Linear Algebra MAT313 Spring 2024
Professor Sormani
Shorter Lesson 6 for students that are behind schedule and did not take Quiz 2 yet.
A Matrix times a Vector

This is an essential lesson and both parts are essential!   There are ten homework problems.

As always: complete your previous lessons before doing this lesson.

Before you start, find your team's project document and submit one line on the group project!

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You will cut and paste the photos of your notes and completed classwork in a googledoc entitled:

MAT313S24-lesson6-lastname-firstname

and share editing of that document with me sormanic@gmail.com.   You will also include your homework and any corrections to your homework in this doc.

If you have a question, type QUESTION in your googledoc next to the point in your notes that has a question and email me with the subject MAT313 QUESTION. I will answer your question by inserting a photo into your googledoc or making an extra video.

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This Lesson has two Parts:

Part I
Matrix Multiplication and Rewriting Systems of Equations
Part II
Checking solutions to systems of linear equations using Matrix Multiplication

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Part I
Matrix Multiplication and Rewriting Systems of Equations
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Watch Playlist 313S24-6-1to4

Rewriting a System of Linear Equations using Matrix Multiplication

Quick Example before the Theory and Definition

System	Augmented Matrix	New Matrix Multiplication
$\boxed{\begin{aligned} 2x_1 + 3x_2 &= 4 \\ 5x_1 + 6x_2 &= 7 \end{aligned}}$	$\left[\begin{array}{cc\|c} 2 & 3 & 4 \\ 5 & 6 & 7 \end{array}\right]$	$\begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$
$\boxed{\begin{aligned} a_{11}x_1 + a_{12}x_2 &= d_1 \\ a_{21}x_1 + a_{22}x_2 &= d_2 \end{aligned}}$	$\left[\begin{array}{cc\|c} a_{11} & a_{12} & d_1 \\ a_{21} & a_{22} & d_2 \end{array}\right]$	$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$

Review:

$$\boxed{\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,m}x_m &= d_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,m}x_m &= d_2 \\ &\cdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,m}x_m &= d_n \end{aligned}}$$

which can be written as

$$\sum_{j=1}^{m} a_{ij} x_j = d_i \quad \text{for } \underset{\text{rows}}{i = 1 \text{ to } n}$$

or as an augmented matrix

$$\left(\begin{array}{cccc|c} a_{1,1} & a_{1,2} & \cdots & a_{1,m} & d_1 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} & d_2 \\ \vdots & & \ddots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} & d_n \end{array}\right)$$

Today we learn to write this with matrix multiplication.

Review:

$$\boxed{\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,m}x_m &= d_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,m}x_m &= d_2 \\ &\cdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,m}x_m &= d_n \end{aligned}}$$

which can be written as

$$\sum_{j=1}^{m} a_{ij}x_j = d_i \quad \text{for } \underbrace{i=1 \text{ to } n}_{\text{rows}}$$

or as an augmented matrix

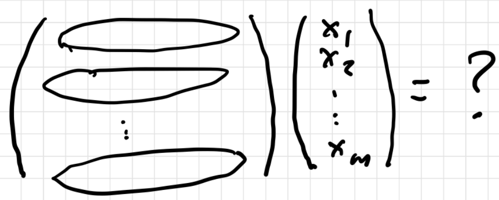
$$\left(\begin{array}{cccc|c} a_{1,1} & a_{1,2} & \cdots & a_{1,m} & d_1 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} & d_2 \\ \vdots & & \ddots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} & d_m \end{array}\right)$$

$n \times m$ matrix (without the column of d_i)

$$\underbrace{\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} \end{pmatrix}}_{n \times m \text{ matrix}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}}_{\vec{d}}$$

$A = [a_{ij}]$ row i

$$\boxed{A\vec{x} = \vec{d}}$$



$n\times m$ matrix (without the column of d_i)

$$\underbrace{\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} \end{pmatrix}}_{n\times m \text{ matrix}} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}$$

$n\times m$ matrix
n rows
and m columns

$\vec{x}\in\mathbb{R}^m$ $\quad$ $\vec{d}\in\mathbb{R}^n$

$$= \begin{pmatrix} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,m}x_m \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,m}x_m \\ \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,m}x_m \end{pmatrix}$$

m

Review:

$$\begin{array}{l} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,m}x_m = d_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,m}x_m = d_2 \\ \cdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,m}x_m = d_n \end{array}$$

which can be written as

$$\sum_{j=1}^{m} a_{i,j}x_j = d_i \text{ for } i = 1 \text{ to } n \text{ rows}$$

or as an augmented matrix

first entry of the answer is the first row of the matrix $\cdot\ \vec{x}$ dot product

second entry is the dot product of the second row with x

Linear Algebra

Abstract_Algebra_1-5 | Modern Algebra &... | Linear Algebra

$n\times m$ matrix (without the column of d_i)

$$\underbrace{\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} \end{pmatrix}}_{n\times m \text{ matrix}} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}$$

