

Linear Algebra

Lesson 17

Determinants and Signed Volumes

This lesson has three parts required for all students.

Be sure to put your notes and homework in a document:

MAT313F21-lesson17-lastname-firstname

If you have a question email me with QUESTION in the subject line.

If you have learned 2x2 or 3x3 determinants before, the formulas you know will be taught in the next lesson. The actual definition of determinant uses row actions. Everyone must learn to take determinants using row actions as taught today.

Part I: 2 dimensional signed areas

Part II: 3 dimensional signed volumes

Part III: n dimensional determinants

For this lesson watch each video one by one as you scroll down through the lesson notes and do homework as you come upon it.

Part I

Watch [Video 313F20-17-1](#) which introduces the signed area and how it changes under skews, scales, and switches.

Lesson 17

Determinants and Signed Volumes

Part I 2 dimensional case

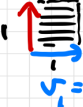
Part II: 3 dimensional case

Part III: n dimensional case


2D volumes are just areas



signed areas can
be + or - or 0.

Explore areas of
parallelograms defined
by pairs of vectors in \mathbb{R}^2

 $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
 $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ area of the
square is 1

skew \vec{v}_2

$\text{new } \vec{v}_2 = \text{old } \vec{v}_2 + k \vec{v}_1$
 new Area = old Area
because it has
the same base + height

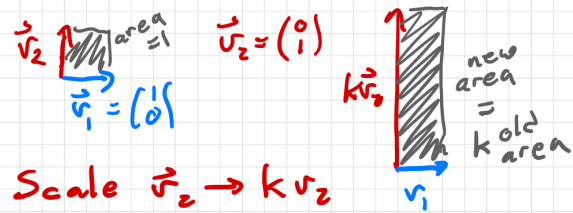
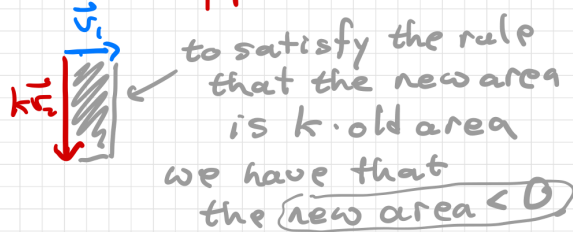
Area  = Area  height
base

Lesson 17

Determinants
and Signed VolumesPart I 2 dimensional case

Part II: 3 dimensional case

Part III: n dimensional case

ThmA skew does not change
the area of a parallelogramThm A scale by k changes
the area to $k \cdot \text{area}$. possibly soScale $\vec{v}_2 \rightarrow k\vec{v}_2$ here above we had $k > 0$ What happens if $k < 0$ 

Orientation

clockwise
then area < 0 If $k = 0$ $\vec{0} \cdot \vec{v}_1 \rightarrow$ no area inside so area $= 0$

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Linear Algebra 2

Lesson 17

Determinants and Signed Volumes

Part I 2 dimensional case

Part II: 3 dimensional case

Part III: n dimensional case

Thm A skew does not change the area of a parallelogram

Thm A scale by k changes the area to $k \cdot \text{area}$. possibly

Thm A switch of vectors changes the area to $-1 \cdot \text{area}$.

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orientation

↺ clockwise then area < 0

If $k=0$

$\vec{0} \cdot \vec{v}_i$ ← no area inside so area = 0

\vec{v}_2 ↑
 \vec{v}_1 →

switch

\vec{v}_2 ↑
 \vec{v}_1 →

old \vec{v}_2 is new \vec{v}_1
 old \vec{v}_1 = new \vec{v}_2

this changes the orientation

And so the $\text{new area} = - \text{old area}$

Watch [Video 313F20-17-2](#) which shows how to compute the signed area:

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Practical Application:
Find the area of the parallelogram formed by $\vec{v}_1 = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\begin{pmatrix} 2 & 8 \\ 2 & 3 \end{pmatrix}$ ← Make a matrix whose rows are the vectors

Do row reduction keeping track of area changes (skew no change, scale k · area, switch -1 area)

$P_1 \rightarrow \frac{1}{2}P_1$
 $\cdot \left(\frac{1}{2}\right) \rightarrow \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$

$\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ ← $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\begin{pmatrix} \text{original area} \end{pmatrix} = 2 \begin{pmatrix} \text{new area} \end{pmatrix}$
 $\begin{pmatrix} \text{new area} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \text{old area} \end{pmatrix}$

