Linear Algebra MAT313 Spring 2023
Professor Sormani
Lesson 4 Dot Products and Hyperplanes and Proofs

Please be sure to mark down the date and time that you start this lesson. Carefully take notes on pencil and paper while watching the lesson videos. Pause the lesson to try classwork before watching the video going over that classwork. If you work with any classmates, be sure to write their names on the problems you completed together. Please wear masks when meeting with classmates even if you meet off campus.

You will cut and paste the photos of your notes and completed classwork and a selfie taken holding up the first page of your work in a googledoc entitled:

MAT313S23-lesson4-lastname-firstname

and share editing of that document with me sormanic@gmail.com. You will also include your homework and any corrections to your homework in this doc.

If you have a question, type QUESTION in your googledoc next to the point in your notes that has a question and email me with the subject MAT313 QUESTION. I will answer your question by inserting a photo into your googledoc or making an extra video.

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This lesson has two parts each with its own playlist:

Part I: Dot Products and Hyperplanes

Part II: Theorems and Proofs

There are five homework problems.

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Part I: Dot Products and Hyperplanes

Watch [Playlist 313S23-L4-P1](). Remember you can watch videos at a higher playback speed but be sure to watch all the videos.

Linear Algebra Lesson 4

Dot Products, Hyperplanes, and Proofs

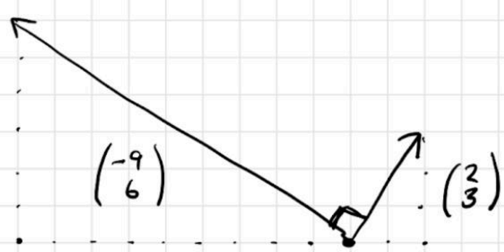
$$\vec{v} \cdot \vec{w} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{pmatrix} = v_1 w_1 + v_2 \cdot w_2 + \ldots + v_m w_m = \sum_{j=1}^{m} v_j w_j$$

dot product

In 2D $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = v_1 w_1 + v_2 w_2 = \sum_{j=1}^{2} v_j w_j$

Theorem: $\vec{v}$ is perpendicular to $\vec{w}$ if $\vec{v} \cdot \vec{w} = 0$

Example $\begin{pmatrix} 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -9 \\ 6 \end{pmatrix} = 2(-9) + 3(6) = -18 + 18 = 0$



If we wish to ~~prove~~ prove this theorem then we need to check all possible pairs of vectors and carefully explain each step.

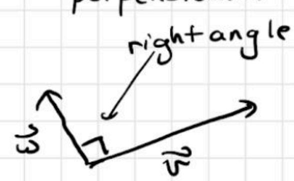
Theorem: Assume $\vec{v}$ and $\vec{w}$ are not zero

$$\vec{v} \cdot \vec{w} = 0 \iff \vec{v} \perp \vec{w}$$

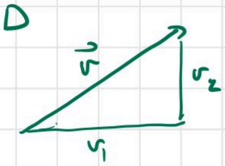
perpendicular

$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0$

$v_1 w_1 + v_2 w_2 = 0$



Before we can do this proof in 2D we need some notation:

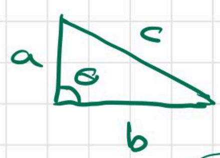


$|\vec{v}|$ = magnitude of $\vec{v}$ (norm, length) $= \sqrt{v_1^2 + v_2^2}$

in higher dim $= \sqrt{v_1^2 + \dots + v_n^2}$

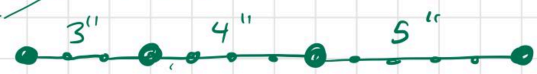
$|\vec{w}| = \sqrt{w_1^2 + w_2^2}$

Also recall the Pythagorean Theorem:

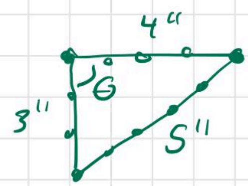


$\theta = 90^\circ$ right angle $\iff a^2 + b^2 = c^2$

Ancient Egyption Method:



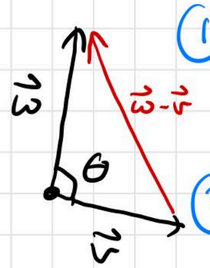
12'' string



make a triangle from the string

$3^2 + 4^2 = 5^2 \Rightarrow \theta = 90^\circ$

Proof that $\vec{v} \cdot \vec{w} = 0 \iff \vec{v} \perp \vec{w}$ in 2D



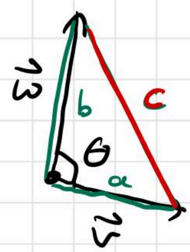
① $\vec{v} \perp \vec{w}$

$\Updownarrow$ by defn of perpendicular

② The triangle formed by $\vec{v}$ and $\vec{w}$ has a right angle at θ

$\Updownarrow$ by Pythagorean Theorem

③ $a^2 + b^2 = c^2$



where $a = |\vec{v}|$ length of $\vec{v}$

$b = |\vec{w}|$ length of $\vec{w}$

$c = |\vec{w} - \vec{v}|$

$\Updownarrow$ substitution

(4) $|\vec{v}|^2 + |\vec{w}|^2 = |\vec{w} - \vec{v}|^2$

$|\vec{v}| = \sqrt{v_1^2 + v_2^2}$ by defn of magnitude

and so $|\vec{v}|^2 = v_1^2 + v_2^2$

$|\vec{w} - \vec{v}|^2 = \left\| \begin{pmatrix} w_1 - v_1 \\ w_2 - v_2 \end{pmatrix} \right\|^2 = (w_1 - v_1)^2 + (w_2 - v_2)^2$

by defn of subtraction of vectors

$\Updownarrow$

(5) $(v_1^2 + v_2^2) + (w_1^2 + w_2^2) = (w_1 - v_1)^2 + (w_2 - v_2)^2$

$\Updownarrow$ by $(a-b)^2 = a^2 - 2ab + b^2$

(6) $v_1^2 + v_2^2 + w_1^2 + w_2^2 = w_1^2 - 2w_1v_1 + v_1^2 + w_2^2 - 2w_2v_2 + v_2^2$

