Linear Algebra
Lesson 17
Determinants and Signed Volumes

This lesson has three parts required for all students.

Be sure to put your notes and homework in a document:
MAT313F21-lesson17-lastname-firstname
If you have a question email me with QUESTION in the subject line.

If you have learned 2x2 or 3x3 determinants before, the formulas you know will be taught in the next lesson. The actual definition of determinant uses row actions. Everyone must learn to take determinants using row actions as taught today.

Part I: 2 dimensional signed areas
Part II: 3 dimensional signed volumes
Part III: n dimensional determinants

For this lesson watch each video one by one as you scroll down through the lesson notes and do homework as you come upon it.

Part I

Watch Video 313F20-17-1 which introduces the signed area and how it changes under skews, scales, and switches.

Lesson 17
Determinants and Signed Volumes

$\boxed{\text{Part I}}$ 2 dimensional case

Part II : 3 dimensional case

Part III : n dimensional case

2D volumes are just areas

signed areas can be + or − or 0.

Explore areas of parallegrams defined by pairs of vectors in $\mathbb{R}^2$

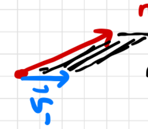
$v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

area of the square is 1

skew $\vec{v}_2$

new $\vec{v}_2$ = old $v_2 + k\vec{v}_1$



new Area = old Area

because it has the same base + height

Lesson 17

Determinants and Signed Volumes

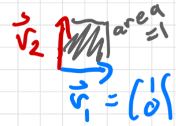
$\boxed{\text{Part I}}$ 2 dimensional case

Part II: 3 dimensional case

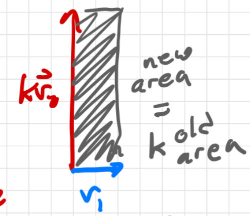
Part III: n dimensional case

Thm
A skew does not change the area of a parallelogram

Thm A scale by k changes the area to $k \cdot$ area. possibly ≤ 0



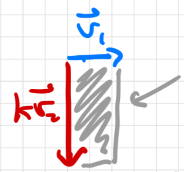
$\vec{v}_2 = \begin{pmatrix}0\\1\end{pmatrix}$



Scale $\vec{v}_2 \to k v_2$

here above we had $k > 0$

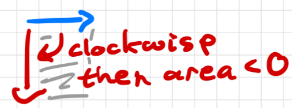
What happens if $k < 0$



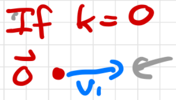
to satisfy the rule that the new area is $k \cdot$ old area

we have that the new area < 0

orientation



If $k = 0$



no area inside so area $= 0$

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Determinants and Signed Volumes

$\boxed{\text{Part I}}$ 2 dimensional case

Part II: 3 dimensional case

Part III: n dimensional case

Thm A skew does not change the area of a parallelogram

Thm A scale by k changes the area to $k \cdot$ area. (possibly ≤ 0)

Thm A switch of vectors changes the area to $-1 \cdot$ area.