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$P_2 \rightarrow P_2 - 2P_1$
 $\cdot (-1) \rightarrow \begin{pmatrix} 1 & 4 \\ 0 & -5 \end{pmatrix}$
 no area change under a skew

$P_2 \rightarrow \frac{-1}{5}P_2$
 $\cdot \left(\frac{-1}{5}\right) \rightarrow \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$
 this area = $-\frac{1}{5}$ (last area) = $-\frac{1}{5} \cdot \frac{1}{2}$ (orig area)

$P_1 \rightarrow P_1 - 4P_2$
 $\cdot (-1) \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
 no change under skew, this area = 1

Thus our original signed area = $(-5)(2)(1) = -10$
 Area = $|-10| = 10$

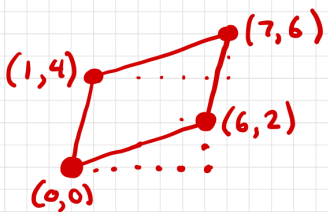
Complete HW1-HW4 using this method:

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[HW1] Find the signed area of $\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$ drawing the vectors at each step and shading the area.
Hint: a switch changes the sign!

[HW2] Find the area of this parallelogram:



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[HW3] Find the area of the parallelogram in HW2 switching which vector is row 1 and doing the reduction a second way.

[HW4] Find the area of the parallelogram formed by $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$ doing row reduction as in Lesson 16 HW 12.

Part II

Watch [Video 313F20-17-3](#) which introduces the signed volume and how it changes under shears, scales, and switches.

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Lesson 17

Determinants and Signed Volumes

Part I 2 dimensional case

Part II 3 dimensional case

Part III n dimensional case

Thm A skew does not change the volume of a parallelopiped

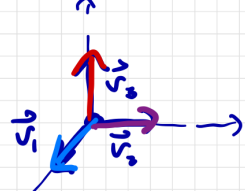
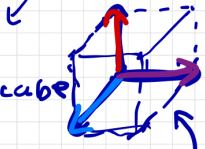
Thm A scale by k changes the volume to $k \cdot \text{volume}$ possibly ≤ 0

Thm A switch of vectors changes the volume to $-1 \cdot \text{volume}$

Linear Algebra 2

$\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^3$


Define the signed volume of the parallelopiped

Example

$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

then the parallelopiped is a cube.



changes orientation and negates signed volume.

Watch [Video 313F20-17-4](#) which computes the signed volume:

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Practical Example: parallelepiped

Find the signed volume of $\begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

First step is to write matrix with these vectors as rows

$$\begin{pmatrix} 0 & 5 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

Second step is to do row reduction keeping track of volume changes

switch $\cdot (-1)$ skew $\cdot (-1)$ scale $\cdot k$

We did this row reduction before \rightarrow

switch (-1) skew (1) scale $(\frac{1}{5})$

(original area) $(-1)(1)(\frac{1}{5}) = (\text{final area})$

Solve for orig area

$(\text{original area}) = (-1)(1)(5)(1) = -5$

\uparrow this is 1 because Red. Ech = $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ a cube

Linear Algebra 2

$\begin{pmatrix} 0 & 5 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{pmatrix} = A$ Find A^{-1}

$\left(\begin{array}{ccc|ccc} 0 & 5 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \xrightarrow[\rho_1 \leftrightarrow \rho_2]{\text{switch } (-1)}$

$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 5 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \xrightarrow[\rho_3 \rightarrow \rho_3 - 2\rho_1]{\text{skew } (1)}$

$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 5 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array} \right) \xrightarrow[\rho_2 \rightarrow \frac{1}{5}\rho_2]{\text{scale } (\frac{1}{5})}$

$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array} \right) \quad A^{-1} \text{ is } B = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{5} & 0 & 0 \\ 0 & -2 & 1 \end{pmatrix}$

Check

$A \times A^{-1} = \begin{pmatrix} 0 & 5 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{5} & 0 & 0 \\ 0 & -2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{5}{5} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$A^{-1} \times A = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{5} & 0 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 5 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{5}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$

Watch [Video 313F20-17-5](#) which computes the signed volume and draws pictures:

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Linear Algebra 2

Ex2

find signed volume of this parallelepiped

$\begin{pmatrix} 4 & 0 & 0 \\ 0 & -1 & 4 \\ 0 & 6 & 0 \end{pmatrix}$

$\cdot \left(\frac{1}{4}\right) \mid p_1 \rightarrow \frac{1}{4} p_1$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 4 \\ 0 & 6 & 0 \end{pmatrix}$

