Definitions and Background

Notations and Conventions

Z: The set of all integers.
Q: The set of all rational numbers.
R: The set of all real numbers.
$X^{+/-}$: The positive/negative set of something (Ex. Z^+ is the set of all ***positive*** integers)

Proofs

1. Implication (if X then Y)
    a. Denoted as X →Y with the meaning that if the equation or condition of X is true, then the equation or condition Y is also true
2. Equivalence (X if and only if Y)
    a. Denoted as X⇔Y or X↔Y with the meaning that [(if X then Y) and (if Y then X)]. Both conditions must be satisfied for the relationship to be true.
3. Disproving
    a. Also called “finding the counterexample”. Given the example, X → Y, find an element that satisfies X but not Y. For instance, if $x^2 > 0$, then $x > 0$. However, we can find an element that satisfies the first condition, but not the second, by saying $x = -1$. $(-1)^2 = 1$ which is greater than 0, however, -1 itself is not greater than 0. Therefore, we disproved the statement ‘if $x^2 > 0$, then $x > 0$’ by finding a counterexample.
4. Contradiction
    a. First, assume that the given theorem (X → Y) is wrong, i.e. X is true but Y is false. You can then find an instance where X is false. However, this contradicts the original statement, proving that the assumption is false and will never happen. This in turn makes the original statement true.
        i. Example: if $x + 1 > 2$, then $x > 1$. To prove by contradiction, we assume there is an x such that if $x + 1 > 2$, then $x <= 1$. From this assumption that $x <= 1$, we can say that $x + 1 <= 1 + 1$

Types of Statements

- Universal Conditional Statement
    - The statement is both universal and conditional
        - Example: For all animals *a*, if *a* is a cat, then *a* is a mammal
- Universal Existential Statement
    - Universal because the first part of the statement describes a situation that is true for every object described, and existential because the second part describes the existence of something
        - Example: Every real number has an additive inverse.

- Existential Universal Statement
    - The first part describes that a specific object exists (existential) and the second part says that the object satisfies a particular property for all things of a certain kind (universal)
        - Example: There is a positive integer that is less than or equal to every positive integer.

Subsets

- If A and B are sets, then A is a subset of B if and only if every element of A is also an element of B; written as $A \subseteq B$.
- A is called a *proper subset* of B if and only if every element of A is also an element of B, but B contains at least one element that is not included in A.

Cartesian Product

- If given sets A and B, then the Cartesian product of A and B, written $A \times B$ and read "A cross B", is the set of all ordered pairs (a, b) where a is in A and b is in B.
- Symbolically, $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$.

Relations

- If given sets A and B, then a relation R from A to B is a subset of $A \times B$. Given an ordered pair (x, y) in $A \times B$, x is related to y by R, written as x R y, if and only if (x, y) is within R.
- In this situation, A is also known as the domain of R, while B is the co-domain of R.
- Symbolically, $x \, R \, y = (x, y) \in R$.

Functions

- A relation F from A to B is a function, if and only if:
    - Every element of A is the first element of an ordered pair of F
    - No two distinct ordered pairs in F have the same first element

Logic Symbols

Symbol	**Logic**
~	not
$\wedge$	and
$\vee$	or

Sample Problems

Set-Roster Notation

Questions

1. Let A = {1, 2, 3, 4}, B = {3, 4, 2, 1}, and C = {2, 2, 1, 4, 3, 2, 3, 1, 4}. What are the elements of A, B, and C? How are these three sets related?
2. Does {0} = 0?
3. How many elements are in the set {2, {2}}?
4. For each non-negative integer *a*, let $R_a = \{a, -a\}$. Find R_2, R_5, and R_0.

Answers

1. Sets A, B, and C have the same elements; namely, 1, 2, 3, and 4. The three sets represent the same elements in different ways.
2. No. {0} is a set with one element in it, the number zero. A set cannot equal a number.
3. Two elements: the element (number) 2 and the set containing another element.
4. $R_2 = \{2, -2\}$, $R_5 = \{5, -5\}$, and $R_0 = \{0, -0\} = \{0, 0\} = \{0\}$.

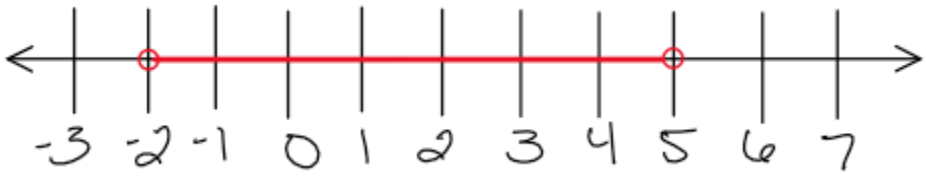
Set-Builder Notation

Questions

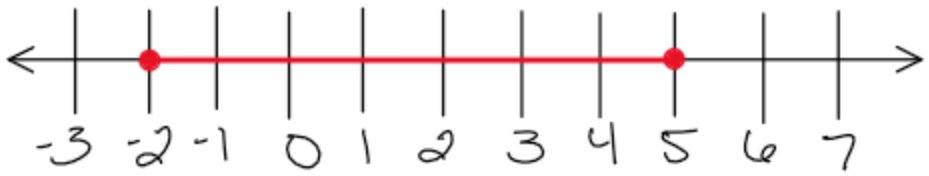
Given that R denotes all real numbers, Z denotes all integers, and Z^+ denotes all positive integers, describe the following sets:

1. $\{x \in R \mid -2 < x < 5\}$
2. $\{x \in R \mid -2 \leq x \leq 5\}$
3. $\{x \in Z \mid -2 < x < 5\}$
4. $\{x \in Z^+ \mid -2 < x < 5\}$

Answers

1. 