$\Uparrow$ add them $\Updownarrow$ subtract $v_1^2 + v_2^2 + w_1^2 + w_2^2$ from both sides $\Downarrow$

(7) $0 = -2w_1v_1 - 2w_2v_2$

$\Updownarrow$ by factoring

(8) $0 = -2(\omega_1 v_1 + \omega_2 v_2)$

$\Updownarrow$ $\Downarrow$ div by -2 $\Uparrow$ mult by -2

(9) $0 = \omega_1 v_1 + \omega_2 v_2$

$\Updownarrow$ by $ab = ba$

(10) $0 = v_1 \omega_1 + v_2 \omega_2$

$\Updownarrow$ by defn of dot product

(11) $0 = \vec{v} \cdot \vec{\omega}$

Notice we justified every step. in green.

QED
"it is proven"
the proof is done

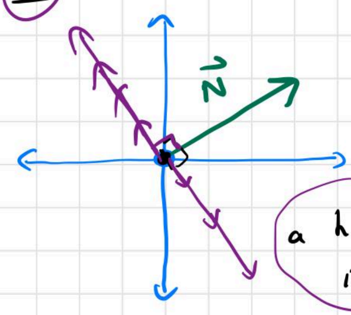
Extra Credit:
write up the proof
for vectors in $\mathbb{R}^3$
and for vectors in $\mathbb{R}^n$
steps 5-10 will be different.
The rest is the same.

$\vec{O}$ = origin $= \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = (0, 0 \ldots 0)$

Defn: A <u>hyperplane</u> through $\vec{O}$ perp to $\vec{N} \neq \vec{O}$ is the set of vectors $\vec{v}$ such that $\vec{v} \cdot \vec{N} = 0$

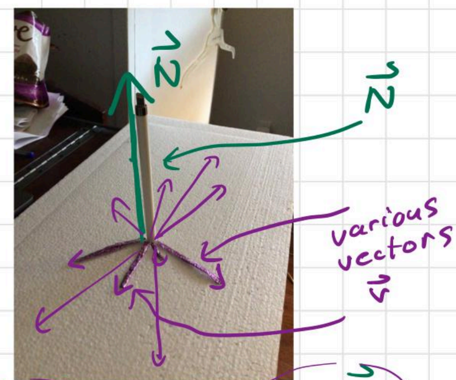
$$\{ \vec{v} \mid \vec{v} \cdot \vec{N} = 0 \} = \text{plane} \perp \text{to } \vec{N} \neq \vec{O}$$

2D



a hyperplane in 2D is a line

3D



The hyperplane $\perp$ to $\vec{N}$ is a plane.

In higher dimensions we cannot draw it.

The solution set of a linear equation

$$a_1 x_1 + a_2 x_2 + \dots + a_m x_m = 0$$

where d is 0

and where $\vec{a} \neq \vec{0}$

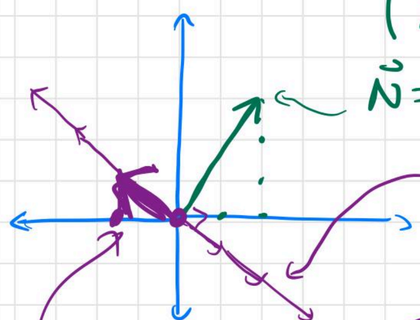
is a hyperplane

$$\{ \vec{x} \mid \vec{a} \cdot \vec{x} = 0 \}$$

Example $\quad 2x + 3y = 0$

$2x_1 + 3x_2 = 0 \quad$ taking $\begin{matrix} x_1 = x \\ x_2 = y \end{matrix}$

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} \cdot \vec{x} = 0$$



$\vec{N} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

all the solutions $\vec{x}$ line on the hyperplane $\perp$ to $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ through the origin

$\begin{pmatrix} -3/2 \\ 1 \end{pmatrix}$

$$\left\{ \vec{x} \mid \begin{pmatrix} 2 \\ 3 \end{pmatrix} \cdot \vec{x} = 0 \right\}$$

Solve this:

$$\boxed{2x_1 + 3x_2 = 0} \xrightarrow{\rho_1 \to \frac{1}{2}\rho_1} \boxed{x_1 + \tfrac{3}{2} x_2 = 0}$$

solve for leader

$x_1 = -\frac{3}{2} x_2$

$x_2 = x_2$ free

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3/2\, x_2 \\ x_2 \end{pmatrix} \;\middle|\; x_2 \in \mathbb{R} \right\}$$

same set

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -3/2 \\ 1 \end{pmatrix} \;\middle|\; x_2 \in \mathbb{R} \right\}$$

all scalar multiples of $\begin{pmatrix} -3/2 \\ 1 \end{pmatrix}$

which is a line with direction

This is $\perp$ to $\vec{N}$

$$\begin{pmatrix} -3/2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \frac{-3}{2} \cdot 2 + 1 \cdot 3 = -3 + 3 = 0 \checkmark$$

Classwork write

$$2x_1 + 3x_2 + 4x_3 = 0$$

as a hyperplane

and as a solution set

$$\left\{ \vec{x} \;\middle|\; \vec{x} \cdot \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = 0 \right\} \qquad \vec{N} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

$$\boxed{2x_1 + 3x_2 + 4x_3 = 0} \xrightarrow{\rho_1 \to \frac{1}{2}\rho_1} \boxed{x_1 + \frac{3}{2}x_2 + 2x_3 = 0}$$

solve for leaders

$$x_1 = -\frac{3}{2}x_2 - 2x_3$$
$$x_2 = x_2 \quad \text{(free)}$$
$$x_3 = x_3 \quad \text{(free)}$$

