ALG1 Unit 7 Lesson Summaries (1-10)

Illustrative Mathematics®

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ALGEBRA 1

Unit 7

STUDENT WORKBOOK

Book 3

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Lesson 1 Summary

The height of a softball, in feet, t seconds after someone throws it straight up, can be defined by $f(t) = -16t^2 + 32t + 5$. The input of function f is time, and the output is height.

We can find the output of this function at any given input. For instance:

- At the beginning of the softball's journey, when $t = 0$, its height is given by $f(0)$.
- Two seconds later, when $t = 2$, its height is given by $f(2)$.

The values of $f(0)$ and $f(2)$ can be found using a graph or by evaluating the expression $-16t^2 + 32t + 5$ at those values of t.

What if we know the output of the function and want to find the inputs? For example:

- When does the softball hit the ground?

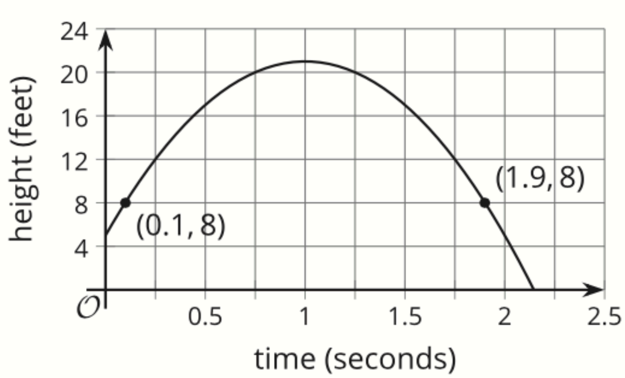
  Answering this question means finding the values of t that make $f(t) = 0$, or solving $-16t^2 + 32t + 5 = 0$.

- How long will it take the ball to reach 8 feet?

  This means finding one or more values of t that make $f(t) = 8$, or solving the equation $-16t^2 + 32t + 5 = 8$.

The equations $-16t^2 + 32t + 5 = 0$ and $-16t^2 + 32t + 5 = 8$ are *quadratic equations*. One way to solve these equations is by graphing $y = f(t)$.

- To answer the first question, we can look for the horizontal intercepts of the graph, where the vertical coordinate is 0.
- To answer the second question, we can look for the horizontal coordinates that correspond to a vertical coordinate of 8.



We can see that there are two solutions to the equation $-16t^2 + 32t + 5 = 8$.

The softball has a height of 8 feet twice, when going up and when coming down, and these occur when t is about 0.1 or 1.9.

Lesson 2 Summary

The height of a potato that is launched from a mechanical device can be modeled by a function, g. Here are two expressions that are equivalent and both define function g.

$$-16x^2 + 80x + 96 \qquad\qquad -16(x-6)(x+1)$$

Notice that one expression is in *standard form* and the other is in *factored form*.

Suppose we wish to know, without graphing the function, the time when the potato will hit the ground. We know that the value of the function at that time is 0, so we can write:

$$-16x^2 + 80x + 96 = 0 \qquad\qquad -16(x-6)(x+1) = 0$$

Let's try solving $-16x^2 + 80x + 96 = 0$, using some familiar moves. For example:

- Subtract 96 from each side: $-16x^2 + 80x = -96$
- Apply the distributive property to rewrite the expression on the left: $-16(x^2 - 5x) = -96$
- Divide both sides by -16: $x^2 - 5x = 6$
- Apply the distributive property to rewrite the expression on the left: $x(x-5) = 6$

These steps don't seem to get us any closer to a solution. We need some new moves!

What if we use the other equation? Can we find the solutions to $-16(x-6)(x+1) = 0$?

Earlier, we learned that the *zeros* of a quadratic function can be identified when the expression defining the function is in factored form. The solutions to $-16(x-6)(x+1) = 0$ are the zeros to function g, so this form may be more helpful! We can reason that:

- If x is 6, then the value of $x - 6$ is 0, so the entire expression has a value of 0.
- If x is -1, then the value of $x + 1$ is 0, so the entire expression also has a value of 0.