Linear Algebra 2

orientation

clockwise then area < 0

If $k = 0$

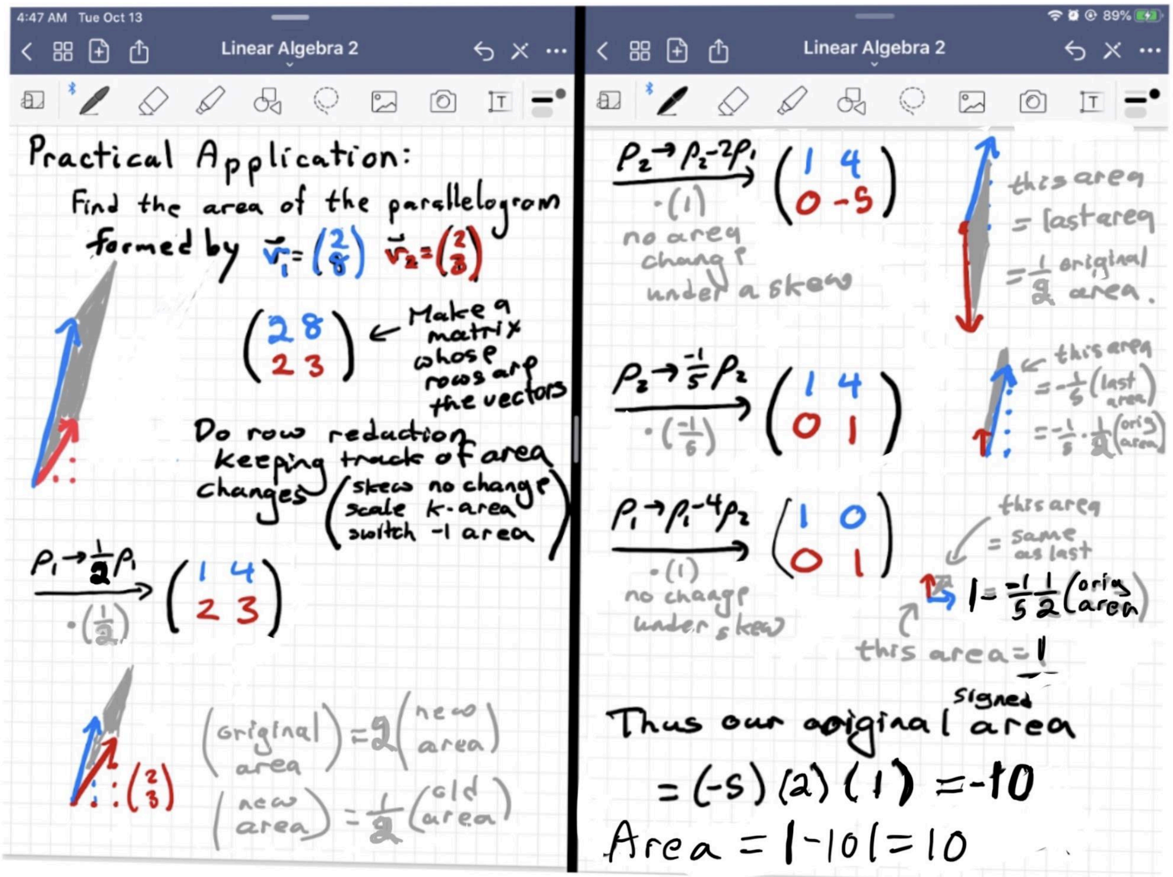
$\vec{0}$ $\vec{v}_1$ ← no area inside so area $= 0$

v_2, v_1 → switch → old v_2 is now v_1, old v_1 = new v_2

this changes the orientation

And so the new area $= -$ old area

Watch [Video 313F20-17-2]() which shows how to compute the signed area:

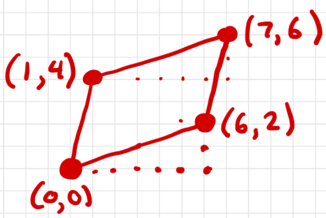


Complete HW1-HW4 using this method:

$\boxed{\text{HW1}}$ Find the signed area of $\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$ drawing the vectors at each step and shading the area.

Hint: a switch changes the sign!

$\boxed{\text{HW2}}$ Find the area of this parallelogram:

