilation and Similarity

What does it mean for two line segments OR triangles OR other subsets of the plane to be "the same shape"?

The concept of "shape", the concept of "similarity"

What does it mean for two figures to be the "same shape but not the same size"? What does "shape" mean? We intuitively know the answer — how do we make it precise, mathematically?

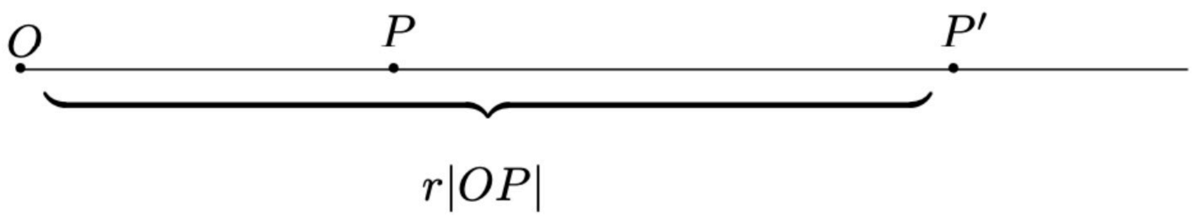
The mathematical term we use for "same shape" is similarity. In TSM, there is a two-pronged approach to defining similarity.

The very intuitive, very general approach: Two figures are similar if they have the same shape but not the same size.
The very precise approach that applies only to triangles: Two triangles are similar if corresponding angles are equal and corresponding sides are proportional.
In order to talk more precisely about similarity, we need a tool to compare two different shapes of different sizes. This tool is a new transformation of the plane, much like the basic isometries (rotation, reflection, translation) we discussed last time, except that this one will NOT preserve distances (is NOT an isometry) — it will "change the size" of objects in the plane.

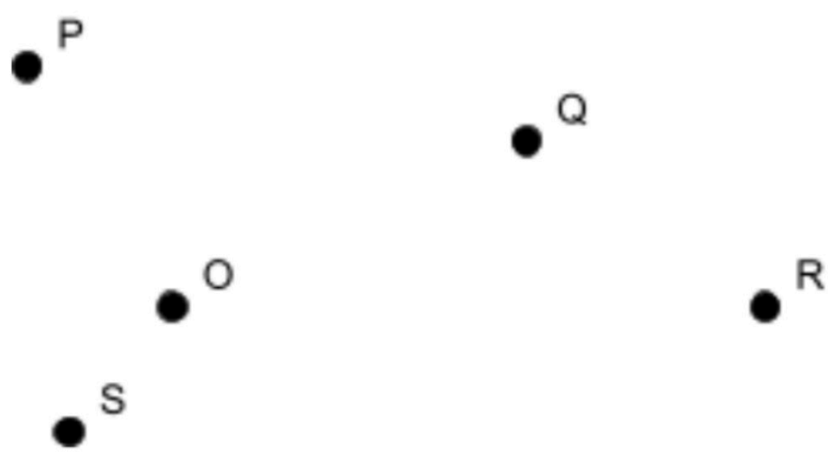
Dilation

Definition. A transformation D of the plane is a **dilation with center O and scale factor** $r (r > 0)$ if

1. $D(O) = O$.
2. If $P \neq O$, the point $D(P)$, to be denoted by P', is the point on the ray R_{OP} so that $|OP'| = r|OP|$.



Example. Consider the dilation D with center O and scale factor $r = 2$ applied to the figure below. Find the image of each point under D.



Basic facts about dilations

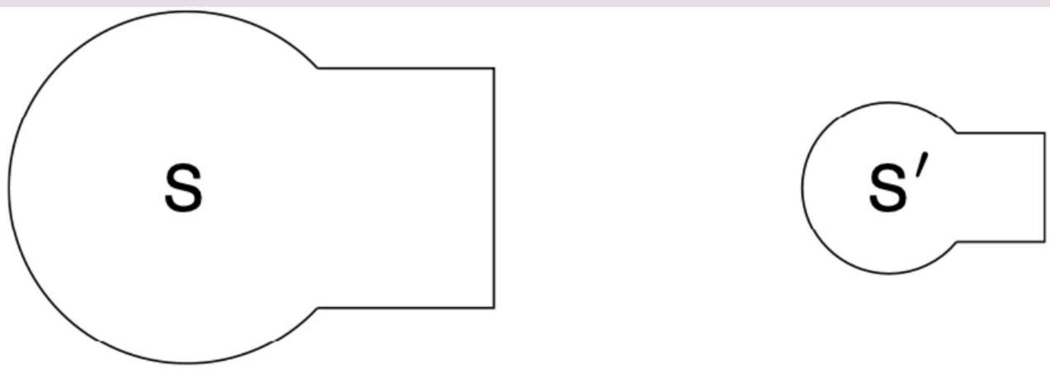
THEOREM. Let D be a dilation with center O and scale factor r. Then:
(a) For any segment AB, $|D(AB)| = r|AB|$.
(b) D maps angles to angles and preserves degrees of angles.

How can we use the idea of Dilation to define when two subsets of the plane are “the same shape” (similar)? (Recall how defined congruence/congruent based on isometries).

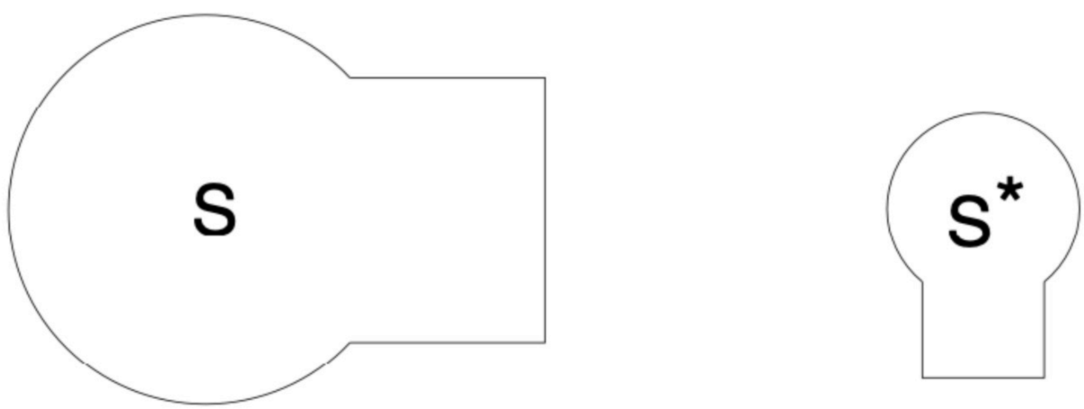
Example. Are the two figures the same shape? Is one a dilation of the other?

Example. In each image, are the two figures the same shape? Can one be obtained from the other by dilation?

a.



b.



Similarity

Definition. A **similarity** is a transformation of the plane that is the composition of a finite number of dilations and congruences.

Similar

A subset S of the plane is **similar** to another subset S' of the plane if there is a similarity F so that $F(S) = S'$. In symbols, we say $S \sim S'$.

Day 8, Exam #1

Day 9: Geometric Constructions I

Math.stackexchange.com is a question-and-answer website that focusses on substantive discussion of mathematical questions, with participants ranging from students to teachers to professional mathematicians. The question What is the (mathematical) point of straightedge and compass constructions? was asked in 2011 and has accumulated a number of interesting responses (representing a wide range of viewpoints). One of the answers argues:

> The beauty of straightedge and compass constructions, as opposed to the use of, say, a protractor, is that you don't measure anything. With ruler and compass you can bisect an angle without knowing its size, whereas with a protractor, you would have to measure the angle and then calculate the result.
>
> In other words, the point of this form of geometry is that it can be done independently of calculations and numbers. I think this is an important idea to teach: mathematics is not about numbers, but about objects adhering to certain rules (axioms).
>
> **MATH.SE USER GREG GRAVITON**

What's the game?

We start by describing how the ruler and compass are intended to be used.

Ruler and Compass Construction Rules

1. Given any two points A and B, the ruler can be used to draw the line L_{AB}.
   *The ruler is used to **draw straight lines** (not for measurement).*
2. Given a point O and a line segment $\overline{AB}$, the compass can be used to draw a circle with center O and radius equal to $\mathrm{length}(\overline{AB})$.
   *The compass is used to **copy distances** (by drawing circles of a specified radius)*

GeoGebra's Ruler and Compass

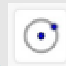
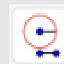
GeoGebra allows us to explore geometric constructions online, rather than with pencil and paper. In the GeoGebra "Ruler and Compass" tool, you'll find you are able to:

1. Add points anywhere on the screen using the point tool 

2. Move points around using the arrow tool 
3. **Ruler:** Create line segments, lines, and rays using the ruler tools 
4. **Compass:** Create circles and copy distances using the circle/compass tools 
5. Zoom in, zoom out, and move the viewing area around using the movement tool 
6. Add text using the Text tool 

7. Undo/Redo recent actions 
8. Reset the tool to a blank page 

ABOUT THE CIRCLE AND COMPASS TOOLS:

- GeoGebra's **circle** tool allow you to draw a circle by clicking two points (the center and a point on the circle). 
- The **compass** tool allows you to *first select a radius by clicking on two points*, then *click a center point* to draw a circle of that radius. 

Give it a try – "ask a friend" constructions

Before we launch into these, give everyone a chance to play around a bit with the GeoGebra Ruler and Compass tool.

Group Activity. Use a ruler and compass in GeoGebra to complete the following constructions. Note: *Every member of the group must complete the construction before the group can move on!*

1. Copy a segment

1. ASK A FRIEND to complete the following step on ***your*** device:
    - Draw a line $\overleftrightarrow{AB}$.
2. Now you complete the following step on your device, using the objects drawn by your friend as a starting point.
    - Construct a point C on the line $\overleftrightarrow{AB}$ so that $\overline{CA} \cong \overline{AB}$.
3. Test it! Use the Move tool (arrow) to drag point A or point B. *If you've completed the construction correctly, the point C should automatically move to keep the two segments congruent*

2. Copy a segment (harder!)

1. ASK A FRIEND to complete the following step on ***your*** device:
   - Draw a line segment, $\overline{AB}$ and a point C not on the line segment.
2. Now you complete the following step on your device, using the objects drawn by your friend.
   - Make a congruent copy of the line segment $\overline{AB}$ starting at the point C.
3. Test it (use the Move tool to drag point A or point B).

3. Construct an equilateral triangle

1. ASK A FRIEND to complete the following step on ***your*** device:
   - Draw two points A and B.
2. Now you complete the following step on your device, using the objects drawn by your friend.
   - Construct $\triangle ABC$ with $\overline{AB} \cong \overline{BC} \cong \overline{CA}$.
3. Test it!