This tells us that 6 and -1 are solutions to the equation, and that the potato hits the ground after 6 seconds. (A negative value of time is not meaningful, so we can disregard the -1.)

Both equations we see here are **quadratic equations**. In general, a quadratic equation is an equation that can be expressed as $ax^2 + bx + c = 0$.

In upcoming lessons, we will learn how to rewrite quadratic equations into forms that make the solutions easy to see.

Lesson 3 Summary

Some quadratic equations can be solved by performing the same operation to each side of the equal sign and reasoning about values of the variable would make the equation true.

Suppose we wanted to solve $3(x+1)^2 - 75 = 0$. We can proceed like this:

- Add 75 to each side: $3(x+1)^2 = 75$
- Divide each side by 3: $(x+1)^2 = 25$
- What number can be squared to get 25? $\left(\boxed{\quad}\right)^2 = 25$
- There are two numbers that work, 5 and -5: $5^2 = 25$ and $(\text{-}5)^2 = 25$
- If $x + 1 = 5$, then $x = 4$.
- If $x + 1 = \text{-}5$, then $x = \text{-}6$.

This means that both $x = 4$ and $x = \text{-}6$ make the equation true and are solutions to the equation.

Lesson 4 Summary

The **zero product property** says that if the product of two numbers is 0, then one of the numbers must be 0. In other words, if $a \cdot b = 0$, then either $a = 0$ or $b = 0$. This property is handy when an equation we want to solve states that the product of two factors is 0.

Suppose we want to solve $m(m+9) = 0$. This equation says that the product of m and $(m+9)$ is 0. For this to be true, either $m = 0$ or $m + 9 = 0$, so both 0 and -9 are solutions.

Here is another equation: $(u - 2.345)(14u + 2) = 0$. The equation says the product of $(u - 2.345)$ and $(14u + 2)$ is 0, so we can use the zero product property to help us find the values of u. For the equation to be true, one of the factors must be 0.

- For $u - 2.345 = 0$ to be true, u would have to be 2.345.
- For $14u + 2 = 0$ or $14u = \text{-}2$ to be true, u would have to be $\text{-}\frac{2}{14}$ or $\text{-}\frac{1}{7}$.

The solutions are 2.345 and $\text{-}\frac{1}{7}$.

In general, when a quadratic expression in factored form is on one side of an equation and 0 is on the other side, we can use the zero product property to find its solutions.

Lesson 5 Summary

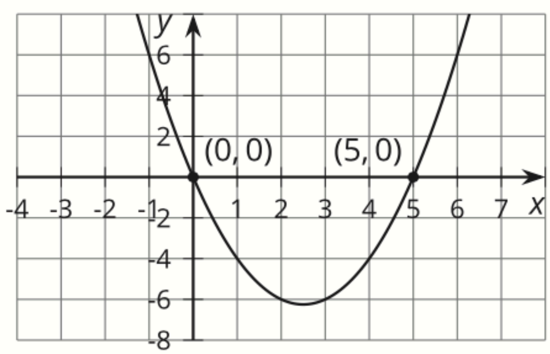
Quadratic equations can have two, one, or no solutions.

We can find out how many solutions a quadratic equation has and what the solutions are by rearranging the equation into the form of $\text{expression} = 0$, graphing the function that the expression defines, and determining its zeros. Here are some examples.

- $x^2 = 5x$

  Let's first subtract $5x$ from each side and rewrite the equation as $x^2 - 5x = 0$. We can think of solving this equation as finding the zeros of a function defined by $x^2 - 5x$.

  If the output of this function is y, we can graph $y = x^2 - 5x$ and identify where the graph intersects the x-axis, where the y-coordinate is 0.



  From the graph, we can see that the x-intercepts are $(0, 0)$ and $(5, 0)$, so $x^2 - 5x$ equals 0 when x is 0 and when x is 5.

  The graph readily shows that there are two solutions to the equation.