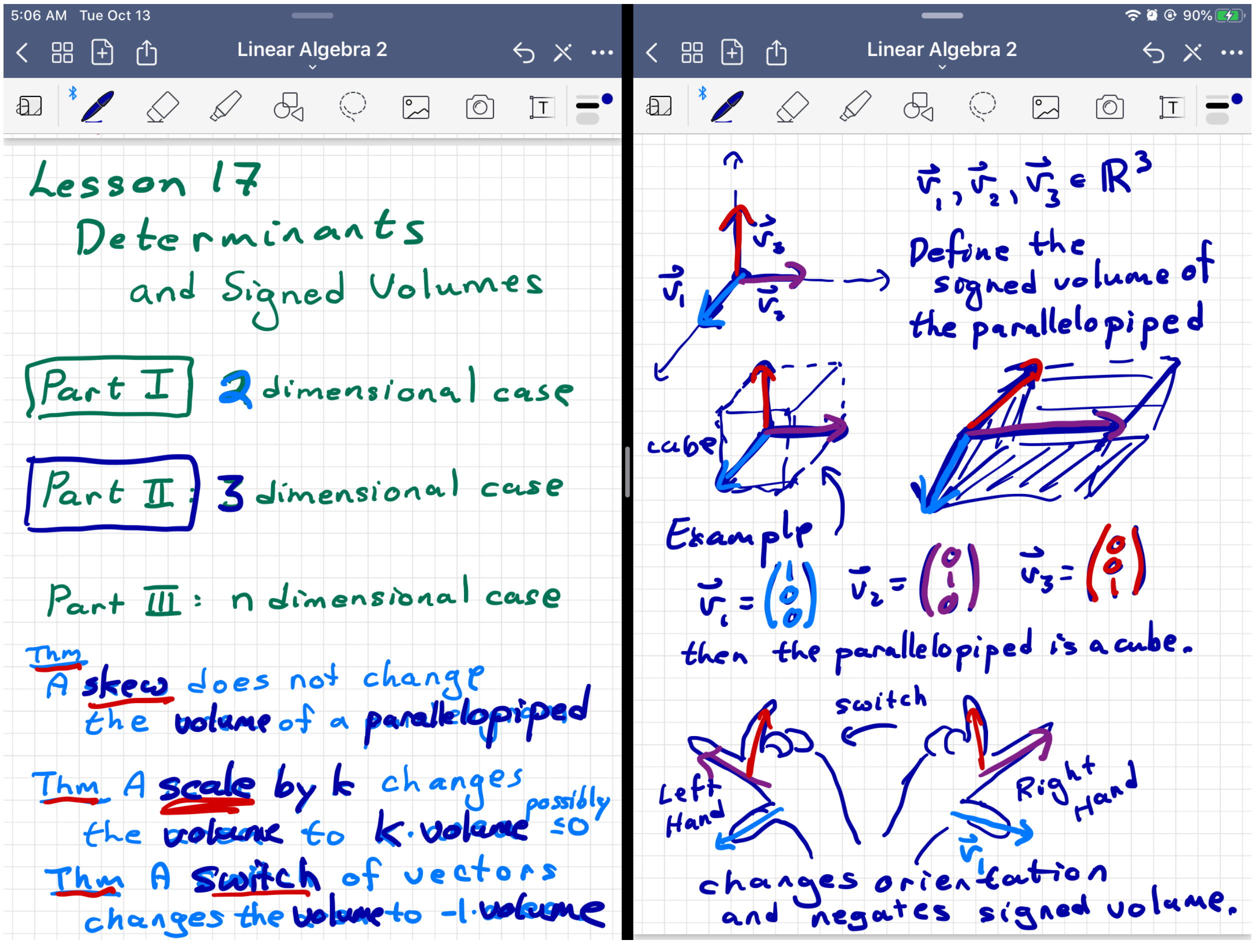


$\boxed{\text{HW3}}$ Find the area of the parallelogram in HW2 switching which vector is row 1 and doing the reduction a second way.

$\boxed{\text{HW4}}$ Find the area of the parallelogram formed by $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$ doing row reduction as in Lesson 16 HW 12.

Part II

Watch [Video 313F20-17-3] which introduces the signed volume and how it changes under skews, scales, and switches.



Lesson 17
Determinants and Signed Volumes

Part I: 2 dimensional case

Part II: 3 dimensional case

Part III: n dimensional case

Thm A skew does not change the volume of a parallelopiped

Thm A scale by k changes the volume to k·volume possibly ≤0

Thm A switch of vectors changes the volume to −1·volume

$\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^3$

Define the signed volume of the parallelopiped

$\vec{v}_1$ $\vec{v}_2$ $\vec{v}_3$

cube

Example

$\vec{v}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$ $\vec{v}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$

then the parallelopiped is a cube.