4. Midpoint and perpendicular bisector

1. ASK A FRIEND to complete the following step on ***your*** device:
   - Draw two points P and Q and the line segment $\overline{PQ}$.
2. Now you complete the following step on your device, using the objects drawn by your friend.
   - Construct a line L that is perpendicular to $\overline{PQ}$ and passes through the midpoint of $\overline{PQ}$
3. Test it!

5. Bisect an angle

1. ASK A FRIEND to complete the following step on ***your*** device:
   - Draw two rays $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ with a common endpoint P to make an angle.
2. Now you complete the following step on your device, using the objects drawn by your friend.
   - Construct another ray $\overrightarrow{PS}$ so that $\angle QPS \cong \angle SPR$
3. Test it!

6. Copy an angle

1. ASK A FRIEND to complete the following step on ***your*** device:
   - Draw two rays $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ with a common endpoint P to make an angle.
2. Now you complete the following step on your device, using the objects drawn by your friend.
   - Make a copy of the angle at another place on the page. Start by drawing another ray $\overrightarrow{ST}$.
   - Now construct the ray $\overrightarrow{SU}$ so that $\angle QPR \cong \angle TSU$
3. Test it!

More Construction Challenges:

Ask a friend to draw a triangle ABC, and a point P not on the triangle. Create a “double size copy of ABC” by constructing the image of ABC under a dilation with center P and scale factor r=2.

Ask a friend to draw a line L and a point P not on the line. Construct a line parallel to L passing through P.

Day 10: Geometric Constructions II - Triangle Centers

As we saw last time, software can easily replicate the capabilities of the ruler and compass – but can also be used to apply those techniques without the same kind of human error that physical tools can generate. More complicated constructions become easier to carry out, and the resulting figures can be manipulated while preserving each element of the construction.

There are a number of geometry tools online, but we will continue to focus on the free Geometry Calculator from GeoGebra. This powerful free tool can be used for all kinds of geometry constructions and demonstrations (no account required, though you can create a free account if you want to save your work). Last time we used a limited version with basic ruler and compass tools only. The full version (link below) has many more tools available, which allow you to carry out complex ruler-and-compass constructions like bisecting an angle or finding the midpoint of a segment with the click of a button.

GeoGebra's online Geometry calculator

- GeoGebra Geometry resources.
- GeoGebra Geometry App: Beginner Tutorials with Lesson Ideas. This series of pages serves a good introduction to the GeoGebra Geometry app.

The Center of a Triangle

DISCUSSION:

How do we find the center of a triangle?

Let's talk about some different ideas of how to find the center – thinking in terms of geometric constructions.

Construct some of the following in GeoGebra. What do you notice?

1. Bisect each angle
2. Center of the biggest possible circle inside the triangle (incenter)
3. Equidistant from each side
4. Connect the midpoint of each side to the opposite vertex (centroid)
5. Equidistant from each vertex
6. Center of the smallest possible circle containing the triangle (circumcenter)
7. Perpendicular bisectors of each side
8. Altitude of each side (orthocenter)

Day 11: Circles

PLANNING: Continue with *some* geometry topics now - - maybe save length/area/volume for later in semester?

NOW:
Circles - geometry theorems, equations (analytic geo)

Circles arise naturally in geometry (they are an immediate consequence of the notion of distance), and we use them extensively in geometric constructions (after all, a compass is essentially a “circle-drawing tool”). Today, we look at some geometrical facts about circles, and then we use the coordinate plane to make the connection between the geometry and algebra (equations) of circles.

Circle

Definition. Given a point P and a positive real number r, the **circle with center P and radius r** consists of all points in the plane of distance r from P.

Circles and Line Segments

A **diameter** of a circle is any line segment with both endpoints on the circle which passes through the center P. *Note: Sometimes we use the word “diameter” to mean the **length** of such a segment.*

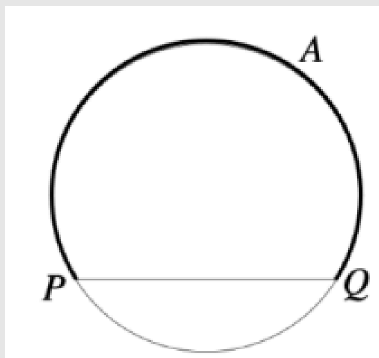
A **radius** of a circle is any line segment with one endpoint at the center P and the other endpoint on the circle. *NOTE: Once again, we sometimes use the word “radius” to mean the **length** of such a segment.*

Chords and Arcs

A **chord** of a circle is any line segment with both endpoints on the circle.

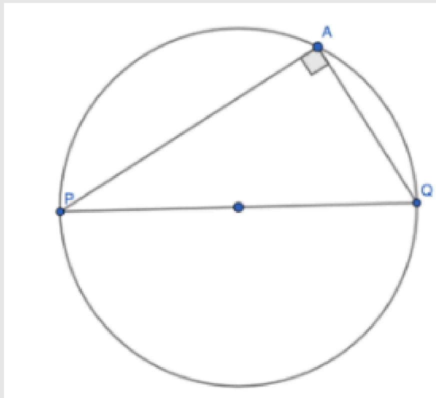
Arc. If P and Q are distinct points on a circle, then we can imagine the chord $\overline{PQ}$ dividing the circle into two parts, called **arcs**. When we use the notation $\widehat{PQ}$ it is usually clear from context which of these two arcs we mean – but sometimes, an additional letter is provided to help the reader.

Example: In the image below $P\widehat{A}Q$ refers to the upper arc.



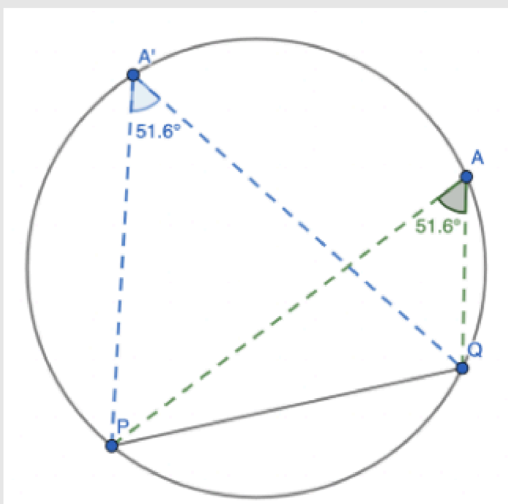
Drawing angles in circles

Theorem. If $\overline{PQ}$ is a diameter of a circle and A is *any* other point on the circle, then the angle $\angle PAQ$ is a right angle.



What if we use a different chord (not a diameter)?

Theorem. If $\overline{PQ}$ is a chord of a circle and A is *any* other point on the circle, then the measure of $\angle PAQ$ does not depend on the choice of A – we call this the **angle subtended by** $\overset{\frown}{PQ}$.



This angle is $90°$ if and only if the chord $\overline{PQ}$ is a diameter.

From Geometry to Algebra

When we start using *x* and *y* axes, we suddenly we gain the ability to refer to points by pairs of numbers, or coordinates. This allows us to use the tools of *algebra* to talk about the shapes from *geometry*. We call this topic *analytic geometry*.

Distance

Definition. Given two points (x_1, y_1) and (x_2, y_2), the **distance** *d* between them is given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Question: Suppose we have a circle of radius *r=5* centered at *(4,6)*. If *(x,y)* is a point on the circle, what is the distance from *(4,6)* to *(x,y)*? Express using the definition of distance.

GROUPS: Find the equation for the circle with center $(-2, \frac{3}{4})$ and radius 5.3. Is the point $(1.1, 4)$ on the circle? Give 3 points on the circle.

GROUPS: Find the center and radius of the circle given by the equation $x^2 + y^2 - 6x + 14y + 54 = 0$.

GROUPS: Find the center and radius of the circle given by the equation $x^2 - \frac{x}{3} + y^2 + \frac{50y}{3} + \frac{737}{36} = 0$

Day 12: Parabolas and Quadratic Functions

Algebra

Definition. A **quadratic function** is a function of the form $f(x) = ax^2 + bx + c$.

Discuss: What does the graph of a quadratic function look like? Major features and how to find them. (Opens up/down, vertex, "width," intercepts)

Geometry

Definition. A **parabola** is the set of points equidistant from a given point P (the focus) and a given line L not containing the point P (the directrix).

Goal: Let's connect these two ideas.

We know the meaning of distance between two points. what is the meaning of "distance between a point and a line?"

Distance to a line

Definition. The distance from a point P to a line L is the length of the line segment PQ, where Q is a point on L and PQ is perpendicular to L.

Group Activity 1: *A parabola with focus* $(2,5)$ *and directrix* $y=-3$. Suppose (x,y) is any point on the plane:

- Find a formula for d_1, the distance from (x,y) to the point $(2,5)$.
- Now find a formula for d_2, the distance from (x,y) to the line $y=-3$.
- Now, suppose we only want to consider points (x,y) for which these two distances are equal. Write an equation expressing this idea. Simplify and solve for y.
- Find the coordinates of the vertex of the parabola.
- Find the x-intercepts of the parabola.

Group Activity 2: Consider the parabola $x^2-6x-4y+45=0$.

- Find the coordinates of the vertex.
- This parabola has focus $(3,10)$. What is the equation of the directrix?
- Verify your work by setting equal the formulas for:
    - the distance from a point (x,y) to the focus $(3,10)$, and
    - from (x,y) to the directrix found above.
- Simplify. Do you get the same equation for a parabola?

Focus and Directrix

I. The equation of a parabola with focus (a, b) and directrix $y = m$ is:

____

II. The focus and directrix of the parabola $y = ax^2 + bx + c$ are:

____

OLD NOTES:

Day 6, Symbolic Notation and Linear Equations

Wu RLE, Chapter 6
OpenLab post: Symbolic Notation and Linear Equations

NOTE: Actually ended up spending most of the day reviewing Regents Geometry problems, which seemed to be a challenge point for most students. Will need to bump this to after the first exam.

The initial part of this chapter, on Symbolic Notation, is important - maybe circle back to it next week?

Basic etiquette in use of symbols (pp299-302)
Variables vs constants:
We have now witnessed the fact that in some symbolic expressions, the symbols stand for elements in an infinite set of numbers,8 e.g., the statement that mn = nm for all real numbers m and n, while in others, the symbols stand for the element in a set consisting of exactly one element (in other words, they stand for a fixed value throughout the discussion), e.g., the numbers a, b, and c in the preceding linear equation ax + b = c. In the former case, the symbols m and n are called variables, and in the latter case, a, b, and c are called constants. Notice that such terminology is no more than an afterthought when we have carefully quantified the symbols in each situation. There is in fact no need for the words variables and constants when such information is already contained in the quantification. However, we will continue to use them not only because they have been in use for over three centuries and are everywhere in the mathematics literature, but also because they are at times an indispensable shorthand.
Expressions, identities

Pedagogical Comments on the teaching of “variable” (p318-320)
This might be a good reading assignment?
Khan Acad:

Basic etiquette in the use of symbols: quantify your variables!
Variables vs constants
Expression
Identity vs equation

What is an identity?
What is an equation?
What is a linear equation?