$\cdot (-1) \mid p_2 \rightarrow -p_2$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 6 & 0 \end{pmatrix}$

Linear Algebra 2

1) skew $p_3 \rightarrow p_3 - 6p_2$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 24 \end{pmatrix} \xrightarrow{\left(\frac{1}{24}\right)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$

$p_3 \rightarrow \frac{1}{24} p_3$

$p_2 \rightarrow p_2 + 4p_3$

$\cdot (-1) \text{ skew}$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

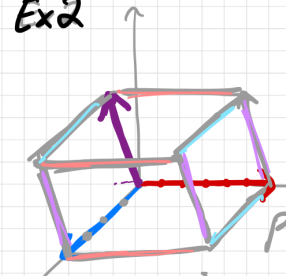
cube of vol = 1

(Orig Vol) = $24(-1)4(1) = \boxed{-96}$

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Linear Algebra 2

Ex2



find signed volume of this parallelepiped

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & -1 & 4 \\ 0 & 6 & 0 \end{pmatrix}$$

$\cdot (-\frac{1}{4}) \mid p_1 \rightarrow \frac{1}{4} p_1$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 4 \\ 0 & 6 & 0 \end{pmatrix}$$

$\cdot (-1) \mid p_2 \rightarrow -p_2$

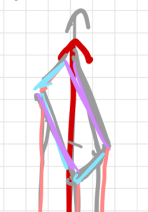
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 6 & 0 \end{pmatrix}$$

Linear Algebra 2

$\cdot (-1) \mid \text{skew} \quad p_3 \rightarrow p_3 - 6p_2$

$$\begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & -4 \\ 0 & 0 & 24 \end{pmatrix} \xrightarrow{(\frac{1}{24})} \begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$$

$p_3 \rightarrow \frac{1}{24} p_3$



$p_2 \rightarrow p_2 + 4p_3$

$\cdot (-1) \mid \text{skew}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

cube of vol = 1

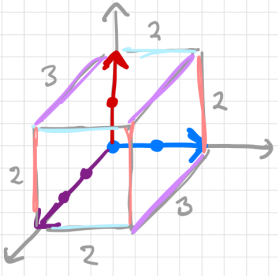
(Orig Vol) = $24(-1)4(1) = \boxed{-96}$

HW5 and HW6

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Linear Algebra 2

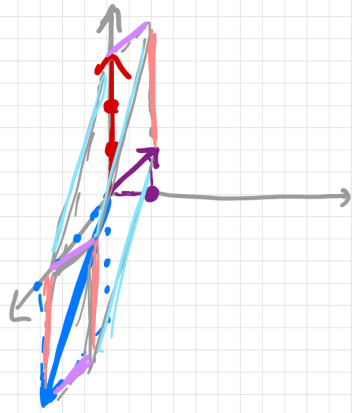
HW5 Find the scaled volume of $\begin{pmatrix} 0 & 2 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ drawing the pictures if you can.



These pictures are not difficult at any step because they are all rectangular blocks.

Linear Algebra 2

HW6 Find the scaled volume of $\begin{pmatrix} 4 & 0 & -4 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}$



Part III

Watch [Video 313F20-17-6](#) which introduces the determinant using skews, scales, and switches.

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Linear Algebra 2

Lesson 17

Determinants and Signed Volumes

Part I 2 dimensional case

Part II 3 dimensional case

Part III n dimensional case

Thm A skew does not change the volume of a parallelepiped

Thm A scale by k changes the volume to $k \cdot \text{volume}$ possibly $\neq 0$

Thm A switch of vectors changes the volume to $-1 \cdot \text{volume}$

Linear Algebra 2

Defn: The determinant of an $n \times n$ matrix $A \in M_{n \times n}$ is a real number found by doing row red. following our algorithm

skew
 $\det(A) = \det(R_{p_i \rightarrow p_i + k p_j} A)$

scale
 $\det(A) = \frac{1}{k} \det(R_{p_i \rightarrow k p_i} A)$

switch
 $\det(A) = -\det(R_{p_i \leftrightarrow p_j} A)$

If A is nonsingular
 $\det(A) = \text{product of these changes.}$

If A singular then $\det(A) = 0$.

Watch [Video 313F20-17-7](#) which goes over examples computing the determinant of a 2×2 matrix using skews, scales, and switches.