switch

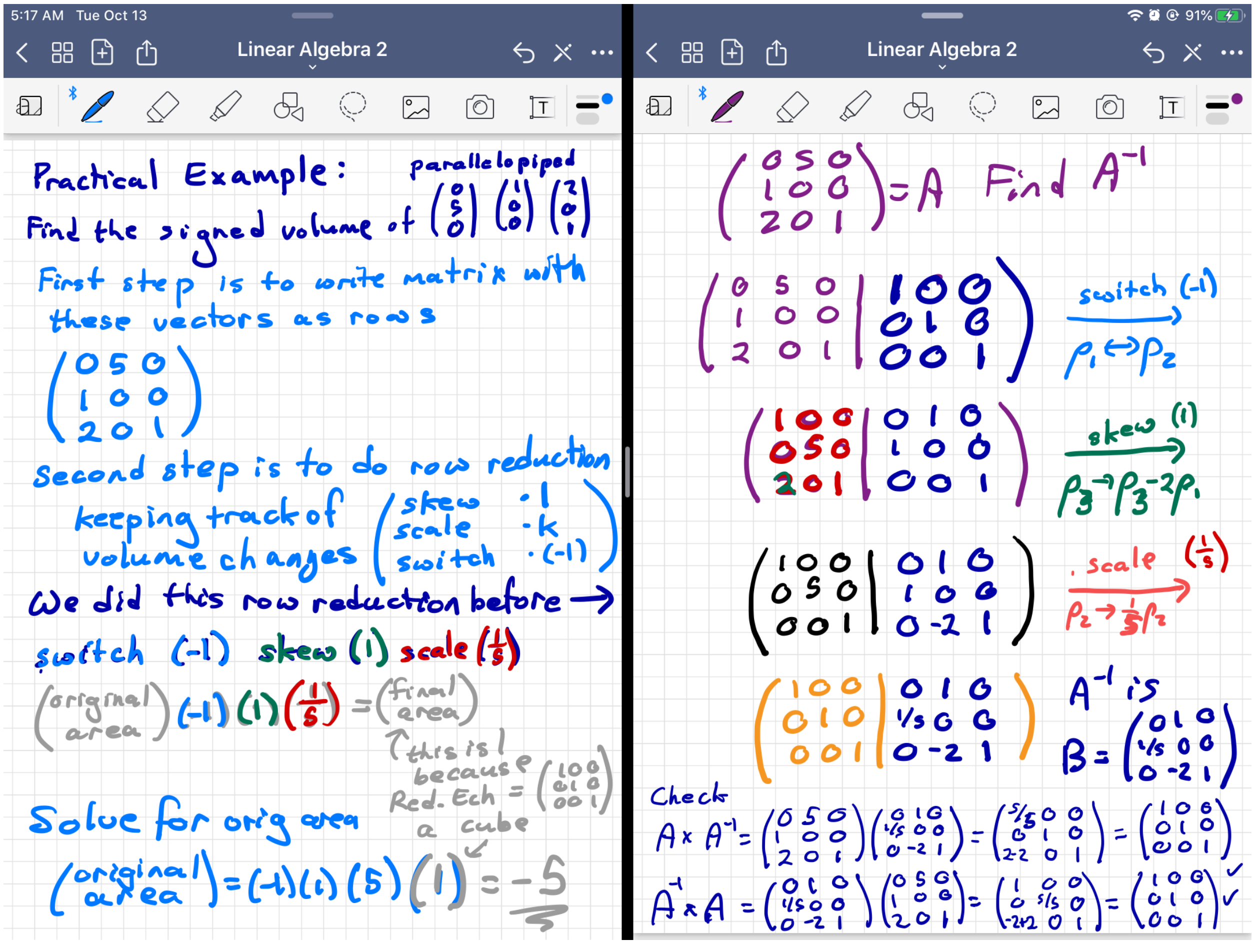
Left Hand

Right Hand

$\vec{v}_1$

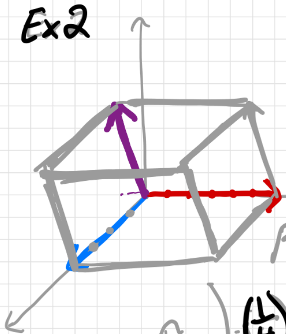
changes orientation and negates signed volume.