What is an equation? (p322)
An equation in one variable x is a question that asks, when two expressions f(x) and g(x) in a number x are given, whether there is a number k so that f(x) is equal to g(x) when x=k.

Pedagogical comments p327 on solving an equation. Alternate approach on p328

SLOPE:
The main goal of this section is to give a detailed exposition of the key concept— the slope of a line—that underlies the proof of the theorem in the next section that the graph of a linear equation in two variables is a line. Students' confusion about the concept of slope is well known. The root cause of this confusion is TSM's failure to give slope a correct definition. The main purpose of this section is to set the record straight regarding slope: first, by defining it correctly and, second, by showing that slope is, above all, a number attached to the line itself that describes the "slant" and the "steepness" of the line (assumed to be nonvertical). There are some subtleties in the definition of slope that should be left out of the typical middle or high school classroom but which every teacher should be aware of nonetheless, and these are duly pointed out (see (‡) on page 344).

Local Slope at O
Local slope at P
Slope of a line

Chapter Structure:
Chapter 6. Symbolic Notation and Linear Equations 297
- 6.1. Symbolic expressions 298
    - The basic etiquette in the use of symbols (p. 299)
    - Expressions and identities (p. 302)
    - An important identity (p. 306)
    - Mersenne primes (p. 307)
    - The finite geometric series (p. 309)
    - Polynomials and "order of operations" (p. 310)
    - Rational expressions (p. 316)

    - Pedagogical comments on the teaching of "variables" (p. 318)
- 6.2. Solving linear equations in one variable 322
    - What is an equation? (p. 322)
    - How to solve an equation (p. 324)
    - A second way to solve an equation (p. 328)
- 6.3. Setting up coordinate systems 331
- 6.4. Lines in the plane and their slopes 337
    - The local slopes of lines at a point (p. 338)
    - The slope of a line (p. 342)
    - A formula for slope (p. 347)
- 6.5. The graphs of linear equations in two variables 351
    - Generalities about graphs of equations (p. 351)
    - The main theorem (p. 354)
    - Some applications of the main theorem (p. 357)
- 6.6. Parallelism and perpendicularity 363
    - Characterization of parallelism (p. 363)
    - Characterization of perpendicularity (p. 366)
    - A coordinate description of translation (p. 368)
- 6.7. Simultaneous linear equations 373
    - The solution set of a linear system: Geometry (p. 373)
    - The solution set of a linear system: Algebra (p. 377)
    - Solution by substitution (p. 380)

Day 8, ~~Symbolic Notation and Linear Equations~~ Linear Equations and Slope

Wu RLE, Chapter 6
OpenLab post: Symbolic Notation and Linear Equations

Lecture: Defining Slope.

Group work: presenting Exam 1 problems
Rachel and Charlotte: problem 14 (then 15)
Anik and Angie: problem 12 (then 8)
Irina and Ihn: problem 9 (then 5)

PLANNING NOTES:
Today: What is a line, what is slope? What is a linear equation? Connecting linear equations to lines.
NEXT TIME: Maybe general discussion about variables (Chp6)? About equations? (Chp 6)? About functions? (Wu-A&G Chp1)?
OR
Connecting geometric translations to corresponding algebraic changes (in quadratic functions)?

- substitution (p. 380)

Day 9, Quadratic Functions and Equations

Wu A&G, Chapter 2

Lecture:

Group work:

Rachel and Charlotte:
Anik and Angie:
Irina and Ihn:

QUADRATIC FUNCTIONS:
Use Desmos, Geogebra calculator -

Geometry
- Defn of parabola (geometric defn - focus/directrix)
- transformations: translation, reflection, rotation, dilation

Algebra
- quadratic equations (standard form ax^2+bx+c)
- form related to translations/etc.
- factored form - connections w/ roots
- completing the square

GROUP CHALLENGES:

1. What happens to the equation of the parabola when we apply a translation? What about a dilation?
- translation of x^2 by (5,2)
    - dilation by factor of 3

2. What is the equation of a parabola with a given focus and directrix?
- formula for parabola with focus/directrix given?
    - focus (0,5), directrix x=-5

3. What is the connection between the roots of a quadratic equation and the formula for the equation?

- GOAL: If the quadratic equation f(x)=ax^2+bx+c has roots r1 and r2, write a formula for f(x) involving the numbers r1 and r2 (you can also use a)

- connection between a,b,c in ax^2+bx+c versus r1, r2 in a(x-r1)(x-r2)

Day 10, Quadratic Functions and Equations

Wu A&G, Chapter 2

Do we continue the activity from last time? Might need some scaffolding if we are going to reach the "conclusion" theorems?

The equation of the parabola with focus (p,q) and directrix y=l is:
(x-p)^2 + p^2-q^2 = 2(q-l)y

Axis of symmetry: x=-b/2a

(Lecture notes here)
GROUP WORK: handouts/problems here

OpenLab assignment: A new way to factor

https://www.nytimes.com/2020/02/05/science/quadratic-equations-algebra.html
The idea behind this method comes down to the following facts, given ax^2 + bx + c = 0

1. *The axis of symmetry (also x-coordinate of vertex) is k=-b/2a*
2. *The roots are symmetric around the axis of symmetry, so r1=k-u and r2=k+u for some u.*
3. *The product of the roots is c/a. This gives k^2 - u^2 = c/a. Solve for u, then find r1 and r2.*

Day 11, Completing the square

Wu A&G, Chapter 3

* Completing the square
* Polynomials and their properties (number of roots)

THINKING QUESTION: Why do we spend so much time on quadratic functions, but relatively little time on cubic functions, quartic functions, quintic functions, and so on?

A syntactic property of solving equations.
Pay attention to *how many times the variable appears in an equation.*
Principle: to solve an equation, we must rearrange it so that the variable appears ONLY ONCE.

For each equation:
1. State how many times the variable appears.
2. Solve the equation.

$2x + 7 = 3$
$7x - 2x + 11 = -20$
$e^{3x} + 5 = 7$
$\sqrt{\frac{2x-5}{3}} = 4$
$(x + 3)(x - 2) = 0$
$(x + 3)(x + 5) = 24$
$(x + 4)^2 = 25$

Observation: If the variable appears only once, we can isolate it using a series of "inverse operations".
If the variable appears more than once, *we will have to use some method (trick) to reduce the number of times the variable appears.*

What kinds of mathematical operations / steps actually reduce the number of times a variable appears in an equation?
* combining like terms
* zero product property
* factoring? Complicated
    ** if one side of eq is zero and we factor the other side, we can use the zero product property
    ** if we factor *part* of the equation into a perfect square

We spend so much time on quadratic functions. You would think we would then spend just as much time on cubic functions, and then quartic functions, and quintic functions, and so on… but we don't! Why?

Wu, A&G, p121: After such a detailed study of linear and quadratic functions, one would expect another long chapter on cubic functions (polynomial functions of degree 3) and yet another on quartic functions (polynomial functions of degree 4), etc. The fact that this does not happen is a consequence of the fact that the theory loses its simplicity when the degree of the polynomial exceeds 2. Recall that the study of quadratic functions was greatly facilitated by the method of completing the square. There is no tool of comparable power and simplicity for polynomial functions of de- gree exceeding 2. Analogs of the quadratic formula for quadratic equations continue to hold for cubic and quartic equations, but they are unwieldy and therefore not particularly useful. It is a famous theorem of Abel and Galois that for equations of degrees exceeding 4, no analog of the quadratic formula exists. Consequently, we know far less about arbitrary polynomial functions. All we can do is to give a general discussion of the most basic properties of polynomial functions and rational functions.

Completing the square.

Wu, A&G, Preface pxiii

Moreover, TSM makes this topic (quadratic functions) more difficult than it needs to be, partly by presenting the technique of **completing the square** as a rote skill for one purpose only: getting the quadratic formula. Consequently, the quadratic formula ends up also being a rote skill and, likewise, the formula for the vertex of the graph of a quadratic function. In Section 2.1, however, we show that completing the square is the major idea that

(a) leads to the proof of the quadratic formula and the formula of the vertex of the graph (see page 75),

(b) proves that the graph of $f(x) = ax^2+bx+c$ is congruent to the graph of $f_a(x) = ax^2$,

(c) exhibits the commonality between the study of linear and quadratic functions, namely, the fact that both revolve around the shape of the graphs of the representative functions ax and ax^2 (see page 73), and

(d) can be approached from a different perspective that becomes generalizable to polynomials of all degrees (see page 80).

By presenting a comprehensive picture that makes sense of the diverse formulas and results, we hope to ease the learning of the subject of quadratic functions and equations.

COMPLETING THE SQUARE.
IDEA: If we give ourselves the power to add a constant to our expression, then we can always turn the sum of an "x^2" term and an "x" term $ax^2 + bx$ into a perfect square.

This uses the wonderful idea of *multiplication as area*.
3*5 (rectangle, 3x5)
x^2 (square)
(x+5)^2 (square)
x^2+6x

CHAPTER STRUCTURE:
Chapter 3. Polynomial and Rational Functions 121
- 3.1. Some basic facts about polynomials 121
  - Intermediate value theorem and odd-degree polynomial functions (p. 122)
  - The number of zeros of a polynomial function (p. 126)
- 3.2. Descartes' rule of signs 128
- 3.3. Rational functions

Day 12, Polynomials

Wu A&G, Chapter 3

IDEA of polynomials: (smallest) collection of functions containing constant functions & identity function, closed under +, -, *
DEFINITION of polynomials: functions of form

Polynomials,
FACT: If a polynomial $f(x)$ has a root *r*, then $f(x)$ can be written in the form $f(x)=(x-r)q(x)$, where $q(x)$ has degree one less than the degree of $f(x)$.

LOOK FOR SOME REGENTS QUESTIONS ABOUT POLYNOMIALS & RATIONAL FUNCTIONS?

Maybe have them work through a couple of problems, then discuss how they would go about presenting this material to a class?

Rational Functions:

Wu A&G, p 131: The interest in rational functions, as far as school mathematics is concerned, is twofold: they are only one step away from polynomials and are therefore worth knowing, and surprisingly, their graphs display features that are genuinely different from those of polynomials—they have asymptotes—and are therefore instructive for that reason.

The functions that come closest to polynomial functions are the rational functions, which are the quotients of polynomial functions; i.e., f is a rational function if there are polynomial functions p(x) and q(x), so that for each number x, f(x) is the division p(x)/q(x). Because we cannot divide by 0, it is understood that the domain of f is outside the (at most) n zeros of q(x), where n is the degree of q (Theorem 3.3 on page 126). Both in form and in substance, rational functions are to polynomial functions as rational numbers are to whole numbers.