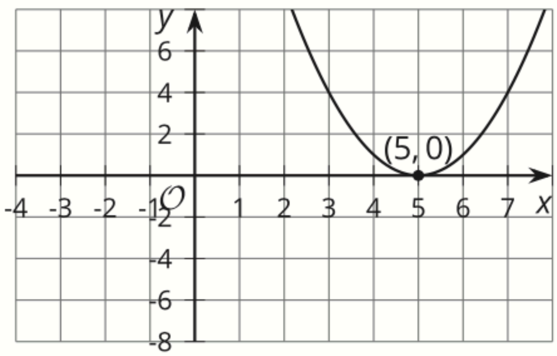
Note that the equation $x^2 = 5x$ can be solved without graphing, but we need to be careful *not* to divide both sides by x. Doing so will give us $x = 5$ but will show no trace of the other solution, $x = 0$!

Even though dividing both sides by the same value is usually acceptable for solving equations, we avoid dividing by the same variable because it may eliminate a solution.

- $(x - 6)(x - 4) = -1$

  Let's rewrite the equation as $(x - 6)(x - 4) + 1 = 0$, and consider it to represent a function defined by $(x - 6)(x - 4) + 1$ and whose output, y, is 0.

  Let's graph $y = (x - 6)(x - 4) + 1$ and identify the x-intercepts.



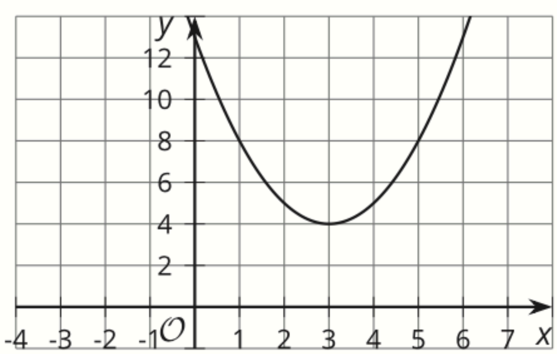
The graph shows one x-intercept at $(5, 0)$. This tells us that the function defined by$(x-6)(x-4)+1$ has only one zero.

It also means that the equation $(x-6)(x-4)+1=0$ is true only when $x=5$. The value 5 is the only solution to the equation.

- $(x-3)(x-3) = \text{-}4$

  Rearranging the equation gives $(x-3)(x-3)+4=0$.

  Let’s graph $y=(x-3)(x-3)+4$ and find the x-intercepts.



The graph does not intersect the x-axis, so there are no x-intercepts.

This means there are no x-values that can make the expression $(x-3)(x-3)+4$ equal 0, so the function defined by $y=(x-3)(x-3)+4$ has no zeros.

The equation $(x-3)(x-3) = \text{-}4$ has no solutions.

We can see that this is the case even without graphing. $(x-3)(x-3) = \text{-}4$ is $(x-3)^2 = \text{-}4$. Because no number can be squared to get a negative value, the equation has no solutions.

Earlier you learned that graphing is not always reliable for showing precise solutions. This is still true here. The x-intercepts of a graph are not always whole-number values. While they can give us an idea of how many solutions there are and what the values may be (at least approximately), for exact solutions we still need to rely on algebraic ways of solving.

Lesson 6 Summary

Previously, you learned how to expand a quadratic expression in factored form and write it in standard form by applying the distributive property.

For example, to expand $(x + 4)(x + 5)$, we apply the distributive property to multiply x by $(x + 5)$ and 4 by $(x + 5)$. Then, we apply the property again to multiply x by x and x by 5, and multiply 4 by x and 4 by 5.

To keep track of all the products, we could make a diagram like this:

	x	4
x		
5		

Next, we could write the products of each pair inside the spaces:

	x	4
x	x^2	$4x$
5	$5x$	$4 \cdot 5$

The diagram helps us see that $(x + 4)(x + 5)$ is equivalent to $x^2 + 5x + 4x + 4 \cdot 5$, or in standard form, $x^2 + 9x + 20$.

- The *linear term*, $9x$, has a *coefficient* of 9, which is the sum of 5 and 4.
- The *constant term*, 20, is the product of 5 and 4.

We can use these observations to reason in the other direction: to start with an expression in standard form and write it in factored form.