Watch Video 313F20-17-4 which computes the signed volume:



Watch Video 313F20-17-5 which computes the signed volume and draws pictures:

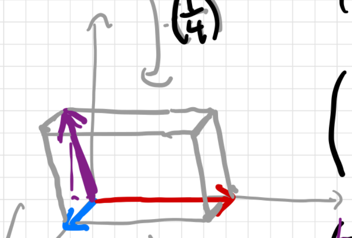
Ex2



$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & -1 & 4 \\ 0 & 6 & 0 \end{pmatrix}$$

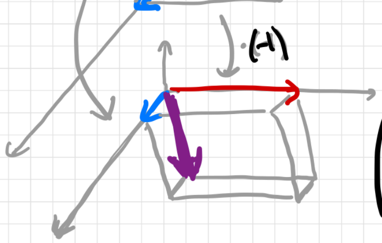
find signed volume of this parallelepiped

$\cdot\left(\frac{1}{4}\right) \downarrow \rho_1 \to \frac{1}{4}\rho_1$



$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 4 \\ 0 & 6 & 0 \end{pmatrix}$$

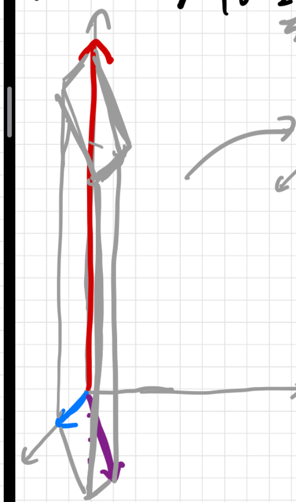
$(-1) \downarrow \rho_2 \to -\rho_2$

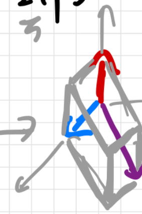


$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 6 & 0 \end{pmatrix}$$

$\cdot(1) \downarrow$ skew $\rho_3 \to \rho_3 - 6\rho_2$

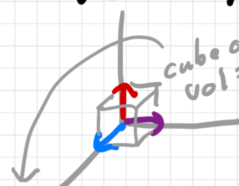
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 24 \end{pmatrix} \xrightarrow[\rho_3 \to \frac{1}{24}\rho_3]{\left(\frac{1}{24}\right)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$$





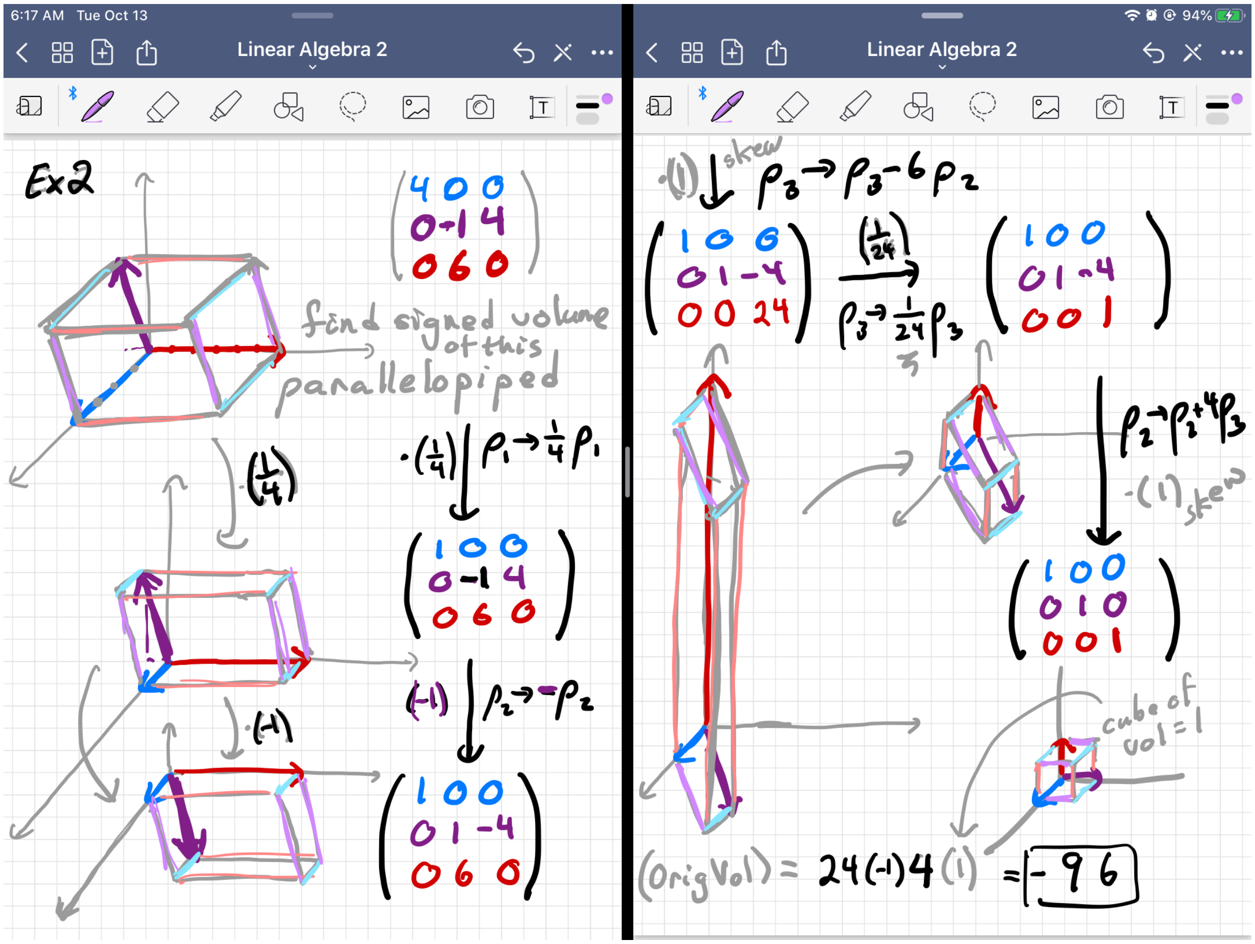
$\downarrow \rho_2 \to \rho_2 + 4\rho_3$ $\cdot(1)$ skew