This picture represents the <u>open</u> interval of all real numbers between -2 and 5 (does not include -2 and 5).

2. 

This picture represents the <u>closed</u> interval of all real numbers between -2 and 5 (including -2 and 5).
3. $\{x \in Z \mid -2 < x < 5\}$ is the set that denotes all integers between and excluding -2 and 5; can also be written as {-1, 0, 1, 2, 3, 4}.
4. $\{x \in Z^+ \mid -2 < x < 5\}$ is the set of all positive integers between and excluding -2 and 5; can also be written as {1, 2, 3, 4} (Note: 0 is not considered a positive integer).

<u>Subsets</u>

Questions

Let $A = Z^+$, $B = \{x \in Z \mid 0 \leq n \leq 100\}$, and C = {100, 200, 300, 400, 500, 600}. Evaluate the following statements as true or false and explain why.

1. $B \subseteq A$
2. $A \subseteq B$
3. C is a proper subset of A
4. C and B have at least one element in common
5. $C \subseteq B$
6. $B \subseteq B$

Answers

1. False. Zero is not considered a positive integer and therefore cannot be contained within A. Thus, there is an element in B that is not A, and $B \not\subseteq A$.
2. False. A is a set that contains all positive integers, while set B is bounded between 0 and 100. Therefore, there are elements in A that are not in B, thus $A \not\subseteq B$.
3. True. Each element of C is contained in A, but there is at least one element in A not contained in C. Therefore, C is considered a proper subset of A.
4. True. The element 100 is contained in both sets B and C, therefore the two sets have at least one element in common.
5. False. There are elements in C not contained in B.
6. True. Every element of B is in B. The definition of a subset implies that all sets are subsets of themselves.

The Difference between ∈ (element of) and ⊆ (subset of)

Questions

Which of the following are true and which are false? Why?

1. $2 \in \{1, 2, 3\}$
2. $\{2\} \in \{1, 2, 3\}$
3. $2 \subseteq \{1, 2, 3\}$
4. $\{2\} \subseteq \{1, 2, 3\}$
5. $\{2\} \subseteq \{\{1\}, \{2\}\}$
6. $\{2\} \in \{\{1\}, \{2\}\}$

Answers

1. True. The number two is an element of the set {1, 2, 3}.
2. False. The set {2} is not an element of the set {1, 2, 3} (*Remember, {2} ≠ 2).
3. False. A number by itself cannot be a subset of another set.
4. True. The set {2} contains the element 2, which is also contained within the set {1, 2, 3}, therefore it is a subset.
5. False. The set {2} contains the element 2 which is not contained within the set {{1}, {2}}, therefore it is not a subset.
6. True. The set {2} is an element of the set {{1}, {2}}.

Cartesian Product

Questions

Given A = {1, 2, 3} and B = {a, b, c}:

1. Find A × B
2. Find B × A
3. Find B × B

Answers

1. A × B = {(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c)}
2. B × A = {(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3), (c, 1), (c, 2), (c, 3)}
3. B × B = {(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)}

Relations

Questions

Given A = {1, 2} and B = {1, 2, 3}, the relation R between A and B is as follows:

$$(x, y) \in R \text{ such that } \frac{(x+y)}{2} \text{ is an integer.}$$

1. Find A × B
2. Find R
3. Is 1 R 3?

4. Is 2 R 1?
5. Is 2 R 3?
6. State the domain and co-domain of R.

Answers

1. $A \times B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}$
2. To find R, check all of the ordered pairs from $A \times B$ to see if they match the definition of R provided above.
    a. $(1, 1) \in R$ because $1 + 1 = 2$, $2 / 2 = 1$ which is an integer.
    b. $(1, 2) \notin R$ because $1 + 2 = 3$, $3 / 2 = 1.5$ which is not an integer.
    c. $(1, 3) \in R$
    d. $(2, 1) \notin R$
    e. $(2, 2) \in R$
    f. $(2, 3) \notin R$
        i. Therefore, $R = \{(1,1), (1, 3), (2, 2)\}$
3. Yes, $(1,3) \in R$
4. No, $(2, 1) \notin R$
5. No, $(2, 3) \notin R$
6. The domain is {1, 2} and the co-domain is {1, 2, 3}

<u>Truth Tables and Values</u>

x	**y**	**~x**	**x ∧ y**	**x ∨ y**	**~x ∧ y**
0	0	1	0	0	0
0	1	1	0	1	1
1	0	0	0	1	0
1	1	0	1	0	0