Day 12, Rational Functions

Wu A&G, Chapter 3

Rational Functions:

Wu A&G, p 131: The interest in rational functions, as far as school mathematics is concerned, is twofold: they are only one step away from polynomials and are therefore worth knowing, and surprisingly, their graphs display features that are genuinely different from those of polynomials—they have asymptotes—and are therefore instructive for that reason.

The functions that come closest to polynomial functions are the rational functions, which are the quotients of polynomial functions; i.e., f is a rational function if there are polynomial functions p(x) and q(x), so that for each number x, f(x) is the division p(x)/q(x). Because we cannot divide by 0, it is understood that the domain of f is outside the (at most) n zeros of q(x), where n is the degree of q (Theorem 3.3 on page 126). Both in form and in substance, rational functions are to polynomial functions as rational numbers are to whole numbers.

ACTIVITY: Maybe a treasure hunt - give a polynomial function satisfying the following?

LOOK FOR SOME REGENTS QUESTIONS ABOUT POLYNOMIALS & RATIONAL FUNCTIONS?

Maybe have them work through a couple of problems, then discuss how they would go about presenting this material to a class?

Day 15, Exponential & Logarithmic functions

Wu A&G, Chapter 4

We have a clear understanding of numbers raised to positive integer powers. How do we extend this to other numbers (rationals, reals?).

INTERPOLATION: The general problem -- what if we have a function C(n) defined on positive integers, for example C(n)=65n.
If we plot the graph of this function, it looks like dots… How do we connect them?
In this case, straight line - so F(x)=65x

This is quite similar to what happens in statistics - we have some data (that's like our C(n)), and we are trying to find a function that "best fits" -- linear regression, exponential regression, etc. However, in statistics, we don't usually get a function that actually perfectly fits our data (we don't have F(n)=C(n) for all n, just that it's "overall as close as possible")

Defn. of a^n for positive integers n (positive real number a).
Laws of exponents apply WITHIN positive integers:
1
2
3

Our goal is to extend the notion of exponents so that we can use any real number as an exponent -- that is, we want F(x) such that for positive integers n, F(n)=C(n).
Is it possible to do it in such a way that we preserve the lawas of exponents?

(Ex: linear interpolation of 2^n -- show why it fails)

Theorem 4.1: Fix a positive number a. There is a unique *continuous* function a^x on reals that
1. extends the exponential function a^n and
2. Satisfies a^s a^t = a^(s+t)

Day 16, Polynomial Forms and Complex Numbers

Wu A&G, Chapter 5

Chapter 5. Polynomial Forms and Complex Numbers
5.1. Polynomial forms
5.2. Complex numbers
5.3. Fundamental theorem of algebra 196
5.4. Binomial theorem 203

Rational Roots.
Thm. If a polynomial f(x) has rational root r=p/q, then (qx-p) is a factor of f(x).
Rational Root thm.

Complex numbers.
Recall: What is a number? (A point on the line).
We define addition/subtraction/multiplication/division of points on the line. This gives us the real numbers R.

Now, in the coordinate plane each point is given by a pair of real numbers (a,b). How can we add/subtract/multiply/divide points in the plane?
Add/Subtract (think in terms of vectors).
Multiplication? We need a rule for multiplying. What about:
(a,b)(c,d)=(ac, bd) ?? (multiply each coordinate, like real numbers?)
- Why isn't this ideal?

GENERAL NOTES

Big threads for this course:
1. do you know how to do the material on the NYS regents exams? (e.g. “operational” - do the problems). “What/How”
2. do you know the “story behind” the operations in (1)? That is, can you present these operations in the context of a coherent story, building up ideas about math, consistent with the Fundamental Properties of Mathematics? “Why”
3. Understand flaws in the Textbook School Mathematics presentation of (2), and explore alternatives.

Quotes from Wu’s RLE

Preface:

Most educators may have suspected that there must be more to school mathematics than TSM (“Textbook School Mathematics”), but without access to an exposition of school mathematics with mathematical integrity, their suspicion remains just that, a suspicion.

Then in 1994, Alan Schoenfeld made a scholarly statement with the clear implication that, while a school mathematics curriculum with mathematical integrity was certainly possible, we did not have it yet. What he wrote was, "Proof is not a thing separable from mathematics, as it appears to be in our curricula . . . . And I believe it can be imbedded in our curricula, at all levels" ([Schoenfeld1994, page 76])

Before proceeding further, we first explain what mathematical integrity is because this concept is coming into focus,. We say a mathematical exposition has mathematical integrity if it embodies the following five qualities:
(a) Definitions: Every concept is clearly and precisely defined so that there is no ambiguity about what is being discussed. (See the quote from Gibson at the beginning of this preface.)
(b) Precision: All statements are precise, especially the hypotheses that guarantee the validity of a mathematical assertion, the reasoning in a proof, and the conclusions that follow from a set of hypotheses.
(c) Reasoning: All statements[4] other than the unavoidable basic assumptions are supported by reasoning.[5]
(d) Coherence: The basic concepts and skills are logically interwoven to form a single fabric, and the interconnections among them are consistently revealed.
(e) Purposefulness: The mathematical purpose behind every concept and skill is clearly brought out so as to leave no doubt about why it is where it is.
These we call the **Fundamental Principles of Mathematics**.

To summarize, if we want students to be taught mathematics that is learn- able, then we must discard TSM and replace it with the kind of mathematics that possesses these five qualities:

1. Every concept has a clear definition.
2. Every statement is precise.
3. Every assertion is supported by reasoning.

4. Its development is coherent.
5. Its development is purposeful.

The foundational documents of the NCTM reform are the two sets of standards: the 1989 [NCTM1989] and the 2000 [PSSM]. … With the hindsight of thirty years, we can see all too clearly the obstacles that confronted the NCTM reform. With students, teachers, and educators completely immersed in TSM, the clarion call for coherence, reasoning, and proof might as well have been stated in a foreign language. Most of them had no conception of what those words meant.

For example, page 96 of [NCTM1989] suggests that the addition of fractions—inscrutable as it is in TSM—has to be approached gingerly, and neither [NCTM1989] nor [PSSM] points out the profound error of using the least common denominator for the addition of fractions (see page 41 below for an explanation of this error).

2010 saw the release of CCSSM, the Common Core State Standards for Mathematics ([CCSSM])

It did not help that the CCSSM agenda also left out the critical component of professional development for teachers, thereby creating the same sense of bewilderment in classrooms across the land (see [Education Week], [Loewus1], [Loewus2], and [Sawchuk]). It would seem that CCSSM is repeating the same mistake as the NCTM reform by not taking seriously the need to offer sustained, large-scale professional development for teachers to help with its implementation. With the publication of these three volumes, at least one complete exposition of school mathematics with mathematical integrity—an exposition that is also consistent in the main with CCSSM—will be available to provide the needed guidance for this kind of professional development, but will these volumes be too little too late? Only time will tell.

IDEA: Have students research a mathematical concept as presented in TSM, then discuss/critique/present an alternative?

To The Instructor:

There is no essential difference between problem solving and theorem proving in mathematics. Each time we solve a problem, we in effect prove a theorem (trivial

To the pre-service teacher:

It is therefore entirely up to you to achieve mastery of everything in the text itself. One way to check is to pick a theorem at random and ask yourself: Can I prove it without looking at the book? Can I explain its significance? Can I convince someone else why it is worth knowing? Can I give an intuitive summary of the proof? These are questions that you will have to answer as a teacher. To the extent possible, these volumes try to provide information that will help you answer questions of this kind. I may add that the most taxing part of writing these volumes was in fact to do it in a way that would allow you, as much as possible, to adapt them for use in a school classroom with minimal changes. (

Therefore, a preliminary suggestion to help you master the content of these volumes is for you to
copy out the statements of every definition, theorem, proposi- tion, lemma, and corollary, along with page references so that they can be examined in detail when necessary,
and also to
form the habit of summarizing the main idea(s) of each proof.
These are good study habits. When it is your turn to teach your students, be sure to pass along these suggestions to them.

IDEA: ELI5 various concepts? Or explain in 5 different levels (or “3 different levels”)?

Resources

SYLLABUS

TEXT:
Wu, H. (2015). Mathematics of the Secondary School Curriculum. UC, Berkeley.
These course notes were later published as three volumes:

- **Wu, Rational Numbers to Linear Equations (RLE)**
- **Wu, Algebra and Geometry (A&G)**
- **Wu, Pre-calculus, Calculus, and Beyond (PCC)**

RECOMMENDED:
Fomin, D., Genkin, S., Itenberg, I. (1996). Mathematical Circles: Russian Experience (Mathematical World, Vol. 7) American Mathematical Society.

RESOURCES:

- https://www.engageny.org/common-core-curriculum - common core curriculum details (math includes grade-by-grade details)
- https://www.illustrativemathematics.org/ - I think this is a vendor offering "IM Certified" math curriculum. Might be worth touching base with Prof. Kennedy & Douglas about this?
- http://ime.math.arizona.edu/progressions/ - The Common Core State Standards in mathematics were built on progressions: **narrative documents describing the progression of a topic across a number of grade levels**, informed both by research on children's cognitive development and by the logical structure of mathematics. These documents were spliced together and then sliced into grade level standards.
- http://map.mathshell.org/ - Mathematics Assessment Project - ASSESSING 21ST CENTURY MATH. The Mathematics Assessment Project is part of the Math Design Collaborative initiated by the Bill & Melinda Gates Foundation. The project set out to design and develop **well-engineered tools for formative and summative assessment that expose students' mathematical knowledge and reasoning**, helping teachers guide them towards improvement and monitor progress. The tools are relevant to any curriculum that seeks to deepen students' understanding of mathematical concepts and develop their ability to apply that knowledge to non-routine problems.

NY Regents Math Exam resources:

- https://www.nysedregents.org/regents_math.html - official site? Has many past exam marking materials: Rubrics, model responses (samples of student work at different levels)
- https://www.engageny.org/resource/regents-exams-mathematics - also official-ish? Lots of resources: common core standards, test guides, course overviews, sample problems

- https://www.nysl.nysed.gov/regentsexams.htm - NY State Library repository of past Regents exams - lots of math regents exams, some dating back 100 years!

GRADING PROCEDURE:
Final course grade will be based on the following:
Homework Assignments 20%
Class Participation/Work 10%
Exam 1 10%
Midterm 25%
Final Exam 35%

Exam 1 will consist of problems selected from the New York State Regents exam. It will tentatively be scheduled for the third week of classes. Students who do not achieve at least 85% on the exam will have two opportunities to pass a make-up exam. A student who does not pass exam #1 with a minimum score of 85% cannot receive a passing grade for the course.

NOTE: Looks like there are three Math Regents exams, Algebra I, Geometry, and Algebra II.

TEACHING AND LEARNING METHODS:
Guided whole-class discussions
Problem solving
Group Work (including hands-on activities)

COURSE OUTLINE:
Technology use (mastery of GeoGebra, Desmos and TI-86 is required).

