By the last time, we saw the goal and process of the algorithm BlockedEdges. We now show its convergence property.

Lemma 4.8: σ such that $z_j \cap d(f(\sigma)) \neq \Phi$ for the selected z_j are much fewer than the currently remaining $\sigma \in \boldsymbol{Q}$ (4-1-14)

We prove it below, and also show Corollary 4.9.

__**Lemma 4.8**__

Thanks to the disjointness between $y_1, y_2, ..., y_q$ in the split **y**, we can focus on the current subspace y_j in the j^{th} step of BlockedEdges. Its Step 1-2 constructs pairs (z_j, σ) such that

1. $|z_j|=n^{11/6}$,
2. the removal of z_j leaves no $n^{1/5}$-clique in y_j, and

¬3. $z_j \cap d(f(\sigma)) \neq \Phi$ (4-1-13)

Step 1-4 determines the blocked edge set z_j in y_j, so that it's incident to the minimum number of σ.

Let's count the total number of (z_j, σ). We have the bound

$\#(z_j, \sigma) \leqq M\ |\boldsymbol{Q}|$, where

$$M = \binom{\binom{|y_j|}{2}-1}{|z_j|-1} |d\left(f(\sigma)\right)| < \frac{|z_j|n^{2\epsilon}}{\binom{|y_j|}{2}} \binom{\binom{|y_j|}{2}}{|z_j|}$$ (4-1-15)

- It's obtained by upper-bounding $\#z_j$ incident to each given $\sigma=(g, g_1, g_2, \alpha) \in \boldsymbol{Q}$
- For the σ and each edge e in $d(f(\sigma))$, there are at most
  { |{y_j \choose 2}| - 1 \choose $|z_j|$ -1 }
  pairs σ such that (4-1-13).
  - Because e fixes one edge in {y_j \choose 2} and one in z_j
  - The above number is $\#z_j$ in y_j containing the particular edge e
- There are $|d(\sigma)|$ edges in $d(\sigma)$, so
  #((z_j, σ) for this particular σ) $\leqq$ M
- Hence $\#(z_j, \sigma) \leqq M\ |\boldsymbol{Q}|$
- Notice the following two for (4-1-15)
  - $| d(f(\sigma)) | < n^{2\varepsilon}$ because $|g_1 \cup g_2| < n^{\varepsilon}$ by (4-1-3) here
  - $\binom{\binom{|y_j|}{2}}{|z_j|} = \frac{\binom{|y_j|}{2}}{|z_j|}\binom{\binom{|y_j|}{2}-1}{|z_j|-1}$ by the identity
    {p \choose q} = {p-1 \choose q-1} p /q

Thus, there exists z_j incident to at most

$$\frac{M\,|\mathcal{Q}|}{\binom{\binom{|y_j|}{2}}{|z_j|}e^{-o(1)}} < \frac{|z_j|n^{2\epsilon}}{\binom{|y_j|}{2}e^{-o(1)}}\,|\mathcal{Q}| = O\left(n^{-\frac{1}{6}+12\epsilon}\,|\mathcal{Q}|\right) \ll |\mathcal{Q}|$$

(4-1-16)

pairs (z_j, σ)

- z_j such that 2 (whose removal leaves no $n^{1/5}$-cliques in y_j) form a majority in y_j
  - As proven last time

  → Their number is M' = { |{y \choose 2}| \choose |z_j| } exp(-o(1))
- There exists at least one z_j incident to ≦M |***Q***| / M' quadruples σ
- Substitute (4-1-15) into this to obtain (4-1-16)
  - Also notice |z_j| =$n^{11/6}$ and |{y_j \choose 2}|∼$n^{2-10ε}$ / 2

The edge set z_j chosen by Step 1-4 satisfies (4-1-16). Thus, #σ such that (4-1-13) is much fewer than the current total number |***Q***| of σ. This confirms (4-1-14).