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2}x_2 - 2x_3 \\ x_2 \\ x_3 \end{pmatrix} \;\middle|\; x_2, x_3 \in \mathbb{R} \right\}$$

This includes a vector where

$x_2 = 1$ and $x_3 = 0$

$$\begin{pmatrix} -\frac{3}{2}(1) - 2(0) \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3/2 \\ 1 \\ 0 \end{pmatrix}$$

Check that vector is $\perp$ to $\vec{N} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

$$\begin{pmatrix} -3/2 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = -\frac{3}{2} \cdot 2 + 1 \cdot 3 + 0 \cdot 4 = -3 + 3 + 0 = 0$$

Also in the solutionset is a vector

where $x_2 = 0$ and $x_3 = 1$

$$\begin{pmatrix} -\frac{3}{2}(0) - 2(1) \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \text{ in the solution set}$$

Check that vector is $\perp$ to $\vec{N} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

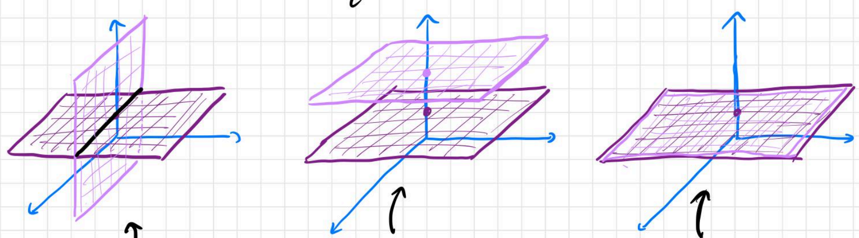
$$\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = -2 \cdot 2 + 0 \cdot 3 + 1 \cdot 4 = -4 + 0 + 4 = 0$$

✓

Also works for $\vec{x} \in \mathbb{R}^m$

m unknowns $x_1, x_2 \ldots x_m$.

In General: 2 equations with 3 unknowns

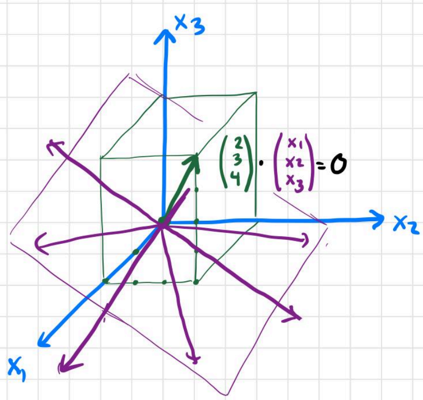


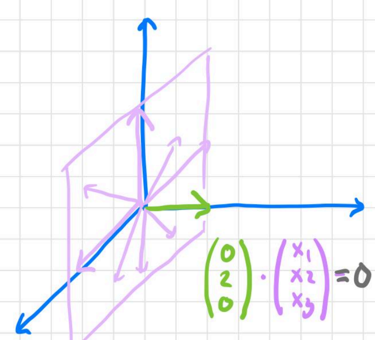
these two planes meet at a line	these two planes never meet	these two planes are the same
Solution Set has one free variable	Solution set is Empty $\emptyset$	Solution Set has two free variables
Example: $\boxed{\begin{aligned} 0x_1+0x_2+1x_3&=1 \\ 0x_1+1x_2+0x_3&=0 \end{aligned}}$	Example: $\boxed{\begin{aligned} 0x_1+0x_2+1x_3&=1 \\ 0x_1+0x_2+1x_3&=3 \end{aligned}}$	Example: $\boxed{\begin{aligned} 0x_1+0x_2+1x_3&=1 \\ 0x_1+0x_2+4x_3&=4 \end{aligned}}$

System of Linear Equations

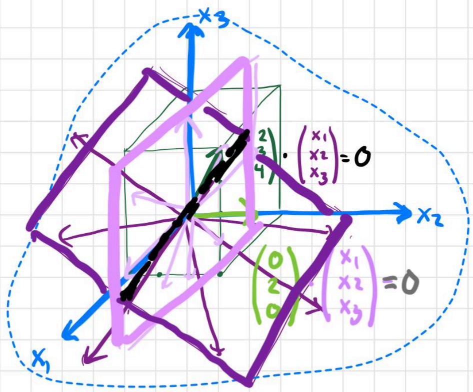
$$\boxed{\begin{aligned} 2x_1+3x_2+4x_3&=0 \\ 0x_1+2x_2+0x_3&=0 \end{aligned}} \leftarrow$$

Find $\vec{x}=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ which solve both equations so they lie on both planes:





Putting the two graphs together:



Which points lie on both planes?

Planes meet along a line.

Try solving the system for classwork.

Augmented Matrix

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right] \xrightarrow{\rho_1 \to \frac{1}{2}\rho_1} \left[\begin{array}{ccc|c} \boxed{1} & 3/2 & 2 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right]$$

box leader

zeroes under leader ✓

$$\xrightarrow{\rho_2 \to \frac{1}{2}\rho_2} \left[\begin{array}{ccc|c} \boxed{1} & 3/2 & 2 & 0 \\ 0 & \boxed{1} & 0 & 0 \end{array}\right]$$

no rows left!