SCHEDULE:

SESSIONS	UNIT	REFERENCES
Week 1	Fractions and Rational Numbers	Chapter 1&2, Wu
Week 2	The Euclidean Algorithm; Basic Isometries and Congruence	Chapter 3&4, Wu
Week 3	Dilation and Similarity; **Exam 1**	Chapter 5, Wu
Week 4	Symbolic Notation and Linear Equations; Linear Functions	Chapter 6, Wu Chapter 7, Wu
Week 5	Quadratic Functions and Equations	Chapter 8, Wu
Week 6	Polynomial and Rational Functions	Chapter 9, Wu
Week 7	Exponential and Logarithmic Functions; **Midterm Exam**	Chapter 10, Wu
Week 8	Polynomial Forms and Complex Numbers	Chapter 11, Wu
Week 9	Basic Theorems of Plane Geometry	Chapter 12, Wu
Week 10	Ruler and Compass Constructions	Chapter 13, Wu
Week 11	Trigonometry	Chapter 15, Wu
Week 12	Length and Area	Chapter 18, Wu
Week 13	3-Dimensional Geometry and Volume	Chapter 19, Wu
Week 14	Combinatorics; Probability	*Fomin et al.*

Week 15	Review; Final Exam	

Wu text

“Mathematics of the Secondary School Curriculum” (pre-publication title - maybe developed as lecture notes?)
3 volumes.
Math151
Math152A

Wu’s subsequently published volumes I&II in this series:

- Teaching School Mathematics: Pre-Algebra
- Teaching School Mathematics: Algebra

And finally produced this three-volume series (which I think is the most recent/authoritative version - I’ll use this for lesson planning etc).

- **Wu, Rational Numbers to Linear Equations (RLE)**
- **Wu, Algebra and Geometry (A&G)**
- **Wu, Pre-calculus, Calculus, and Beyond (PCC)**

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Fomin, D., Genkin, S., Itenberg, I. (1996). Mathematical Circles: Russian Experience (Mathematical World, Vol. 7) American Mathematical Society.

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Mathematics of the Secondary School Curriculum, III
Chapter 15: Trigonometry
~~Chapter 16: The Concept of Limit~~
~~Chapter 17: The Decimal Expansion of a Number~~
Chapter 18: Length and Area
Chapter 19: 3-Dimensional Geometry and Volume
~~Chapter 20: Derivative and Integral~~
~~Chapter 21: Exponents and Logarithms, Revisited~~

Wu comments/questions/errata

Wu RLE:

- P10, definition of fractions: *"The collection of all the sequences of n-ths, as n runs through the nonzero whole numbers 1, 2, 3, . . . , is called the fractions."* Some weirdness here about saying "collection of sequences" while intending "collection of points in sequences". This defines a fraction as "a sequence of n-ths" -- I think we really want something like "the collection of all points in the sequences of n-ths"
- P121, properties of \leq:

  Reflexive property of "$\leq$". *If $x \leq y$ and $y \leq x$, then $x = y$.*

  I'd say this property is antisymmetry. (reflexive says that every x in the domain is related to itself by \leq, i.e. "forall x, x \leq x").
- P147, Goldbach Conjecture. "*The Goldbach Conjecture, dating back to 1742, states that there are in fact an infinite number of twin primes; this conjecture remains unsolved.*" The Goldbach Conjecture states that every even number is the sum of two primes (the stated conjecture is the Twin Primes Conjecture).

Wu A&G:

- P122. defn of polynomial: "*Recall that a function of the form f(x) = anxn + an−1xn−1 + ··· + a1x + a0 for fixed numbers an, ..., a0 (an ≠ 0) is called a polynomial function (see page 4) and n is called its degree."* seems to exclude the constant polynomial y=0, as it has no nonzero coefficients (also, what is the degree of y=0?). But without y=0, the polynomials are not closed under addition.

Meeting with Nadia Kennedy 2/8/22 re: Regents requirement

- First time we are teaching the course @ City Tech
- Relying on it to collect data for the accreditation.
- That's purpose of the regents-material exam. The idea is that the test there will be approximating the **content specialty exam** (NY State - effectively Pearson).
- At stonybrook: give the students the exam once, if they fail, give them a makeup a few weeks later.
- Need to track exam results by categories.
    - NADIA: Will send a list of areas from content specialty exam.
- Heads up: They struggle more with Geometry - had a student who had never taken Geometry before.
- On the content specialty test, some calculus may appear (NOT my responsibility)
- There is also an open-ended response to a pedagogical item (NOT my responsibility).
- Right now, they are taking a CST workshop on the content specialty exam w/ Armando Cosmo (strongly recommended to all students that they attend - Fridays in PM).
- Nadia's ambition - these folks will get vouchers so they can do the exam - could mention this to them.
- Test creation: in previous years, have used a test that was similar to what Andrew Douglas created.
- Discussion about course goals etc:
    - Nadia: would be helpful to have THEM explain topics.
    - Are we doing group work?
    - When N would teach this course - would sacrifice breadth of content.

Regents Content exam planning

The tables w/ content emphases on each exam are taken from EngageNY "Test Guides".

In-class exam: 24 multiple choice (8 alg i, 8 alg ii, 8 geo), 6-8 short answer

Summary of "Major Cluster Emphases" per exam:

Algebra I			Regent Problems
Number & Quantity 2-8%			
Algebra 50-56%	Seeing Structure in Expressions	Interpret the structure of expressions	3
	Arithmetic with Polynomials and Rational Expressions	Perform arithmetic operations on polynomials	13

	Creating Equations	Create equations that describe numbers or relationships	17
	Reasoning with Equations and Inequalities	Understand solving equations as a process of reasoning and explain the reasoning.	29, 37
		Solve equations and inequalities in one variable.	18
		Represent and solve equations and inequalities graphically.	19
Functions 32-38%	Interpreting Functions	Understand the concept of a function and use function notation.	5
		Interpret functions that arise in application in terms of the context.	
Statistics & Probability 5-10%	Interpreting Categorical and Quantitative Data	Interpret linear models.	2
Algebra II			
Number & Quantity 5-12%	The Real Number System	Extend the properties of exponents to rational exponents.	12
Algebra 35-44%	Seeing Structure in Expressions	Interpret the structure of expressions.	3
		Write expressions in equivalent forms to solve problems.	14
	Arithmetic with Polynomials and Rational Expressions	Understand the relationship between zeros and	4

		factors of polynomials.	
	Reasoning with Equations and Inequalities	Understand solving equations as a process of reasoning and explain the reasoning.	33
		Represent and solve equations and inequalities graphically.	16
Functions 30-40%	Interpreting Functions	Interpret functions that arise in application in terms of the context.	22
	Building Functions	Build a function that models a relationship between two quantities.	
Statistics & Probability 14-21%	Making Inferences and Justifying Conclusions	Make inferences and justify conclusions from sample surveys, experiments, and observational studies.	18
Geometry			
Congruence 27% - 34%	Understand congruence in terms of rigid motions		2, 28
	Prove geometric theorems		10
Similarity, Right Triangles, & Trigonometry 29% - 37%	Understand similarity in terms of similarity transformations		23
	Prove theorems involving similarity		12
	Define trigonometric ratios and solve problems involving right triangles		6

Circles 2% - 8%			18
Expressing Geometric Properties with Equations 12% - 18%	Use coordinates to prove simple geometric theorems algebraically		17, [28]
Geometric Measurement & Dimensions 2% - 8%			
Modeling with Geometry 8% - 15%	Apply geometric concepts in modeling situations		(6)

ALGEBRA 1 CONTENT EMPHASES

Conceptual Category	Domain	Cluster	Cluster Emphasis	Standard	Shared with Algebra II
Number & Quantity 2% - 8%	Quantities	Reason quantitatively and use units to solve problems.	Supporting	N-Q.1	
				N-Q.2	x
				N-Q.3	
	The Real Number System	Use properties of rational and irrational numbers.	Additional	N-RN.3	
Algebra 50% - 56%	Seeing Structure in Expressions	**Interpret the structure of expressions.**	**Major**	A-SSE.1	
				A-SSE.2	x
		Write expressions in equivalent forms to solve problems.	Supporting	A-SSE.3	x (c)
	Arithmetic with Polynomials and Rational Expressions	**Perform arithmetic operations on polynomials.**	**Major**	A-APR.1	
		Understand the relationship between zeros and factors of polynomials.	Supporting	A-APR.3	x
	Creating Equations	**Create equations that describe numbers or relationships.**	**Major**	A-CED.1	x
				A-CED.2	
				A-CED.3	
				A-CED.4	
	Reasoning with Equations and Inequalities	**Understand solving equations as a process of reasoning and explain the reasoning.**		A-REI.1	x
		Solve equations and inequalities in one variable.		A-REI.3	
				A-REI.4	x (b)
		Represent and solve equations and inequalities graphically.		A-REI.10	
				A-REI.11	x
				A-REI.12	
		Solve systems of equations.	Additional	A-REI.5	
				A-REI.6	x
Functions 32% - 38%	Interpreting Functions	**Understand the concept of a function and use function notation.**	**Major**	F-IF.1	
				F-IF.2	
				F-IF.3	x
		Interpret functions that arise in application in terms of the context.		F-IF.4	x
				F-IF.5	
				F-IF.6	x
		Analyze functions using different representations.	Supporting	F-IF.7 (a,b)	
				F-IF.8 (a)	
				F-IF.9	x
	Building Functions	Build a function that models a relationship between two quantities.		F-BF.1 (a)	x
		Build new functions from existing functions.	Additional	F-BF.3	x
	Linear, Quadratic and Exponential Models	Construct and compare linear, quadratic, and exponential models and solve problems.	Supporting	F-LE.1	
				F-LE.2	x
				F-LE.3	
		Interpret expressions for functions in terms of the situation they model.		F-LE.5	x
Statistics & Probability 5% - 10%	Interpreting Categorical and Quantitative Data	**Interpret linear models.**	**Major**	S-ID.7	
				S-ID.8	
				S-ID.9	
		Summarize, represent and interpret data on two categorical and quantitative variables.	Supporting	S-ID.5	
				S-ID.6	x (a)
		Summarize, represent and interpret data on a single count or measurement variable.	Additional	S-ID.1	
				S-ID.2	
				S-ID.3	