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



cube of vol = 1

$(\text{Orig Vol}) = 24(-1)4(1) = \boxed{-96}$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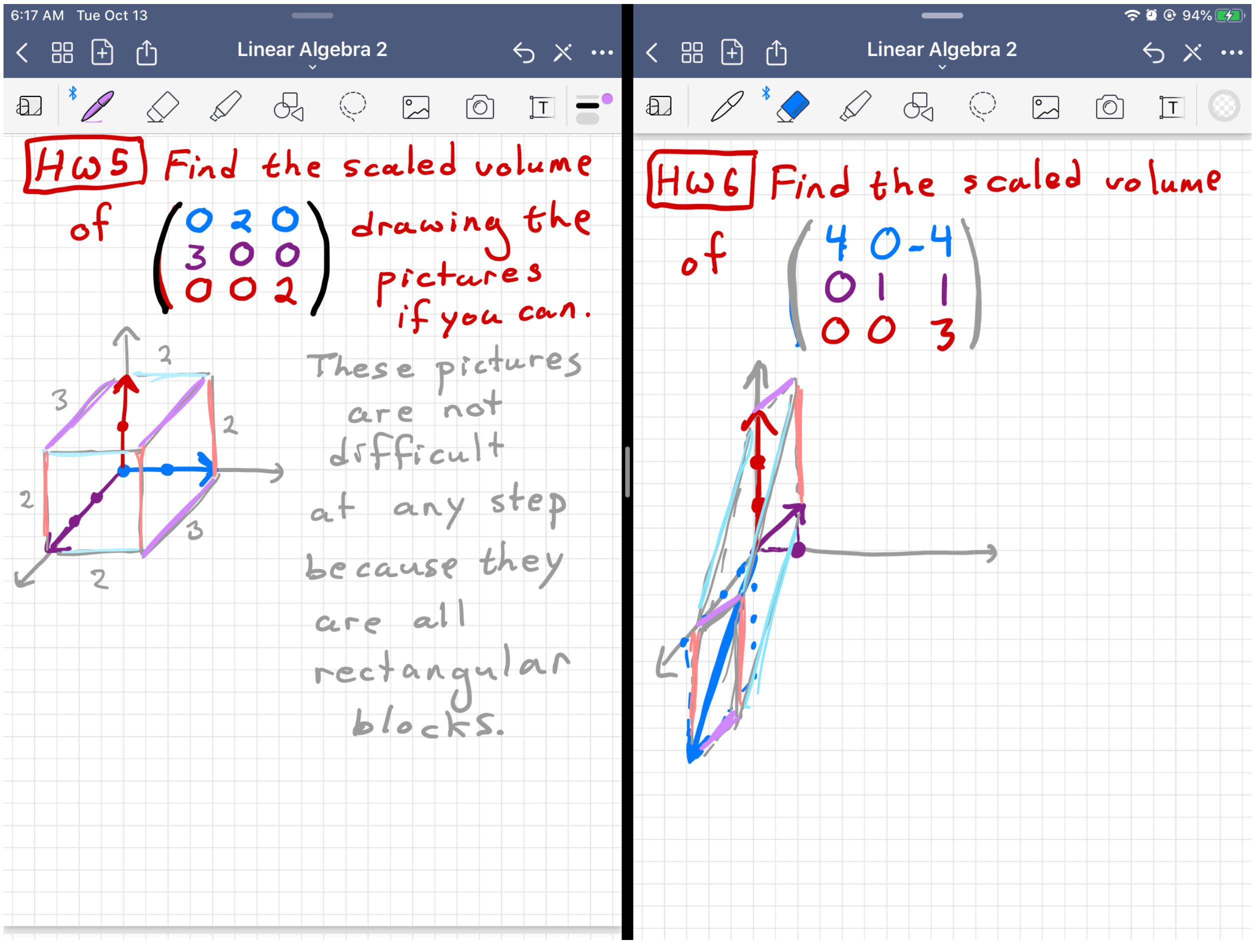
Linear Algebra 2
94%
Ex2
find signed volume of this parallelopiped
400
0 -1 4
060
$\cdot(\frac{1}{4})$ $\rho_1 \to \frac{1}{4}\rho_1$
100
0 -1 4
060
(-1) $\rho_2 \to -\rho_2$
100
0 1 -4
060
$\cdot(1)$ skew $\rho_3 \to \rho_3 - 6\rho_2$
100
0 1 -4
0 0 24
$(\frac{1}{24})$ $\rho_3 \to \frac{1}{24}\rho_3$
100
0 1 -4
001
$\rho_2 \to \rho_2 + 4\rho_3$
$\cdot(1)$ skew
100
010
001
cube of vol = 1
(Orig Vol) = 24(-1)4(1) = -96