Echelon Form

Pause + Try

$$\boxed{\begin{aligned} \boxed{1 x_1} + \tfrac{3}{2} x_2 + 2x_3 &= 0 \\ 0x_1 + \boxed{1 x_2} + 0x_3 &= 0 \end{aligned}}$$

← Rewrite as a system

Solve for leaders

$x_1 = 0 - \frac{3}{2}x_2 - 2x_3 = -\frac{3}{2}(0) - 2x_3 = -2x_3$

$x_2 = 0 - 0x_3 = 0$ sub up x_2

free $x_3 = x_3$

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_3 \\ 0 \\ x_3 \end{pmatrix} \;\middle|\; x_3 \in \mathbb{R} \right\}$$

Pause + Try

Now write the solution set as a line:

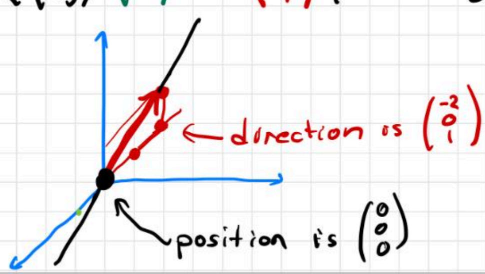
Pause + Try

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 - 2x_3 \\ 0 + 0x_3 \\ 0 + 1x_3 \end{pmatrix} \;\middle|\; x_3 \in \mathbb{R} \right\}$$

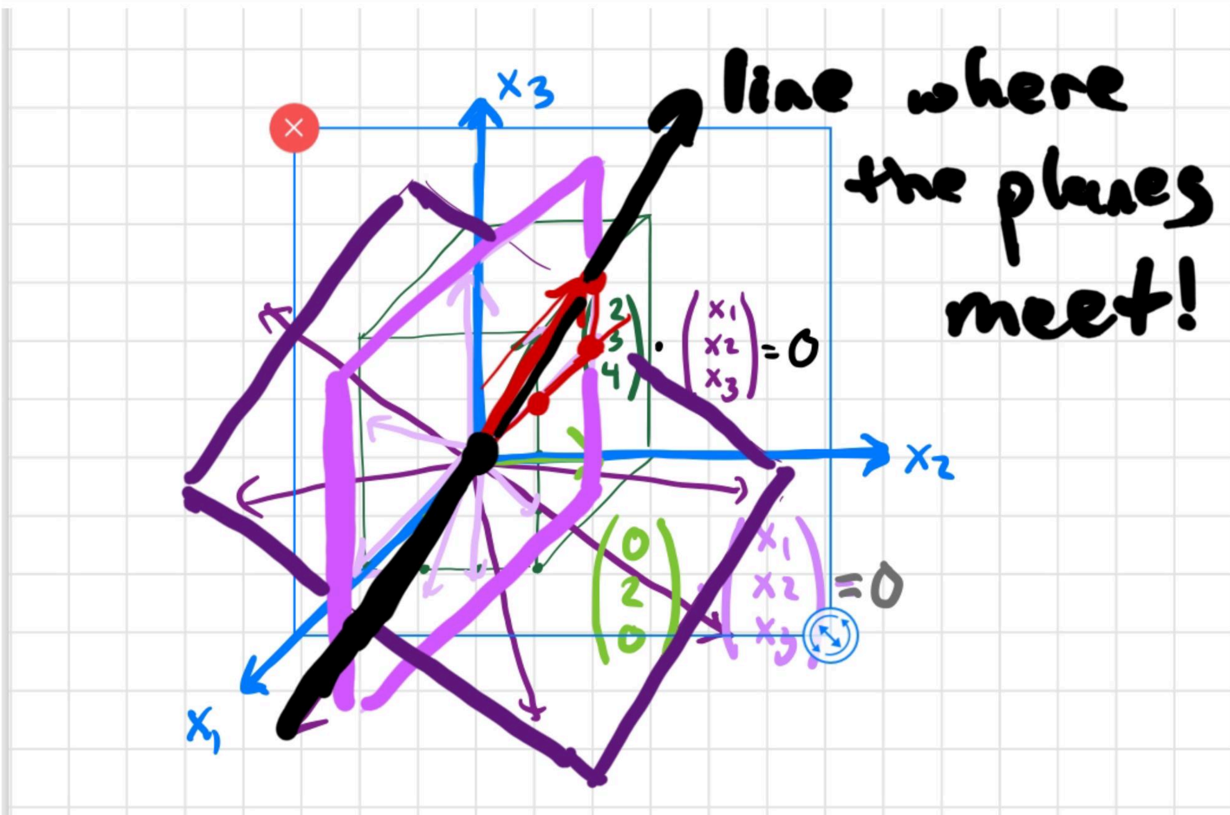
rewrote each term with const + coeffic x_3

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; x_3 \in \mathbb{R} \right\}$$

← This is the line where the planes intersect



Adding this line to the graphs of the two planes together:



You will not be required to make 3D graphs in this course.

Part II: Theorems and Proofs

Watch [Playlist 313S23-L4-P2](). This is essential for the Group Project.

Useful Theorems about Dot Products

Commutativity Thm

$$\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$$

Proof use $a \cdot b = b \cdot a$ for $a, b \in \mathbb{R}$

Distribution over + Thm

$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$$

Proof uses $a(b+c) = ab + ac$ for $a, b, c \in \mathbb{R}$

Distribution over subtraction:

$$\vec{x} \cdot (\vec{y} - \vec{z}) = \vec{x} \cdot \vec{y} - \vec{x} \cdot \vec{z}$$

Proof uses $a(b-c) = ab - ac$

Dot products and Magnitudes Thm

$$\vec{x} \cdot \vec{x} = |\vec{x}|^2$$

Proof uses $a \cdot a = |a|^2$ for $a \in \mathbb{R}$

Distribution over addition

$$(\vec{x} + \vec{y}) \cdot \vec{z} = \vec{x} \cdot \vec{z} + \vec{y} \cdot \vec{z}$$

Proof uses $(a+b)c = ac + bc$

Distribution over − Thm:

$$(\vec{x} - \vec{y}) \cdot \vec{z} = \vec{x} \cdot \vec{z} - \vec{y} \cdot \vec{z}$$

$(a-b)c = ab - ac$ for $a, b, c \in \mathbb{R}$

To prove these theorems we need to also use "the defn of dot product", "defn of $\mathbb{R}^m$" or $\mathbb{R}^3$ or $\mathbb{R}^2$

Distribution over + Thm

$$\vec{x}\cdot(\vec{y}+\vec{z}) = \vec{x}\cdot\vec{y} + \vec{x}\cdot\vec{z}$$

Proof uses $a(b+c) = ab + ac$ for $a, b, c \in \mathbb{R}$

we will prove this in detail

You can prove the rest using the same method.

