People say to save your money and retire. It seems intuitively obvious to do this, but let's do the math.

Allocation

There's some evidence that the quality of your experience is some constant times the logarithm of your earnings. I'll assume money is spent continuously according to the function $\$$ of time T and that experiential utility depends continuously on your current spending according to the function $U(\$) = c \times log(\$)$. Yes, this point is contended, so the rest of this doc could be meaningless. Furthermore, I'll assume money grows, when invested in the market, at a perfect exponential r_1^T and the price of goods similarly follows the curve r_2^T. The latter means that experiential utility is really $c \times log(\frac{\$}{r_2^T})$. For simplicity I'll use the consequence that money's value after investment is $\left(\frac{r_1}{r_2}\right)^T$, I'll use $R = \frac{r_1}{r_2}$, and I'll use inflation-adjusted dollars throughout this doc. Finally, I'll assume that present mortality tables will remain reasonably consistent over time, with mortality at a particular age improving as it has historically.

Now, suppose you have some small amount of money d to allocate at time 0. Think of time 0 as now, even if you're not 0 when reading this. One way to formulate the problem is that one's utility function (of an entire life) can be computed by dynamic programming, starting from old age and working backward. Your options at any step are to invest or save. A simpler way to think about it is that you can either spend this money immediately or save it for a particular later time T. If you spend it immediately it will pay off $d \times \frac{dU}{d\$} = d \times \frac{c}{\$(0)}$. If you spend it at T it will first grow to $d \times R^T$, then pay off $R^T \times d \times \frac{c}{\$(T)}$. However, there's a chance you'll die and it will be worthless. Let's say your chance of being alive at time T is $M(T)$.

Suppose for a moment that you get a large pile of cash at the start of your life to allocate as you see fit. You have the flexibility to spend money at time 0 or save it to spend at time T, and will pick whichever yields more utility. Thus, we set the above utility payoffs equal: $d \times \frac{c}{\$(0)} = M(T) \times R^T \times d \times \frac{c}{\$(T)}$. Solving, we get simply $\$(T) = R^T \times M(T) \times \$(0)$.

In other words, you should **spend money over the course of your life proportionally to how much it will grow during that time and also proportional to how likely you are to still be alive**. Knowing how much of your current money to spend or save makes this fact actionable. Preparing to spend $\$(T)$ in year T requires saving $\$(T) \times R^{-T}$ at time 0, since money invested until T will grow by a factor of R^T. I'll call this the present value from now on. Combining with the above, we see that $\$(T) = R^T \times M(T) \times \$(0) \times R^{-T} = M(T) \times \$(0)$ and the **present value is just proportional to how likely you are to still be alive**. You will have to save the same amount for every year in your remaining life expectancy. Projecting mortality trends, this is about 55 years for men at age 25 and 59 for women. I'll lazily say 55 from now on. So if you're 25 and living off an inheritance, spend $\frac{1}{55}$ of it this year and save the rest.

In the real world we must remove the assumption that all money arrives at the start of time. Suppose instead one is given a continuous function for when money arrives, representing payments for jobs worked. Money should still be saved if it will be more useful later, so the allocation function derived above will in some sense bound the optimal function from below. More concretely, $\frac{d\ log\$}{dT}$ will be bounded below by $\frac{d(log(R^{T} \times M(T)))}{dT}$. You will always want to spend the most at age 77, or 80 for women. I'll say 77 from now on.

Income

Most people have to work to earn their optimal amount of money. This introduces the additional optimization parameter of deciding when additional work is worth additional money. First, let's answer the question "What are the best years to work?". The simplest possible model here is that you earn a fixed salary (inflation adjusted, as everything here) for any times you decide to work. Under this model, working earlier in life is preferred so long as $R^{T} > \frac{1}{M(T)}$. One problem with the model is that working later is preferred when the equation is reversed. You shouldn't save any money for years beyond a certain age - just start working again if you make it there!

Another irksome result is that, because your spending steadily increases over life, it might be worth putting up with extremely poor circumstances to secure high spending late in life. In this case a life which has negative experiential utility for the first 30 years, for example, can still be worthwhile overall.

Now suppose you've picked how many years to work. If that number is a small k we can assume the present value of all your earnings will be about $k \times salary$. **If you're 25 and currently have no savings then, using the result from before, you should spend about k/55 of your salary this year.** If k is instead large**, the present value of all your earnings will never exceed 21*salary, and I encourage you to do the math. Everyone without savings should save at least 61% of their income.**

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*** A mathematical model for how work impacts utility would be nice.
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Salary Growth

If you expect rapid salary growth, your best earning years may not start right away. If $\frac{salary\ next\ year}{salary\ this\ year} \times \frac{1}{M(next\ year)} > R$, you might prefer not to work in this year and instead work in some later year. In most cases you'll want to work enough years that these early years will still be among the optimal years.

If they're not, and you have to work full-time anyway, you'll want to spend a greater fraction of your income than one would naively expect. In extreme cases of very rapid salary growth and a very short desired working time, it becomes optimal to spend all income and only start saving when your earnings become high enough. As an example, **if you expect 4x as high a salary in 8 years, followed by 25 more years of work, then spend all your money this year. This is about the smallest increase possible before saving becomes a bad idea, so probably you should be saving.** In reality the model breaks when you have 0 savings and you'd want to keep a few months of living expenses.

In the real world[1], salary growth tends to slow down over a career. I commonly assume $R = 1.0512$, so I'll estimate that 40 years of work produces a net present value of 28 times this year's salary.

A comprehensive picture of optimal spending now looks like this:

- The best years to earn are around one's time of peak present value earning.
- For salary growth of 4x or more, there may be a few years at the start when you shouldn't try to save at all. You should even borrow money if the interest is low enough!
- After you retire you should spread out your savings roughly evenly over your remaining life.
- Expect a rapid decrease in spending after age 77.
- At some old age the model suggests you start working again. In this hypothetical you're like 97, so you'd probably instead rely on social security or an equivalent.

Part of my motivation for this whole thought experiment was to determine the value of my time and decide between a software engineer salary or an academic salary. This requires further work.

This doc is also a response to many other thoughts and conversations. Special thanks to Chen Xu for adding equations.

[1] see pages 15, 19, 40, 42, 44