Corollary 4.9

Step 1-5 of BlockedEdges puts all the σ violating Rule 3 into ***Q***(z_j)

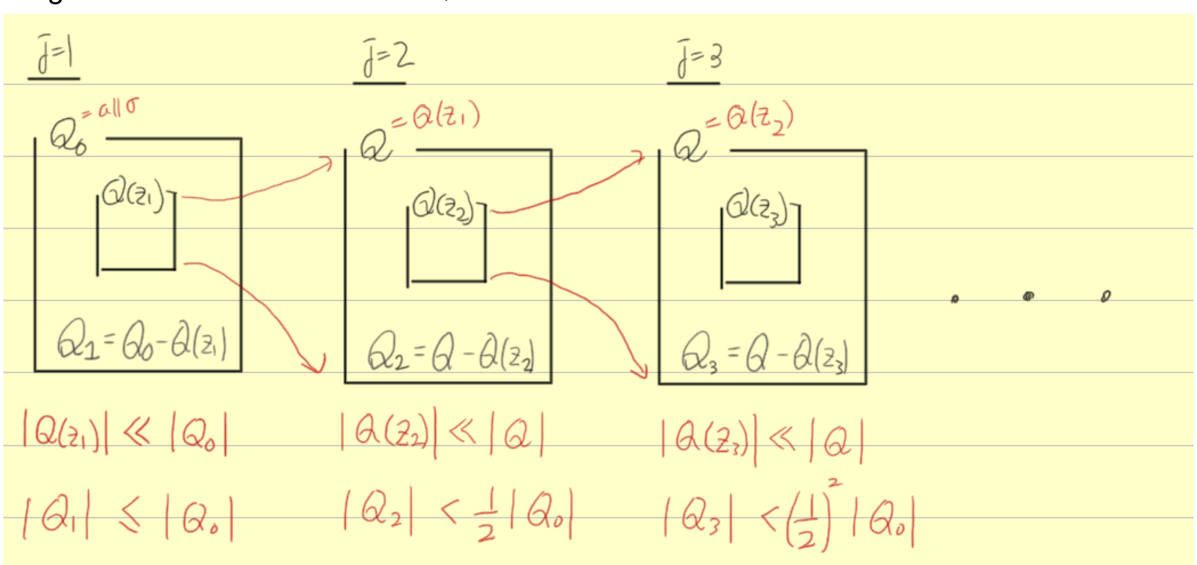
$$\text{1-5.}\;\; \mathcal{Q}(z_j) \leftarrow \{\sigma \in \mathcal{Q}\;;\; d\,(f(\sigma)) \cap z_j \neq \emptyset\};$$

For the next value of j, this will be the current ***Q*** where the rest

Q$_j$ = ***Q*** - ***Q***(z_j)

is the family of σ whose y(σ) has been just determined as y_j.

The figure below summarizes how ***Q*** reduces its size



This is what the corollary claims.

Corollary 4.9 *For each $j \in [q]$,*

$$\mathcal{Q}_j = \{\sigma \in \mathcal{Q}_0 \; ; \; y(\sigma) = y_j\} \quad \text{and} \quad |\mathcal{Q}_j| < 2^{-j} |\mathcal{Q}_0|,$$

after BLOCKEDEDGES *terminates.*

(2^{-j} should be 2^{-j+1}.)

We found in Quadruples σ that the total number of σ is $\exp(O(n^{\varepsilon}))$. The number of y_j in **y** is $q=n^{5\varepsilon}$ that is asymptotically larger than ln $|\boldsymbol{Q}_0| = O(n^{\varepsilon})$. By the corollary,

$$Q_q = \Phi \qquad (4\text{-}1\text{-}17)$$

Let's check if $\boldsymbol{Q}_j$ becomes really empty for sufficiently large j. While ***Q*** is large, the family keeps reducing its size exponentially by Lemma 4.8. Before j reaches q, there is a point where there are at most constant σ in the current ***Q***. BlockedEdges still chooses z_j with the same logic for this small number of σ: so that #σ violating Rule 3 is minimum. It is zero. i.e., the algorithm chooses z_j incident to no σ. Every remaining σ∈***Q*** satisfies Rule 3: $z_j \cap d(f(\sigma)) = \Phi$. In the next step, Thus ***Q*** becomes empty.