Proof:

LHS Left Hand side (LHS)

① $LHS = \vec{x}\cdot(\vec{y}+\vec{z})$

② $= \ldots$

③ $= \ldots$

simplify

final $= - - - -$

We must number each step of our proof

We must justify each step explaining why the step is true.

RHS Right Hand Side (RHS)

① $RHS = \vec{x}\cdot\vec{y} + \vec{x}\cdot\vec{z}$

② $=$

③ $=$

simplify

MATCH the final lines.

final try to match the final line of LHS

Thus $LHS = RHS$ $\quad \vec{x}\cdot(\vec{y}+z) = \vec{x}\cdot\vec{y} + \vec{x}\cdot\vec{z}$ QED

Distribution over + Thm

$$\vec{x}\cdot(\vec{y}+\vec{z}) = \vec{x}\cdot\vec{y} + \vec{x}\cdot\vec{z}$$

Proof uses $a(b+c)=ab+ac$ for $a,b,c\in\mathbb{R}$

(for HW you will do a very similar proof) (This will also be on exams)

Proof for $\vec{x},\vec{y},\vec{z}\in\mathbb{R}^2$ (later do $\mathbb{R}^3$ and $\mathbb{R}^m$)

LHS Left Hand side (LHS)

① $\vec{x}\cdot(\vec{y}+\vec{z}) = \begin{pmatrix}x_1\\x_2\end{pmatrix}\cdot\left(\begin{pmatrix}y_1\\y_2\end{pmatrix}+\begin{pmatrix}z_1\\z_2\end{pmatrix}\right)$ ① by defn of $\mathbb{R}^2$

② $= \begin{pmatrix}x_1\\x_2\end{pmatrix}\cdot\begin{pmatrix}(y_1+z_1)\\(y_2+z_2)\end{pmatrix}$ ② by defn of addition of vectors in $\mathbb{R}^2$

③ $= x_1(y_1+z_1) + x_2(y_2+z_2)$ ③ by defn of dot product

④ $= \underline{x_1y_1} + \underline{x_1z_1} + x_2y_2 + x_2z_2$ ④ $a(b+c)=ab+ac$ for $a,b,c\in\mathbb{R}$

RHS Right Hand Side (RHS)

① $\vec{x}\cdot\vec{y}+\vec{x}\cdot\vec{z} = \begin{pmatrix}x_1\\x_2\end{pmatrix}\cdot\begin{pmatrix}y_1\\y_2\end{pmatrix} + \begin{pmatrix}x_1\\x_2\end{pmatrix}\cdot\begin{pmatrix}z_1\\z_2\end{pmatrix}$ ① by defn of $\mathbb{R}^2$

② $= \underline{x_1y_1} + x_2y_2 + \underline{x_1z_1} + x_2z_2$ ② by defn of dot product

③ $= x_1y_1 + x_1z_1 + x_2y_2 + x_2z_2$ ③ $a+b+c+d = a+c+b+d$ for $a,b,c,d\in\mathbb{R}$

MATCH! Step 4 of LHS = Step 3 of RHS

Thus LHS = RHS $\vec{x}\cdot(\vec{y}+\vec{z}) = \vec{x}\cdot\vec{y}+\vec{x}\cdot\vec{z}$

proof is done $\longrightarrow$ QED $\square$

Distribution over + Thm

$$\vec{x}\cdot(\vec{y}+\vec{z}) = \vec{x}\cdot\vec{y} + \vec{x}\cdot\vec{z}$$

Proof uses $a(b+c) = ab+ac$ for $a,b,c \in \mathbb{R}$

(for HW you will do a very similar proof)
(This will also be on exams)

Proof for $\vec{x},\vec{y},\vec{z} \in \mathbb{R}^3$ (later do $\mathbb{R}^3$ and $\mathbb{R}^m$)

LHS Left Hand side (LHS)

① $\vec{x}\cdot(\vec{y}+\vec{z}) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \left(\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \right)$ ① by defn of $\mathbb{R}^3$

② $= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} (y_1+z_1) \\ (y_2+z_2) \\ (y_3+z_3) \end{pmatrix}$ ② by defn of addition of vectors in $\mathbb{R}^3$

③ $= x_1(y_1+z_1) + x_2(y_2+z_2) + x_3(y_3+z_3)$ ③ by defn of dot product in $\mathbb{R}^3$

④ $= x_1y_1 + x_1z_1 + x_2y_2 + x_2z_2 + x_3y_3 + x_3z_3$ ④ $a(b+c) = ab+ac$ for $a,b,c \in \mathbb{R}$

RHS Right Hand Side (RHS) pause + try

① $\vec{x}\cdot\vec{y} + \vec{x}\cdot\vec{z} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$ ① by defn of $\mathbb{R}^3$

② $= x_1y_1 + x_2y_2 + x_3y_3 + x_1z_1 + x_2z_2 + x_3z_3$ ② by defn of dot product

③ $= x_1y_1 + x_1z_1 + x_2y_2 + x_2z_2 + x_3y_3 + x_3z_3$ ③ $a+b+c+d = a+c+b+d$ for $a,b,c,d \in \mathbb{R}$