HW5 and HW6



Part III

Watch [Video 313F20-17-6]() which introduces the determinant using skews, scales, and switches.

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Linear Algebra 2

Lesson 17
Determinants
and Signed Volumes

$\boxed{\text{Part I}}$ 2 dimensional case

$\boxed{\text{Part II}}$: 3 dimensional case

$\boxed{\text{Part III}}$: n dimensional case

Thm
A skew does not change the volume of a parallelopiped

Thm A scale by k changes the volume to k·volume possibly ≤0

Thm A switch of vectors changes the volume to −1·volume

Linear Algebra 2

Defn: The determinant of an $n\times n$ matrix $A\in M_{n\times n}$ is a real number found by doing row red. following our algorithm

Skew
$$\det(A)=\det(R_{\rho_i\to\rho_i+k\rho_j} * A)$$

Scale
$$\det(A)=\tfrac{1}{k}\det(R_{\rho_i\to k\rho_i}\, A)$$

Switch
$$\det(A)=-\det(R_{\rho_i\leftrightarrow\rho_j}\, A)$$

If A is nonsingular
$\det(A)=$ product of these changes.
If A singular then $\det(A)=0$.

Watch Video 313F20-17-7 which goes over examples computing the determinant of a 2x2 matrix using skews, scales, and switches.

Defn: The determinant of an $n \times n$ matrix $A \in M_{n \times n}$ is a real number found by doing row red. following our algorithm

Skew
$$\det(A) = \det(R_{\rho_i \to \rho_i + k\rho_j} * A)$$

scale
$$\det(A) = \frac{1}{k} \det(R_{\rho_i \to k\rho_i} A)$$

switch
$$\det(A) = -\det(R_{\rho_i \leftrightarrow \rho_j} A)$$

If A is nonsingular $\det(A) =$ product of these changes.

If A singular then $\det(A) = 0$.