ALGEBRA II CONTENT EMPHASES

Conceptual Category	Domain	Cluster		Cluster Emphasis	Standard	Shared with Algebra I
Number & Quantity 5% - 12%	The Real Number System	**Extend the properties of exponents to rational exponents.**	**N-RN.A**	**Major**	N-RN.1	
					N-RN.2	
	Quantities	Reason quantitatively and use units to solve problems.	N-Q.A	Supporting	N-Q.2	x
	The Complex Number System	Perform arithmetic operations with complex numbers.	N-CN.A	Additional	N-CN.1	
					N-CN.2	
		Use complex numbers in polynomial identities and equations.	N-CN.C		N-CN.7	
Algebra 35% - 44%	Seeing Structure in Expressions	**Interpret the structure of expressions.**	**A-SSE.A**	**Major**	A-SSE.2	x
		Write expressions in equivalent forms to solve problems.	**A-SSE.B**		A-SSE.3	x (c)
					A-SSE.4	
	Arithmetic with Polynomials and Rational Expressions	**Understand the relationship between zeros and factors of polynomials.**	**A-APR.B**		A-APR.2	
					A-APR.3	x
		Use polynomial identities to solve problems.	A-APR.C	Additional	A-APR.4	
		Rewrite rational expressions.	A-APR.D	Supporting	A-APR.6	
	Creating Equations	Create equations that describe numbers or relationships.	A-CED.A	Supporting	A-CED.1	x
	Reasoning with Equations and Inequalities	**Understand solving equations as a process of reasoning and explain the reasoning.**	**A-REI.A**	**Major**	A-REI.1	x
					A-REI.2	
		Solve equations and inequalities in one variable.	A-REI.B	Supporting	A-REI.4	x (b)
		Solve systems of equations.	A-REI.C	Additional	A-REI.6	x
					A-REI.7	
		Represent and solve equations and inequalities graphically.	**A-REI.D**	**Major**	A-REI.11	x
	Expressing Geometric Properties with Equations*	Translate between the geometric description and the equation for a conic section.	G-GPE.A*	Additional	G-GPE.2*	
Functions 30% - 40%	Interpreting Functions	Understand the concept of a function and use function notation.	F-IF.A	Supporting	F-IF.3	x
		Interpret functions that arise in application in terms of the context.	**F-IF.B**	**Major**	F-IF.4	x
					F-IF.6	x
		Analyze functions using different representations.	F-IF.C	Supporting	F-IF.7 (c,e)	
					F-IF.8 (b)	
					F-IF.9	x
	Building Functions	**Build a function that models a relationship between two quantities.**	**F-BF.A**	**Major**	F-BF.1 (a)	x
					F-BF.1 (b)	
					F-BF.2	
		Build new functions from existing functions.	F-BF.B	Additional	F-BF.3	x
					F-BF.4 (a)	
	Linear, Quadratic and Exponential Models	Construct and compare linear, quadratic, and exponential models and solve problems.	F-LE.A	Supporting	F-LE.2	x
					F-LE.4	
		Interpret expressions for functions in terms of the situation they model.	F-LE.B	Additional	F-LE.5	x
	Trigonometric Functions	Extend the domain of trigonometric functions using the unit circle.	F-TF.A	Additional	F-TF.1	
					F-TF.2	
		Model periodic phenomena with trigonometric functions.	F-TF.B		F-TF.5	
		Prove and apply trigonometric identities.	F-TF.C		F-TF.8	
Statistics & Probability 14% - 21%	Interpreting Categorical and Quantitative Data	Summarize, represent and interpret data on a single count or measurement variable.	S-ID.A	Additional	S-ID.4	
		Summarize, represent and interpret data on two categorical and quantitative variables.	S-ID.B	Supporting	S-ID.6 (a)	x
	Making Inferences and Justifying Conclusions	Understand and evaluate random processes underlying statistical experiments.	S-IC.A	Supporting	S-IC.1	
					S-IC.2	
		Make inferences and justify conclusions from sample surveys, experiments, and observational studies.	**S-IC.B**	**Major**	S-IC.(3-6)	
	Conditional Probability and the Rules of Probability	Understand independence and conditional probability and use them to interpret data.	S-CP.A	Additional	S-CP.(1-5)	
		Use the rules of probability to compute probabilities of compound events in a uniform probability model.	S-CP.B	Additional	S-CP.(6,7)	

GEOMETRY CONTENT EMPHASES:

Conceptual Category	Domain	Cluster	Cluster Emphasis	Standard
Geometry	Congruence 27% - 34%	Experiment with transformations in the plane	Supporting	G-CO.1 G-CO.2 G-CO.3* G-CO.4 G-CO.5
		Understand congruence in terms of rigid motions	**Major**	G-CO.6 G-CO.7 G-CO.8
		Prove geometric theorems		G-CO.9* G-CO.10* G-CO.11*
		Make geometric constructions	Supporting	G-CO.12* G-CO.13
	Similarity, Right Triangles, & Trigonometry 29% - 37%	**Understand similarity in terms of similarity transformations**	**Major**	G-SRT.1a G-SRT.1b G-SRT.2 G-SRT.3
		Prove theorems involving similarity		G.SRT.4* G.SRT.5*
		Define trigonometric ratios and solve problems involving right triangles		G.SRT.6 G.SRT.7 G.SRT.8
	Circles 2% - 8%	Understand and apply theorems about circles	Additional	G.C.1 G.C.2* G.C.3
		Find arc lengths and areas of sectors of circles		G.C.5
	Expressing Geometric Properties with Equations 12% - 18%	Translate between the geometric description and the equation for a conic section	Additional	G.GPE.1
		Use coordinates to prove simple geometric theorems algebraically	**Major**	G.GPE.4 G.GPE.5 G.GPE.6 G.GPE.7
	Geometric Measurement & Dimensions 2% - 8%	Explain volume formulas and use them to solve problems	Additional	G.GMD.1 G.GMD.3
		Visualize relationships between two-dimensional and three-dimensional objects		G.GMD.4
	Modeling with Geometry 8% - 15%	**Apply geometric concepts in modeling situations**	**Major**	G.MG.1 G.MG.2 G.MG.3

UNUSED MATERIAL/IDEAS: PRIOR SEMESTERS & DRAFTS

Class 1

Some concepts of fraction (maybe this could be day 1 group activity? Followup on day 2 with the actual definition of fractions a la TSM?)

- Division. Top number divided by bottom number. Top number divided into "bottom number" of pieces - size of one piece.
  - Canonical Examples:: dividing up 3 lbs flour among 5 people
  - Could I do this with sand? And kitchen scale?
- Multiplication. Take one, divide into denominator-many pieces. Then take top-number-many of these.
  - Divide 2 bananas among 3 people
  - Could do this with candy bars?
- Ratio. Top number forms a certain portion, or percentage, of bottom number. When we take "fraction *of* number", we are asking for the same portion, or percentage, of the number.
  - 2/7 of 20? Think of a practical example? The age of A man's son is 1/4 of the age of the man. If the man is 35, how old is the son?
  - What's a nice "hands on" version?
  -

"Natural numbers were created by God, everything else is the work of men."

> - Leopold Kronecker "a German mathematician who worked on number theory and algebra" in a lecture for the Berliner Naturforscher Versammlung (1886).

READING/WRITING assignment: READ Wu on TSM(?).

ASSIGNMENT: Do Algebra Regents problems 1-12 (mult choice)? (grade your responses? Maybe write a written response?)

Discussion (think/pair/share? Or general discussion?): How was your own math education grades 6-12 (middle + high school)? What flaws did it have? What did you like about it? What could be improved?

TSM - Textbook School Mathematics (OPENLAB ASSIGNMENT)

- TSM - what it is, what are its flaws, how to address them?
- maybe point people to this blog post: https://scottbaldridge.net/2013/02/06/eradicating-textbook-school-mathematics-tsm-2/
- I also like these slides from a talk Wu gave on these ideas: https://math.berkeley.edu/~wu/Stony_Brook_2014.pdf
- and here's a 16 minute video of Wu speaking on TSM: https://www.youtube.com/watch?v=nUalvG0EUEU
- maybe some thoughts on reading Wu? (i.e. it's not always clear to me what his criticisms refer to!)

The fundamental principles of mathematics (see discussion of these wrt TSM above)

1. Every concept has a clear definition.
2. Every statement is precise.
3. Every assertion is supported by reasoning.
4. Its development is coherent.
5. Its development is purposeful.

- Prerequisites - where are we starting from? (what assumed knowledge, what assumed sophistication?). SHARE THIS PAGE OF WU?

- QUOTES:

By guiding teachers and educators systematically through the correction of TSM errors on a case-by-case basis, we believe they will gain a new and deeper understanding of school mathematics.

There is no essential difference between problem solving and theorem proving in mathematics. Each time we solve a problem, we in effect prove a theorem (trivial as that theorem may sometimes be).

Nevertheless, our most urgent task—the fundamental task—in the mathematical education of teachers and educators as of 2020 has to be the reconstruction of their mathemat- ical knowledge base. This is not about judiciously tinkering with what teachers and educators already know or tweaking their existing knowledge here and there. Rather, it is about **the hard work of replacing their knowledge of TSM with mathematics that is consistent with the fundamental principles of mathematics from the ground up**. The primary goal of these three volumes is to give a detailed exposition of school mathematics in grades 9–12 to help educators and teachers achieve this reconstruction.

----

challenges of "forgetting" stuff you know - if we are going to talk about fractions "from scratch", you'll have to (temporarily) disable the fraction knowledge in your brain - what are you taking for granted?)

Class 2

NOTE: Didn’t cover the material below in this class - next time?

Why Fractions?

Why start with fractions? Our students are exposed to fractions in grades 3-6, so much of this should be review. However, fractions are difficult topic for many students, especially as presented in TSM (Textbook School Mathematics), and you will no doubt be asked to explain various aspects of fractions many times in your work with secondary school students. Having a coherent presentation, or story, of what the fractions are is essential!

The following short excerpt is from Wu’s text in a section called Leaving the Past Behind.
Beyond pizzas, the most common definition TSM has to offer for fractions is "parts of a whole". For students, the difficulty with the conception of a fraction as "parts of a whole" is multifaceted:
(1) The concept of a "whole" is elusive. TSM never defines what a "whole" is. It is many things, and thus a moving target. A concept this nebulous cannot serve as a solid foundation for learning fractions.
(2) A fraction is a number that you compute with, but TSM does not explain in what sense "parts of a whole" is a "number" and how to compute with "parts of a whole". For example, how does "parts of a whole" logically lead to invert and multiply in the division of fractions?

(3) "Parts of a whole" is at least two things: the "parts" and then there is the "whole". It is difficult to conceptualize a single number as two separate pieces.
(4) There is a psychological issue. Since "the whole" means "the whole thing", how can you have more than the whole thing as in the case of 3/2?

Quotes:

TSM seems never to discuss how teachers and their students should deal with real numbers. Its main instructional strategy seems to be one that exploits students' passive willingness to accept orders from authority figures about what to do, reasonable or not, and since this strategy has been implemented since kindergarten, TSM has a high expectation of compliance. Therefore, having drilled students on formal computations with fractions—whose numerators and denominators are whole numbers—TSM is comfortable asking students to simply believe that the same computations can also be carried out when the numerators and denominators are real numbers. No explanation given, of course.