MATCH! Step 4 of LHS = Step 3 of RHS

Thus LHS = RHS $\quad \vec{x}\cdot(\vec{y}+\vec{z}) = \vec{x}\cdot\vec{y} + \vec{x}\cdot\vec{z}$

proof is done $\rightarrow$ QED $\square$

Distribution over + Thm

$$\vec{x}\cdot(\vec{y}+\vec{z}) = \vec{x}\cdot\vec{y} + \vec{x}\cdot\vec{z}$$

Proof uses $a(b+c) = ab+ac$ for $a,b,c \in \mathbb{R}$

(for HW you will do a very similar proof)
(This will also be on exams)

Proof for $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^m$ (later do $\mathbb{R}^3$ and $\mathbb{R}^m$) $\mathbb{R}^m$

LHS Left Hand side (LHS)

① $\vec{x}\cdot(\vec{y}+\vec{z}) = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \cdot \left(\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} + \begin{pmatrix} z_1 \\ \vdots \\ z_m \end{pmatrix} \right)$ ① by defn of $\mathbb{R}^m$

② $= \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \cdot \begin{pmatrix} y_1+z_1 \\ \vdots \\ y_m+z_m \end{pmatrix}$ ② by defn of addition of vectors in $\mathbb{R}^m$

③ $= x_1(y_1+z_1) + \dots + x_m(y_m+z_m)$ ③ by defn of dot product

④ $= (x_1y_1 + x_1z_1) + \dots + (x_my_m + x_mz_m)$ ④ $a(b+c) = ab+ac$ for $a,b,c \in \mathbb{R}$

RHS Right Hand Side (RHS) pause + try RHS

① $\vec{x}\cdot\vec{y} + \vec{x}\cdot\vec{z} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} + \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ \vdots \\ z_m \end{pmatrix}$ ① by defn of $\mathbb{R}^m$

② $= x_1y_1 + \dots + x_my_m + x_1z_1 + \dots + x_mz_m$ ② by defn of dot product

③ $= x_1y_1 + x_1z_1 + \dots + x_my_m + x_mz_m$ ③ $a+b+c+d = a+c+b+d$ for $a,b,c,d \in \mathbb{R}$

MATCH! Step 4 of LHS = Step 3 of RHS

Thus LHS = RHS $\vec{x}\cdot(\vec{y}+\vec{z}) = \vec{x}\cdot\vec{y} + \vec{x}\cdot\vec{z}$

QED $\square$

proof is done

LHS

Use sum notation for $\mathbb{R}^m$

① same

② same

③ $\sum_{j=1}^{m} x_j(y_j+z_j)$

④ $\sum_{j=1}^{m} (x_jy_j + x_jz_j)$

RHS

① keep as is

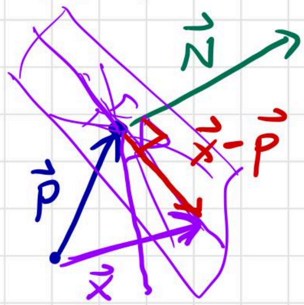
② $\sum_{j=1}^{m} x_jy_j + \sum_{j=1}^{m} x_jz_j$

③ $\sum_{j=1}^{m} (x_jy_j + x_jz_j)$

same justifications

same

A Hyperplane $\perp$ to $\vec{N}$ through position $\vec{p}$



$$(\vec{x}-\vec{p})\cdot\vec{N} = 0$$

$$\vec{x}\cdot\vec{N} - \vec{p}\cdot\vec{N} = 0$$

$$\vec{x}\cdot\vec{N} = \vec{p}\cdot\vec{N}$$

Solution to a Linear equation

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = d$$

$$\vec{N} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \qquad \vec{N}\cdot\vec{x} = \vec{x}\cdot\vec{N} = d$$

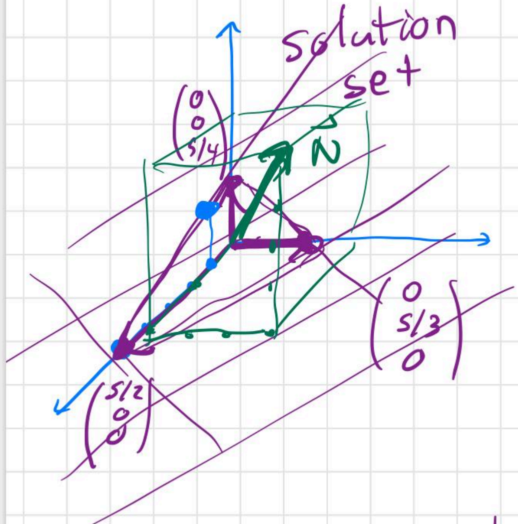
If we call $\vec{p}$ a particular solution to the system. $d = \vec{p}\cdot\vec{N}$

As long as $\vec{N} \neq 0$ there is a solution so we just find one, and that is $\vec{p}$ and so we see the solution to a linear equation with $d \neq 0$ is also a hyperplane.

Example $2x_1 + 3x_2 + 4x_3 = 5$

particular solution $\begin{pmatrix} 5/2 \\ 0 \\ 0 \end{pmatrix} = \vec{P}$

So this is a hyperplane through $\begin{pmatrix} 5/2 \\ 0 \\ 0 \end{pmatrix}$ with normal $\vec{N} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$



To find a particular solution

Just take $x_2 = 0 \quad x_3 = 0$

$2x_1 + 3 \cdot 0 + 4 \cdot 0 = 5$

$2x_1 = 5$

$x_1 = 5/2$

$\begin{pmatrix} 5/2 \\ 0 \\ 0 \end{pmatrix}$

To find another particular solution

try $x_1 = 0 \quad x_3 = 0$

$2 \cdot 0 + 3x_2 + 4 \cdot 0 = 5$

$x_2 = 5/3$

$\begin{pmatrix} 0 \\ 5/3 \\ 0 \end{pmatrix}$

Classwork

$x_1 = 0 \quad x_2 = 0$ find another particular solution

pause + try

$2 \cdot 0 + 3 \cdot 0 + 4x_3 = 5$

$x_3 = 5/4$

$\begin{pmatrix} 0 \\ 0 \\ 5/4 \end{pmatrix}$

Classwork find the solution set.