Defn: The determinant of an $n \times n$ matrix $A \in M_{n \times n}$ is a real number found by doing row red. following our algorithm

skew

$$\det(A) = \det(R_{p_i \rightarrow p_i + k p_j} A)$$

scale

$$\det(A) = \frac{1}{k} \det(R_{p_i \rightarrow k p_i} A)$$

switch

$$\det(A) = -\det(R_{p_i \leftrightarrow p_j} A)$$

If A is nonsingular

$\det(A) = \text{product of these changes.}$

If A singular then $\det(A) = 0$.

Go through lesson looking at examples + HW using determinants

Det is signed area

[HW1] Find the signed area.

of $\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$

drawing the vectors at each step and shading the area

$$\det \begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix} = -\det \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} \text{ by } p_1 \leftrightarrow p_2$$

$$= (-1)(5) \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad p_1 \rightarrow \frac{1}{5} p_1$$

$$= (-1)(5) \cdot (1)$$

$$= -5$$



Area = 5 signed area = -5

Do HW7 and HW8

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Linear Algebra 2

HW 4 Find the area of the parallelogram formed by $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$ doing row reduction as in Lesson 16 HW 12. *assume $a \neq 0$*

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \det \begin{pmatrix} 1 & b/a \\ c & d \end{pmatrix} \quad \rho_1 \rightarrow \frac{1}{a} \rho_1$$

$$= a \cdot (1) \det \begin{pmatrix} 1 & b/a \\ 0 & \frac{ad-bc}{a} \end{pmatrix} \quad \rho_2 \rightarrow \rho_2 - c\rho_1$$

$$= a \cdot 1 \cdot \left(\frac{ad-bc}{a}\right) \det \begin{pmatrix} 1 & b/a \\ 0 & 1 \end{pmatrix} \quad \rho_3 \rightarrow \frac{a}{ad-bc} \rho_3$$

skew $\rho_1 \rightarrow \rho_1 - \frac{b}{a} \rho_2$ *assume $ad-bc \neq 0$*

$$= a \cdot 1 \cdot \left(\frac{ad-bc}{a}\right) \cdot 1 \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad-bc) 1 \cdot 1 = ad-bc$$

Linear Algebra 2

HW 7 Show that $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad-bc$ when $a=0$ and $ad-bc \neq 0$ by looking at $\det \begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$ and starting with a switch etc

HW 8 Show that if $ad-bc=0$ then A is singular and

$$\det(A) = (1)(3)(?) (1) \underbrace{\det \begin{pmatrix} ? & ? \\ 0 & 0 \end{pmatrix}}_0$$

In HW8 above I have not written specific numbers in the parentheses. You are doing the row actions and finding the formulas to replace the scribbles I wrote.

Watch [Video 313F20-17-8](#) which goes over an example computing the determinant of a 3x3 matrix using skews, scales, and switches.

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Linear Algebra 2

Ex 2 again: (quicker)

$$\det \begin{pmatrix} 4 & 0 & 0 \\ 0 & -1 & 4 \\ 0 & 6 & 0 \end{pmatrix} = 4 \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 4 \\ 0 & 6 & 0 \end{pmatrix}$$

$p_1 \rightarrow \frac{1}{4} p_1$

$$= 4(-1) \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 4 \\ 0 & 6 & 0 \end{pmatrix}$$

$p_2 \rightarrow -p_2$

$$= 4(-1)(1) \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 6 & 0 \end{pmatrix}$$

$p_3 \rightarrow p_3 - 6p_2$ skew

$$= 4(-1)(1) \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 24 \end{pmatrix}$$

$p_3 \rightarrow \frac{1}{24} p_3$

$$= 4(-1)(1)(24) \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$$

$p_2 \rightarrow p_2 + 4p_3$ skew

$$= 4(-1)(1)(24)(1) \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$= 4(-1)(1)(24)(1)(1) \rightarrow I$

$$= -96$$

Linear Algebra 2

HW9 Find $\det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 5 \end{pmatrix}$ using this quicker method.

HW10 Find the following quickly

$$\det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \det \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \det \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \det \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

HW11 Find $\det \begin{pmatrix} 0 & a & 0 & 0 \\ b & 0 & 0 & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & d & 0 \end{pmatrix}$
for $a=2$ $b=3$
 $c=4$ $d=5$

HW12 Find $\det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}$
for $a=2$ $b=3$
 $c=4$ $d=5$

Do HW9-HW12

Homework should have been completed as you watched the videos above. Please check that you watched the complete [Playlist 313F21-17-1to8](#).