Go through lesson looking at examples + HW using determinants

$\boxed{\text{Det is signed area}}$

$\boxed{\text{HW1}}$ Find the signed area of $\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$ drawing the vectors at each step and shading the area

$$\det\begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix} = -\det\begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} \text{ by } \rho_1 \leftrightarrow \rho_2$$

$$= (-1)(5)\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \rho_1 \to \frac{1}{5}\rho_1$$

$$= (-1)(5) \cdot (1)$$

area

$$= -5$$

Area $= 5$ Signed area $= -5$

Do HW7 and HW8

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$\boxed{HW4}$ Find the area of the parallelogram formed by $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$ doing row reduction as in Lesson 16 HW12.

assume $a \neq 0$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \det\begin{pmatrix} 1 & \frac{b}{a} \\ c & d \end{pmatrix} \quad \rho_1 \to \tfrac{1}{a}\rho_1$$

$$= a \cdot (1) \det\begin{pmatrix} 1 & b/a \\ 0 & \frac{ad-bc}{a} \end{pmatrix} \quad \rho_2 \to \rho_2 - c\rho_1$$

$$= a \cdot 1 \cdot \left(\frac{ad-bc}{a}\right) \det\begin{pmatrix} 1 & b/a \\ 0 & 1 \end{pmatrix} \quad \rho_3 \to \frac{a}{ad-bc}\rho_3$$

assume $ad-bc \neq 0$

skew $\rho_1 \to \rho_1 - \frac{b}{a}\rho_2$

$$= \cancel{a} \cdot 1 \left(\frac{ad-bc}{\cancel{a}}\right) \cdot 1 \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad-bc) 1 \cdot 1 = ad-bc$$

Linear Algebra 2 — 95%

$\boxed{HW7}$ Show that $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad-bc$ when $a=0$ and $ad-bc \neq 0$ by looking at $\det\begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$ and starting with a switch etc

$\boxed{HW8}$ Show that if $ad-bc=0$ then A is singular and

$$\det(A) = (?)(?)(?)(?)\underbrace{\det\begin{pmatrix} ? & ? \\ 0 & 0 \end{pmatrix}}_{0}$$

In HW8 above I have not written specific numbers in the parentheses. You are doing the row actions and finding the formulas to replace the scribbles I wrote.

Watch Video 313F20-17-8 which goes over an example computing the determinant of a 3x3 matrix using skews, scales, and switches.

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Linear Algebra 2

Ex 2 again: (quicker)

$$\det\begin{pmatrix}4&0&0\\0&-1&4\\0&6&0\end{pmatrix} \underset{\rho_1\to\frac{1}{4}\rho_1}{=} 4\det\begin{pmatrix}1&0&0\\0&-1&4\\0&6&0\end{pmatrix}$$

$\rho_2 \to -\rho_2$

$$= 4(-1)\det\begin{pmatrix}1&0&0\\0&1&-4\\0&6&0\end{pmatrix}$$

$\rho_3 \to \rho_3 - 6\rho_2$ skew

$$= 4(-1)(1)\det\begin{pmatrix}1&0&0\\0&1&-4\\0&0&24\end{pmatrix}$$

$\rho_3 \to \frac{1}{24}\rho_3$

$$= 4(-1)(1)(24)\det\begin{pmatrix}1&0&0\\0&1&-4\\0&0&1\end{pmatrix}$$

$\rho_2 \to \rho_2 + 4\rho_3$ skew

$$= 4(-1)(1)(24)(1)\det\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$= 4(-1)(1)(24)(1)(1) \quad I$$

$$= -96$$

$\boxed{\text{HW9}}$ Find $\det\begin{pmatrix}1&2&3\\0&1&4\\0&0&5\end{pmatrix}$ using this quicker method.

$\boxed{\text{HW10}}$ Find the following quickly

$$\det\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \quad \det\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix} \quad \det\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$$

$$\det\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix} \quad \det\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix} \quad \det\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$$

$\boxed{\text{HW11}}$ Find $\det\begin{pmatrix}0&a&0&0\\b&0&0&0\\0&0&0&c\\0&0&d&0\end{pmatrix}$ for $a=2$ $b=3$ $c=4$ $d=4$

$\boxed{\text{HW12}}$ Find $\det\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&a&b\\0&0&c&d\end{pmatrix}$ for $a=2$ $b=3$ $c=4$ $d=5$

Do HW9-HW12

Homework should have been completed as you watched the videos above. Please check that you watched the complete Playlist 313F21-17-1to8.