Classwork find the solution set.

$$\boxed{\boxed{2x_1} + 3x_2 + 4x_3 = 5}$$

divide by 2 to coefficient one

$$\boxed{\boxed{1x_1} + \frac{3}{2}x_2 + \frac{4}{2}x_3 = \frac{5}{2}}$$ Reduced Echelon Form

$$x_1 = \frac{5}{2} - \frac{3}{2}x_2 - \frac{4}{2}x_3$$

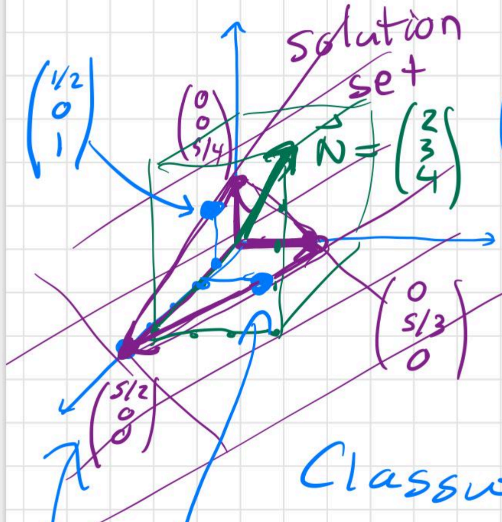
$x_2 = x_2$ (free) $\quad x_3 = x_3$ (free)

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} - \frac{3}{2}x_2 - \frac{4}{2}x_3 \\ x_2 \\ x_3 \end{pmatrix} \middle| \, x_2, x_3 \in \mathbb{R} \right\}$$

notice that $x_2 = 0$ and $x_3 = 1$ gives

a particular solution

$$\begin{pmatrix} \frac{5}{2} - \frac{3}{2}(0) - \frac{4}{2}(1) \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} - \frac{4}{2} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$$



$$\begin{pmatrix} 1/2 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \frac{1}{2} \cdot 2 + 0 \cdot 3 + 1 \cdot 4 = 5$$

5 is correct

$$2x_1 + 3x_2 + 4x_3 = 5 \checkmark$$

Classwork find particular solutions when $x_2 = 1$ and $x_3 = 0$

pause & try.

$$\begin{pmatrix} 5/2 - \frac{3}{2}(1) - \frac{4}{2}(0) \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{5-3}{2} \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Check dot product $\begin{pmatrix}1\\1\\0\end{pmatrix} \cdot \begin{pmatrix}2\\3\\4\end{pmatrix} = 1\cdot 2 + 1\cdot 3 + 0\cdot 4 = 5$ ✓

What is the position?
when all free variables are 0.

$$\begin{pmatrix} 5/2 - \frac{3}{2}(0) - \frac{4}{2}(0) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5/2 \\ 0 \\ 0 \end{pmatrix}$$

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5/2 - \frac{3}{2}x_2 - \frac{4}{2}x_3 \\ 1x_2 + 0x_3 \\ 0x_2 + 1x_3 \end{pmatrix} \,\middle|\, x_2, x_3 \in \mathbb{R} \right\}$$

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5/2 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -3/2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -4/2 \\ 0 \\ 1 \end{pmatrix} \,\middle|\, x_2, x_3 \in \mathbb{R} \right\}$$

position   direction for x_2 direction for x_3

We discuss more in the next lesson.

Homework for Lesson 4:

Hint for HW1: Imitate the proof done in class for distribution over addition.

Lesson 4 Homework

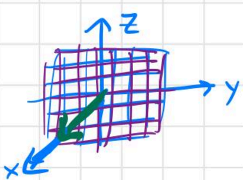
HW1 Prove $\vec{x}\cdot(\vec{y}-\vec{z}) = \vec{x}\cdot\vec{y} - \vec{x}\cdot\vec{z}$ in $\mathbb{R}^2$

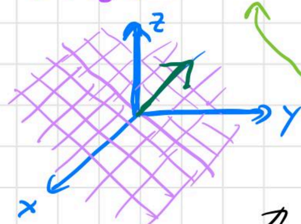
Extra Credit: do the proof for $\mathbb{R}^m$

HW2 Consider the planes

$\left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \middle| \, 1x+0y+0z=0\right\}$ and $\left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \middle| \, 0x+y+z=0\right\}$.

$\vec{N}=?$





$\vec{N}=?$

Find their normals.

Solve the system $\boxed{\begin{matrix}1x+0y+0z=0\\0x+1y+1z=0\end{matrix}}$

to find the line where they intersect.

What is the direction of that line?

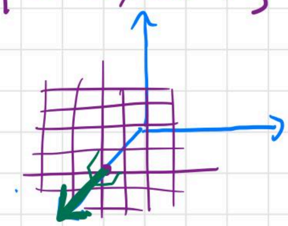
Check it is $\perp$ to both normals.

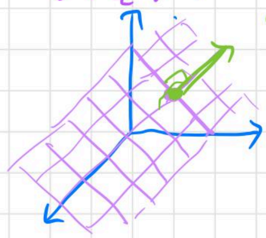
Hint for HW2: it is already in Echelon form, so solve for the leaders and set free variables equal to themselves. After you write the solution set, answer the questions.

$\boxed{HW3}$ Consider the planes

$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \; 1x + 0y + 0z = 1 \right\}$ and $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \; 0x + y + z = 2 \right\}$.

$\vec{N} = ?$





$\vec{N} = ?$

Find their normals.