In order to make school mathematics learnable, it is essential that we stop such underhanded maneuvers and be explicit about what we ask students to assume. This is what we have come to call the Fundamental Assumption of School Mathematics (FASM).

…..

Finally, in reading these two chapters (Fractions and Rational Numbers), please keep in mind that the emphasis here will not be on individual facts or skills. For example, it is taken for granted that you are entirely comfortable with identifying the fractions and rational numbers as certain points on the x-axis (called "the number line" in school mathematics) and are fluent in the four arithmetic operations with rational numbers. Rather, the emphasis will be on t**he reconstruction of familiar facts about fractions and rational numbers into a new body of knowledge that is logical and coherent,** so that it can provide a framework for you to communicate with high school students who need help. You can be certain that, as of 2020, you will often be called upon to explain fractions and rational numbers to your students. We hope that you will take the time and trouble to become thoroughly familiar with these two chapters so that you will be able to make sense of fractions and rational numbers when explaining them to your students. This will be an important first step toward improving school mathematics education.

2022 LESSON OUTLINE:

- What issues is Wu addressing?
    - FASM
    - Leaving the Past Behind - pages 4&5

- WHAT ARE FRACTIONS
  - The number line (segments etc)
  - Defn fractions. Fractions are points on the number line!
- Equivalent fractions
  - Theorem on equivalent fractions ("The" fundamental fact about fractions!) - *"proof" - using the definition to offer explanation!*
  - Cross-multiplication algorithm
  - Fundamental Fact of Fraction-Pairs
- Adding and Subtracting Fractions
  - DEFN: concatenate segments
  - Addition of fractions
  - Cross-multiplication inequality
  - Subtraction of fractions
    - Comments re: LCD
- Multiplication of fractions
  - Defn of "1/2 of 3/4"
  - DEFN: IDEA of multiplication
  - Formal defn of multiplication
  - Product formula
- Dividing fractions
  - Definition

Class 3

GROUP ACTIVITY?

Maybe give different groups different questions, ask them to give a short presentation to the class afterwards? (maybe give different questions to each person - each person should be responsible for one question, present it to the class?)

1. What are numbers? Human mathematical thought began with countin (the natural numbers). Over time, we have gradually come to add more numbers to our toolbox, until we reach the number line (the real numbers).
   - What are the four main types of numbers, from counting up to the number line?
   - Each time the idea of numbers expands, it allows us to do more.
   - For each of the first three types of numbers:
     - Give an example of a number that is "missing", which leads us to the next type of number.
     - Give an example of a problem that couldn't be solved before, which can be solved with this new type of number.
2. What are the three basic building blocks that we rely on in our students to help us define and understand the "real numbers"?

- What is the definition of number? (what is the definition of number line?)
- How does the number line (just 2 points and line) define the location of every number?
  - Natural numbers
  - Integers
  - Fractions (?)

LECTURE/DISCUSSION
- symbol vs thing it represents?
- Given our definition of numbers as points on a number line (and our 3 basic building blocks), how do we do xxx (?? do this in groups?)

- Types of numbers. What distinguishes natural numbers, integers, rational numbers, real numbers?
  - This could be an activity? Or a solo writing prompt?
    - Give an example of an integer that is not a natural number. What can we achieve in the integers that we cannot achieve in the natural numbers?
    - Give an example of a fraction that is not an integer. What can we achieve in the fractions that we cannot achieve in the integers?
    - Give an example of a real number that is not a rational number. What can we achieve in the real numbers that we cannot achieve in the rational numbers?/What makes the real numbers different from the rational numbers?
- MIGHT BE WORTH DISCUSSING: The differences between a number (a point on the number line), and the way we write a number.
  - **Difference between a symbol and the thing it represents** (signifier and signified)
  - What is a number? *(point on the number line!!!)*
  - Are these numbers the same? *Give a series of examples. Ultimately, the guiding principle is "do they represent the same point on the number line?"*
  - *Whole number, rational number, decimal, infinite decimal.*
  - COULD DISCUSS: Are these two fractions the same "3/4" and "6/8"?
  - COULD DISCUSS: Is there a difference between a number, and the way we write the number?
- Numbers as points on a number line. RECALL DEFINITION.
- Given our definition of numbers as points on a number line (and our 3 basic building blocks), how do we: *NOTE: Maybe do one or two of these as a group??*
  - Define length of an interval. "Finding the length" means starting with an interval [a,b] and giving me back a *number* (recall: what is a number?)
    - Start with: an interval [a,b] between two numbers.

        - End with: a particular point *c* (a number) on the number line (the length of [a,b])
        - *Describe a step-by-step process*
    - Define addition of two numbers (in terms of the number line)
        - Start with: two numbers, a and b
        - End with: a number c, which is equal to the sum a+b
        - Ex: start with 5 and 3. What's the procedure (using just the number line and our basic building blocks)?
    - Define subtraction of two numbers (in terms of the number line)
        - Start with two numbers, a and b
        - End with: a number c which is equal to a-b.

Recall:

- BASIC BUILDING BLOCKS:
    - 1. Counting 1, 2, 3, 4, … This is one of the very first mathematical activities we learn – we learn it at an early age, and it sticks.
    - 2. Basic ideas about space. We live in physical world, and our intuitions about physical object and how they behave are hardwired into our brains. For example:
        - Two objects can't occupy the same place at the same time.
        - If I pick an object up and move it, the size doesn't change.
    - 3. Points and Lines. Even though we can't find actual physical examples of these in the world (they come from the more abstract world of geometry), we can still use our basic ideas about space to reason about them.
        - If two points occupy the sample place, they are the same point.
        - I can make a line segment by giving two points on a line.
        - I can slide a line segment along the line without changing its length (it sometimes helps to think about this as "making a copy of you line segment in a different place"
        - I can divide a line segment up into any number of equal pieces.
        - A line doesn't have any holes in it.

OpenLab: To the Pre-Service Teacher

Assignment, due Monday, 2/14/22: Read the following 3-page excerpt from the introduction of Wu's text. In it, he is speaking directly to you, pre-service teachers who are working through the secondary school math curriculum. Respond to the questions below by leaving a comment on this post.

Wu-To-the-Pre-Service-Teacher - Download

What do you think is the main purpose of the reading?
What is a quote that stood out for you in the reading? Explain.
Emotional self-awareness: Name an emotion you felt while you were reading the text. Describe what triggered that emotion. Do you think that emotion was useful? Why?
Extra Credit. Respond in some way to one of your fellow students' comments. Do you agree? Disagree? Did their comments make you think or provoke additional questions? Reminder: Be respectful, be kind.

Class 4

2022 NOTE: This semester at least I will skip Chapter 2: Rational Numbers, as I think much of the conceptual underpinnings are provided by Chapter 1 (the missing link is "how to deal with negatives"). Instead, I'll focus today on Chapter 3, Euclidean Algorithm - starting with a little discussion from Chapter 1 about LCD.

2022 NOTE TO SELF: Todays lesson went a little long, and seemed quite "technical" -- not sure how much it has done to enlighten the presentation of this material. How to address this? Would like to give a "sense" of the connection between reduced fractions, GCD, division with remainder, Euclidean Algorithm, and Fundamental Thm of Arithmetic

MAYBE RE-FRAME THE LESSON IN TERMS OF TWO BASIC IDEAS:
- linear combinations of a and b
- GCD of a and b

INTRO:
This section proves that every fraction has a unique reduced form, i.e., a fraction equal to the original fraction so that its numerator and its denominator have no common factor other than 1, and that there is a sequence of explicit steps to get the reduced form (Theorem 3.1 on page 139).
The proof of this theorem is as important as the theorem itself because it proves the Euclidean algorithm along the way.
The proof of the latter is, in turn, most interesting because it uncovers the mathematical potential of division-with-remainder, a mundane tool usually taught as a rote skill in TSM.

Pedagogical Comments pp 154/155 (why cover these proofs, when we wouldn't actually carry them out in a secondary school classroom?)

- Discussion about LCD(?)

- 3.1. The reduced form of a fraction 137
  - Divisors, GCD, and the reduced form of a fraction (p. 138)
  - The Euclidean algorithm (p. 139)
- 3.2. The fundamental theorem of arithmetic 147
  - Primes (p. 148)
  - Fundamental theorem of arithmetic (p. 149)

LESSON: The Euclidean Algorithm
Every fraction has a unique reduced form, and there is a sequence of explicit steps to obtain the reduced form.

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Why not use LCD in defining addition of fractions?
Why not use the Least Common Denominator to define the addition of fractions? Wu's discussion of this can be found here (look at both the Pedagogical Comments and the Mathematical Aside that follows):

Wu-Pedagogical-Comments-on-LCD
Download
Divisors, GCD, and the reduced form of a fraction
First, some terms we use when talking about integers and divisibility.

Basic Concepts of Divisibility
Definition. A nonzero integer d is a divisor or factor of an integer a if
 for some integer c. (NOTE: A divisor is nonzero by definition!)

We also say d divides a, written
.
We also say a is a multiple of d (in particular, a is an integral multiple of d).
We call an expression of a as a product,
, a factorization of a.

Definition. Consider two whole numbers a and b (not both 0). An integer d is a common divisor of a and b if d divides a and d divides b.

Definition. An integer d is the GCD (greatest common divisor) of whole numbers a and b if d is a common divisor of a and b and, among all common divisors of a and b, d is the greatest.
Notation:
.

Definition. Two whole numbers a and b are relatively prime if
.

Definition. A fraction

is said to be the reduced form of a given fraction

if

and m and n are relatively prime. We say such an

is in lowest terms, or reduced.

NOTE: Is GCD well-defined (that is, is it true that any two whole numbers have a greatest common divisor)? Proof idea – all whole numbers have at least one divisor d=1, and the set of divisors of a is bounded above by a, so there must be a largest number that divides both.

QUESTION: Is

reduced?

NOTE: unless the numerator and denominator are relatively small, it's not always easy to tell! If classroom instruction focusses on single-digit numerator/denominator, we get the impression we can always tell "just by looking" if a fraction is reduced.

QUESTION: What's
? The GCD of any nonzero integer and 0 is the nonzero integer itself.

NOTE: In the past (in our own primary and secondary education, for example, as well as in MAT 2571 Intro to Proofs), we simply accepted that every fraction has a reduced form. Now let's look at "why".

Reduced form of a fraction
Theorem 3.1. Every nonzero fraction has a unique reduced form. Furthermore, this reduced form can be obtained by an algorithm.

FIRST: Let’s see if we can find a reduced form of a fraction.
Observe: If

is a fraction and
. Then
and
for some integers m,n, and the reduced form of

is

.

QUESTION: So, how do we find the GCD? (What is an algorithm?)

Division with remainder
Given positive integers
and
, the division-with-remainder of
by
is given by the equation:
where
and
.

Where q is the quotient, d is the divisor, a is the dividend, and r is the remainder.