For example, suppose we wish to write $x^2 - 11x + 24$ in factored form.

Let's start by creating a diagram and writing in the terms x^2 and 24.

We need to think of two numbers that multiply to make 24 and add up to -11.

	x	
x	x^2	
		24

After some thinking, we see that -8 and -3 meet these conditions.

The product of -8 and -3 is 24. The sum of -8 and -3 is -11.

	x	-8
x	x^2	$-8x$
-3	$-3x$	24

So, $x^2 - 11x + 24$ written in factored form is $(x - 8)(x - 3)$.

Lesson 7 Summary

When we rewrite expressions in factored form, it is helpful to remember that:

- Multiplying two positive numbers or two negative numbers results in a positive product.
- Multiplying a positive number and a negative number results in a negative product.

This means that if we want to find two factors whose product is 10, the factors must be both positive or both negative. If we want to find two factors whose product is -10, one of the factors must be positive and the other negative.

Suppose we wanted to rewrite $x^2 - 8x + 7$ in factored form. Recall that subtracting a number can be thought of as adding the opposite of that number, so that expression can also be written as $x^2 + \text{-}8x + 7$. We are looking for two numbers that:

- Have a product of 7. The candidates are 7 and 1, and -7 and -1.
- Have a sum of -8. Only -7 and -1 from the list of candidates meet this condition.

The factored form of $x^2 - 8x + 7$ is therefore $(x + \text{-}7)(x + \text{-}1)$ or, written another way, $(x - 7)(x - 1)$.

To write $x^2 + 6x - 7$ in factored form, we would need two numbers that:

- Multiply to make -7. The candidates are 7 and -1, and -7 and 1.
- Add up to 6. Only 7 and -1 from the list of candidates add up to 6.

The factored form of $x^2 + 6x - 7$ is $(x + 7)(x - 1)$.

Lesson 8 Summary

Sometimes expressions in standard form don't have a linear term. Can they still be written in factored form?

Let's take $x^2 - 9$ as an example. To help us write it in factored form, we can think of it as having a linear term with a coefficient of 0: $x^2 + 0x - 9$. (The expression $x^2 - 0x - 9$ is equivalent to $x^2 - 9$ because 0 times any number is 0, so $0x$ is 0.)

We know that we need to find two numbers that multiply to make -9 and add up to 0. The numbers 3 and -3 meet both requirements, so the factored form is $(x + 3)(x - 3)$.

To check that this expression is indeed equivalent to $x^2 - 9$, we can expand the factored expression by applying the distributive property: $(x + 3)(x - 3) = x^2 - 3x + 3x + (\text{-}9)$. Adding $\text{-}3x$ and $3x$ gives 0, so the expanded expression is $x^2 - 9$.

In general, a quadratic expression that is a difference of two squares and has the form:

$a^2 - b^2$	can be rewritten as:	$(a + b)(a - b)$

Here is a more complicated example: $49 - 16y^2$. This expression can be written $7^2 - (4y)^2$, so an equivalent expression in factored form is $(7 + 4y)(7 - 4y)$.

What about $x^2 + 9$? Can it be written in factored form?

Let's think about this expression as $x^2 + 0x + 9$. Can we find two numbers that multiply to make 9 but add up to 0? Here are factors of 9 and their sums:

- 9 and 1, sum: 10
- -9 and -1, sum: -10
- 3 and 3, sum: 6
- -3 and -3, sum: -6

For two numbers to add up to 0, they need to be opposites (a negative and a positive), but a pair of opposites cannot multiply to make positive 9, because multiplying a negative number and a positive number always gives a negative product.

Because there are no numbers that multiply to make 9 and also add up to 0, it is not possible to write $x^2 + 9$ in factored form using the kinds of numbers that we know about.

Lesson 9 Summary

Recently, we learned strategies for transforming expressions from standard form to factored form. In earlier lessons, we have also seen that when a quadratic expression is in factored form, it is pretty easy to find values of the variable that make the expression equal zero. Suppose we are solving the equation $x(x+4)=0$, which says that the product of x and $x+4$ is 0. By the zero product property, we know this means that either $x=0$ or $x+4=0$, which then tells us that 0 and -4 are solutions.