Solve the system $\boxed{\begin{matrix} 1x + 0y + 0z = 1 \\ 0x + 1y + 1z = 2 \end{matrix}}$

to find the line where they intersect.

What is the direction of that line?

Check it is $\perp$ to both normals.

Note that z is free. Plug in $z=0$ to find a point $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ on the line. Check

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = ? \quad \text{and} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = ?$$

Now plug in $z=1$ to find another $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ on the line. Check:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = ? \quad \text{and} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = ?$$

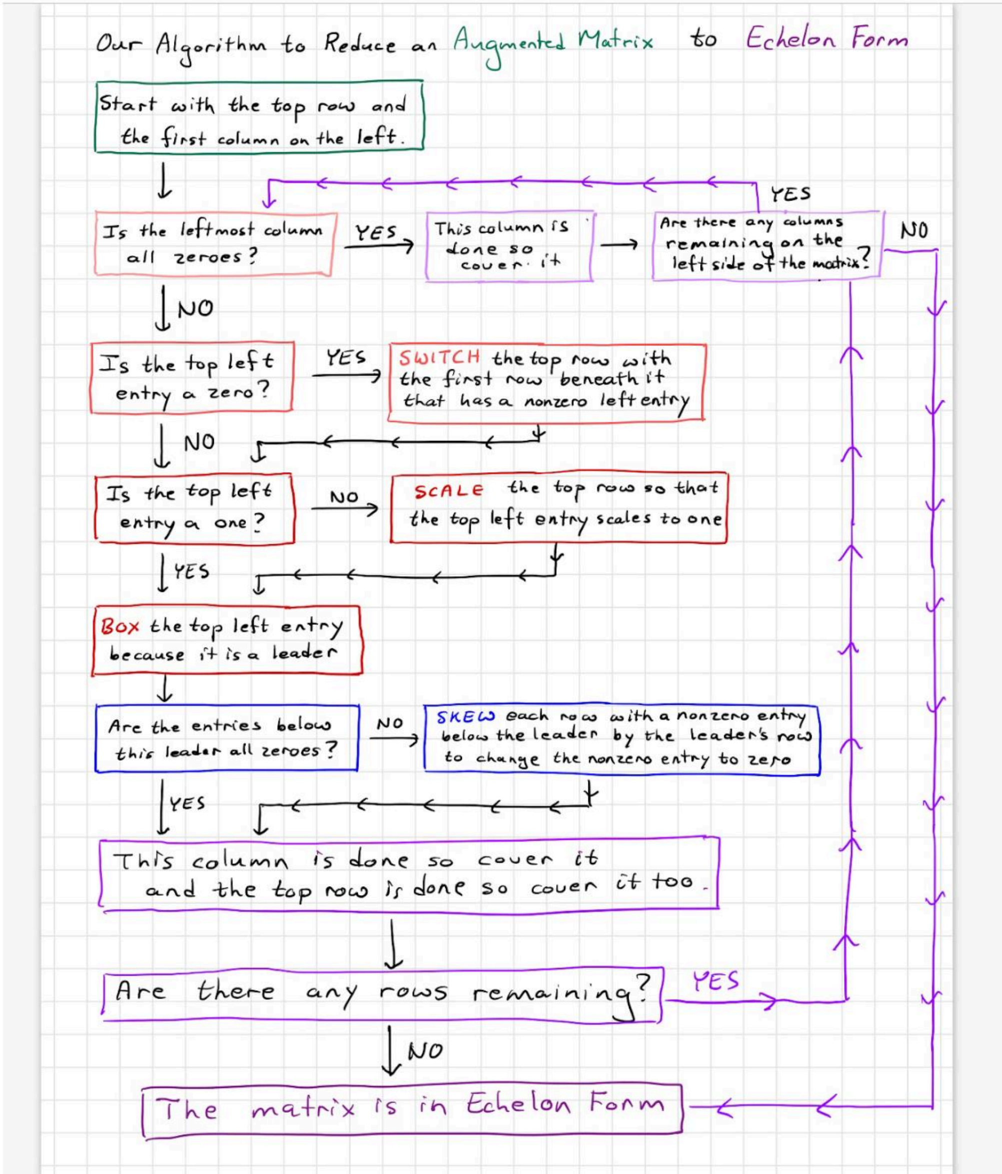
Hint the answers should be 1 and 2

HW3: be sure to take all the dot products requested here.

HW4-HW5 are not required. They are just more practice solving systems and checking them for those who want more practice. Do not fall behind schedule.

HW4-HW5 must be done following the method from class:

A quick glance at our algorithm:



Before doing HW4 and HW5 it is recommended to do this Example.

HW4 Find the normals for these:

$$2x_1 + 4x_2 + 6x_3 + 8x_4 = 12$$

$$x_1 + 3x_2 + 0x_3 + 4x_4 = 12$$

$$2x_1 + 6x_2 + 0x_3 + 5x_4 = 12$$

Solve the system. Note that this is a line with x_3 free.

What is the direction?

Check the direction is $\perp$ to all three normals.

Find the position $\vec{x}$ when $x_3 = 0$

What is $\vec{x} \cdot \vec{N}$ for each normal.

Find the solution $\vec{x}$ when $x_3 = 1$;

What is $\vec{x} \cdot \vec{N}$ for each normal.

Hint answers should be 12

HW5 Repeat HW4 for:

$$2x_1 + 4x_2 + 6x_3 + 8x_4 = 0$$

$$x_1 + 3x_2 + 0x_3 + 4x_4 = 0$$

$$2x_1 + 6x_2 + 0x_3 + 5x_4 = 0$$

Note you are checking all your homework yourself using dot products before submitting it. I will then check if you have done the problems and the checks correctly and will help you with your proof.

If a check doesn't work out, fix your work or type QUESTION in your google doc next to the photo with the problem and email me at sormanic@gmail.com so that I can help.