Linear combinations
An integer k is a linear combination of integers a and b if

for some integers m and n. (Also called an integral linear combination since m,n are required to be integers).

The Euclidean Algorithm
Theorem 3.2 (Euclidean algorithm). If a and d are positive integers, then
can be obtained by a finite number of applications of division-with- remainder. Furthermore,
is a linear combination of a and d.

Some facts about GCD
Lemma. If m is any integer, then

LEMMA 3.3. Given whole numbers
, and
so that

and
, the following equality holds:

Example: Find
.

Example: Find
and write the GCD as a linear combination of
and
.

QUESTION: Does this work for any pair of integers? How do we know the process will eventually terminate?

3.2 The Fundamental Theorem of Arithmetic
Primes
Definition. A proper divisor of a whole number a is an is a divisor of a which is strictly between 1 and a, (that is, a divisor d of a with
).

Definition. A whole number >1 with no proper divisors is called a prime number.

Definition. A whole number >1 that is not prime is called composite.

The following theorem shows that prime numbers are the basic building blocks of whole numbers when it comes to multiplication.

Fundamental theorem of arithmetic
THEOREM 3.6 (Fundamental theorem of arithmetic). Every whole number
is the product of a finite number of primes:
(the
‘s are not necessarily distinct). Moreover, this collection of primes
, counting the possible repetitions, is unique.

We call
the prime decomposition of n.

NOTE: This theorem has two parts, existence of a prime decomposition and uniqueness of the prime decomposition.
SKETCH of existence: start with n, if it’s prime we are done, if not it is composite so it has at least one prime factor p. Factor out p and repeat with what remains.
For Uniqueness, why do we care so much about uniqueness? Consider non-uniqueness of factorizations without the prime requirement:

What is it about primes that forces the prime decomposition to be unique??

The Fundamental Theorem of Arithmetic gives us a new perspective which, if we know the prime decompositions of the two numbers, is extremely useful for extracting information from the two numbers..

Example: Consider the numbers 129212216 and 213541888.

Are they relatively prime?
Find their Greatest Common Divisor.
Find their Least Common Multiple.
Example (cont'd): What if we are given the prime factorizations of the two numbers,
and
? Does this help us? How?

Finally, I'd like to look at some remarks by Wu on the content of this chapter – in particular, the fact that we present several proofs that would not realistically appear in a secondary school classroom

Wu-Pedagogical-Comments-on-GCD-EuclideanAlgo-FunThmArithmetic - Download

Day 6 - Geometry, Congruence

Chapter structure:

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  - Betweenness and definition of a segment (p. 166) Line separation (p. 172)
  - Plane separation (p. 175)
- 4.2. The basic vocabulary, Part 2 180
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  - Degrees of angles (p. 186)
  - Polygonal regions (p. 193)
- 4.3. Transformations of the plane 199
  - Why transformations (p. 199)
  - Rotations (p. 201)

    - Generalities about transformations (p. 205)
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- 4.4. The basic isometries: Rotations 216
    - Assumptions about rotations and first consequences (p. 217)
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- 4.5. The basic isometries: Reflections and translations 229
    - Reflections (p. 229)
    - Translations (p. 231)
    - Assumption (L7) about basic isometries (p. 236)
- 4.6. Congruence, SAS, and ASA 239
    - The definition and basic properties of congruence (p. 240)
    - Two congruence criteria for triangles (p. 244)
    - The crossbar axiom (p. 250)
- 4.7. A brief pedagogical discussion of proofs 252

Geometry – Basic Isometries and Congruence
The main issues [with TSMs treatment of geometry] may be summarized as follows:

The high school geometry course cannot be the only place in the K–12 curriculum where definitions, theorems, and proofs are taken seriously. To the extent that we are trying to teach students mathematics with mathematical integrity rather than some latter-day concoction that purports to be “mathematics”, we have to make the rest of the school mathematics curriculum take definitions, theorems, and proofs seriously too. The use of an axiomatic system in the high school geometry course is a more complex issue and will have to be handled with some care.
Congruence and similarity are not only the bedrock of the high school geometry course and a foundation of high school algebra, they are also a mainstay of the whole school geometry curriculum. We must find definitions for these two concepts so that they are usable in both middle school and high school.
The curricular decision by TSM to make students work with slope in introductory algebra before teaching them about similar triangles has caused great harm in students’ learning of slope (see [PG], for example). We have to provide students with the necessary mathematical knowledge about similar triangles before the introduction of slope. This will improve students’ ability to learn about slope and strengthen the relevance of the high school geometry course to the school mathematics curriculum.
– Wu, RLE

The concept of distance or length
Our geometric intuition is grounded in our experience of the real world. For many basic concepts in geometry, that intuition is a great (indeed, fantastic) tool! However, it can be helpful

to recognize some of the underlying assumptions we make about geometry and geometric objects that come from our real-world experience.

What does it mean for two line segments OR triangles OR other subsets of the plane to be "the same"?

Many basic concepts will not be defined here — for example, point, line, plane, angle, and so on (you can take a look at Wu RLE Chapters 4 and 5 for a detailed exposition of basic concepts). It's also important to note at the outset that the geometry curriculum does NOT make use of a coordinate plane (complete with x- and y- axes, etc), but just a "plain plane" – so things like coordinates of points, and the use of equations to describe lines and other figures, will not appear.

The concept of the length of a segment in the plane is far from simple; it can only be fully understood in the context of rotations, translations, and reflections of the plane.

WU, RLE
The basic notion that will describe so much of our work in the plane is called a transformation – a function from the plane to itself.

Transformation
Let
denote the plane. A transformation
of
is a rule that assigns to each point
of
a unique point
(read: "
of
") in
.

We also say
maps
to
or, sometimes,
moves
to
. Indeed, it is intuitively appealing to think of a transformation as a way of "moving" the points of the plane around.

Two examples of transformations, very different in character:

I. The identity transformation I which maps every point to itself (the “do nothing” transformation).

II. For a point
, the constant transformation
which sends every point in the plane to
.

Isometry
A transformation
is an isometry if it preserves distances/lengths. That is, if
for all
in
.

Is the identity transformation an isometry? How about a constant transformation?

NOTE: Preserving distance is a very strong property – in fact, any isometry also preserves angles, maps lines to lines and circles to circles, and so on.

The basic isometries
Rotation around a point
by angle
(
is measured in degrees, if
is positive then the rotation is counterclockwise, if
is negative the rotation is clockwise).

Reflection across a line
.

Translation along a vector

.

Note that we can do multiple transformations in a row, and the result is a new transformation – this exactly corresponds to composition of functions.

Question: If we do multiple isometries in a row, is the result an isometry?

Example

Suppose
is the rotation of

about the point
, and
is the reflection across the line

.

Locate
(referred to as the image of B, C, and A under the transformation
).
Locate
.
Now suppose we *first* do
, followed by
.

Locate
.
Locate
.
Congruence
Definition. A congruence is a transformation of the plane that is the composition of a finite number of basic isometries.

Congruent
A subset
of the plane is congruent to another subset
of the plane if there is a congruence
so that
. In symbols, we say
.

Homework 4: Algebra II part 2

Answers to the previous homework, Homework 3: Algebra 2 part 1 are here:

Assignment, due Monday, 2/14/22. Complete the 12 multiple choice problems and 8 short answer problems in this document (these problems are from a past version of the Algebra II Regents exam):

AlgII-Problems13-32
Download
Record your answers to the 12 multiple choice problems below.
Complete the 8 short answer problems on paper, bring them with you to class on Monday.

Record your confidence level for each problem (including the short answer problems) below. Finally, use the boxes at the bottom of the form to leave a comment or question about three different problems (multiple choice or short answer).
Your grade for this assignment will be based on your participation, NOT on the number you get correct – so don't panic if you don't know how to do something!

Record your answers here:

Class 7:

Similarity for Triangles

In the case of triangles, the notation we use for similarity carries additional information - in particular, it tells us which points in the first triangle correspond with which points in the second triangle.

We say $\triangle ABC \sim \triangle A^\prime B^\prime C^\prime $ if there is a similarity F so that $F(A)=A^\prime$, $F(B)=B^\prime$, $F(C)=C^\prime$.

What does it mean for triangles to be similar?

Similar Triangles

THEOREM G 20. Given two triangles $A B C$ and $A^{\prime} B^{\prime} C^{\prime}$, then $\triangle A B C \sim \triangle A^{\prime} B^{\prime} C^{\prime}$, if and only if
$$
|\angle A|=\left|\angle A^{\prime}\right|, \quad|\angle B|=\left|\angle B^{\prime}\right|, \quad|\angle C|=\left|\angle C^{\prime}\right|
$$
and
$$
\frac{|A B|}{\left|A^{\prime} B^{\prime}\right|}=\frac{|A C|}{\left|A^{\prime} C^{\prime}\right|}=\frac{|B C|}{\left|B^{\prime} C^{\prime}\right|}
$$

Note: The first part says "corresponding angles are equal", the second part says "corresponding sides are proportional."

AA theorem

Theorem G22 (AA for similarity). Two triangles with two pair of equal angles are similar.

Chapter Structure:
Chapter 5. Dilation and Similarity 255

- 5.1. The fundamental theorem of similarity 255
    - Statement of the theorem (p. 256)
    - Two characterizations of a parallelogram (p. 259)
    - Proof of FTS when r = 12 (p. 263)
- 5.2. Dilation 268
    - Definition of a dilation (p. 268)
    - Basic properties of dilations (p. 269)
    - Effects of dilations on lengths and degrees (p. 275)
- 5.3. Similarity
    - Definition of similarity, its symmetry, and its transitivity (p. 283)
    - Two criteria for triangle similarity (p. 287)
    - The Pythagorean theorem and its proof (p. 290)

Day ?? - Parabolas:

Much of the work we do with quadratics consists of making connections between geometric and algebraic features. When we change the picture, what happens to the formula? When we change the formula, what happens to the picture?

Resources: Feel free to use an online calculator, like Desmos or GeoGebra to help test hypotheses and explore different formulas and graphs.

Question A: What is the relationship between each of the basic geometric transformations (translation, reflection, rotation, dilation) and the formula for a parabola?

- For starters, look at a translation. Begin with the basic graph of $y=x^2$. Choose a translation (like "shift up 5 and shift right 2"). What is the formula for the parabola obtained by applying the translation?

Question B: What is the connection between the geometric definition of parabola (focus, directrix) and the formula for the parabola?

- For starters, choose a point P on the y-axis to be the focus, and choose a *horizontal line* to be the directrix. Sketch the parabola. Can you find the equation?

Question C: What is the connection between the roots of a quadratic function and the formula for the function?

- Make up a quadratic function. Find the roots. Can you re-write the formula for the quadratic function using only the roots as constants?

Question D: How do the following *algebraic* processes relate to the questions above?

- Factoring
- Completing the square