Together, these two skills—writing quadratic expressions in factored form and using the zero product property when a factored expression equals 0—allow us to solve quadratic equations given in other forms. Here is an example:

$$\begin{aligned} n^2-4n&=140 && \text{Original equation}\\ n^2-4n-140&=0 && \text{Subtract 140 from each side so the right side is 0}\\ (n-14)(n+10)&=0 && \text{Rewrite in factored form} \end{aligned}$$

$$\begin{aligned} n-14=0 \quad &\text{or} \quad n+10=0 && \text{Apply the zero product property}\\ n=14 \quad &\text{or} \quad n=\text{-}10 && \text{Solve each equation} \end{aligned}$$

When a quadratic equation is written as $\text{expression in factored form}=0$, we can also see the number of solutions the equation has.

In the example earlier, it was not obvious how many solutions there would be when the equation was $n^2-4n-140=0$. When the equation was rewritten as $(n-14)(n+10)=0$, we could see that there were two numbers that could make the expression equal 0: 14 and -10.

How many solutions does the equation $x^2-20x+100=0$ have?

Let's rewrite it in factored form: $(x-10)(x-10)=0$. The two factors are identical, which means that there is only one value of x that makes the expression $(x-10)(x-10)$ equal 0. The equation has only one solution: 10.

Lesson 10 Summary

Only some quadratic equations in the form of $ax^2 + bx + c = 0$ can be solved by rewriting the quadratic expression into factored form and using the zero product property. In some cases, finding the right factors of the quadratic expression is quite difficult.

For example, what is the factored form of $6x^2 + 11x - 35$?

We know that it could be $(3x + \square)(2x + \square)$, or $(6x + \square)(x + \square)$, but will the second number in each factor be -5 and 7, 5 and -7, 35 and -1, or -35 and 1? And in which order?

We have to do some guessing and checking before finding the equivalent expression that would allow us to solve the equation $6x^2 + 11x - 35 = 0$.

Once we find the right factors, we can proceed to solving using the zero product property, as shown here:

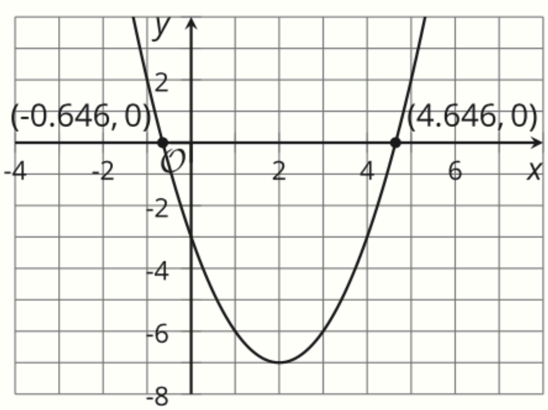
$$6x^2 + 11x - 35 = 0$$
$$(3x - 5)(2x + 7) = 0$$

$$3x - 5 = 0 \quad \text{or} \quad 2x + 7 = 0$$
$$x = \frac{5}{3} \quad \text{or} \quad x = -\frac{7}{2}$$

What is even trickier is that most quadratic expressions can't be written in factored form!

Let's take $x^2 - 4x - 3$ for example. Can you find two numbers that multiply to make -3 and add up to -4? Nope! At least not easy-to-find rational numbers.

We can graph the function defined by $x^2 - 4x - 3$ using technology, which reveals two x-intercepts, at around $(-0.646, 0)$ and $(4.646, 0)$. These give the approximate zeros of the function, -0.646 and 4.646, so they are also approximate solutions to $x^2 - 4x - 3 = 0$.



The fact that the zeros of this function don't seem to be simple rational numbers is a clue that it may not be possible to easily rewrite the expression in factored form.

It turns out that rewriting quadratic expressions in factored form and using the zero product property is a very limited tool for solving quadratic equations.

In the next several lessons, we will learn some ways to solve quadratic equations that work for any equation.