$\vec{x}$ $\quad$ $\vec{d}$

$$= \begin{pmatrix} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,m}x_m \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,m}x_m \\ \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,m}x_m \end{pmatrix}$$

Linear Algebra

$$A\vec{x} = \begin{pmatrix} (\text{row 1 of } A)\cdot\vec{x} \\ (\text{row 2 of } A)\cdot\vec{x} \\ \vdots \\ (\text{row } n \text{ of } A)\cdot\vec{x} \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{m} a_{1,j}x_j \\ \sum_{j=1}^{m} a_{2,j}x_j \\ \vdots \\ \sum_{j=1}^{m} a_{n,j}x_j \end{pmatrix}$$

i^{th} component

$$\left[A\vec{x}\right]_i = d_i = \sum_{j=1}^{m} a_{i,j}x_j$$

Example

$$\begin{pmatrix}1&2&3\\4&8&9\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$=\begin{pmatrix}1x+2y+3z\\4x+8y+9z\end{pmatrix}$$

Example

$$\begin{pmatrix}1&4\\5&8\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix}=\begin{pmatrix}1\cdot2+4\cdot3\\5\cdot2+8\cdot3\end{pmatrix}$$

$$=\begin{pmatrix}2+12\\10+24\end{pmatrix}=\begin{pmatrix}14\\34\end{pmatrix}$$

Classwork

$$\begin{pmatrix}1&3\\5&7\end{pmatrix}\begin{pmatrix}2\\0\end{pmatrix}=\begin{pmatrix}1\cdot2+3\cdot0\\5\cdot2+7\cdot0\end{pmatrix}=\begin{pmatrix}2\\10\end{pmatrix}$$

m columns
n rows

$\vec{v}$ is in $\mathbb{R}^m$

answer is in $\mathbb{R}^n$

$$\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1x+0y+0z\\0x+1y+0z\end{pmatrix}$$

$$=\begin{pmatrix}x\\y\end{pmatrix}$$

3 columns
2 rows

in $\mathbb{R}^3$

answer is in $\mathbb{R}^2$

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}1a+0b+0c\\0a+1b+0c\\0a+0b+1c\end{pmatrix}$$

$$=\begin{pmatrix}a\\b\\c\end{pmatrix}$$

Cannot multiply

$$\begin{pmatrix}1 & 0 & 1\\ 2 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

can't be done

Try $1x + 0y + 1$?

> If a matrix has m columns that it can only be multiplied by a vector in $\mathbb{R}^m$
> If the matrix has n rows then the answer is in $\mathbb{R}^n$

HW1 $\begin{pmatrix}1 & 4 & 8\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = ?$

HW2 Rewrite the system

$1x + 2y + 4z = 0$

$2x + 5y + 0z = 1$

as a matrix: $A\vec{v} = \vec{d}$

$\begin{pmatrix} & & \end{pmatrix}\begin{pmatrix} \end{pmatrix} = \begin{pmatrix} \end{pmatrix}$

HW3 $\begin{pmatrix}b_{1,1} & b_{1,2} & b_{1,3}\\ b_{2,1} & b_{2,2} & b_{2,3}\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix} =$

write the i^{th} entry of the answer using sum notation.

HW4 $\begin{pmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix} = ?$

When you do the homework, show your work, do not just write the answer.

Check your answers to HW1 - HW4 here

HW1 $\begin{pmatrix}1 & 4 & 8\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \Big(1x + 4y$ stuck! Cannot multiply! No answer

HW2 $\begin{pmatrix}1 & 2 & 4\\ 2 & 5 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix}$ Check $\begin{pmatrix}1 & 2 & 4\\ 2 & 5 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}1x + 2y + 4z\\ 2x + 5y + 0z\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix}$ ✓

coefficient matrix — variables

HW3 $\begin{pmatrix}b_{1,1}v_1 + b_{1,2}v_2 + b_{1,3}v_3\\ b_{2,1}v_1 + b_{2,2}v_2 + b_{2,3}v_3\end{pmatrix}$

HW4 $\begin{pmatrix}1a + 0b + 0c\\ 0a + 2b + 0c\\ 0a + 0b + 3c\end{pmatrix} = \begin{pmatrix}1a\\ 2b\\ 3c\end{pmatrix}$

HW5 Rewrite the system using Matrix Mult

$$\boxed{\begin{cases} 5x + 2y - 3z = 11 \\ 2x + 8y + 10z = -2 \\ x - y - 2z = 0 \end{cases}} \qquad \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

Then check by multiplying:

$$\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} =$$

HW6 $$\begin{pmatrix} 2 & -3 & 4 & 5 & 6 \\ -7 & 8 & 9 & 0 & 3 \\ 1 & 0 & -1 & 2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 9 \\ 8 \\ 7 \end{pmatrix}$$

Multiply out the left side and then rewrite as a system.

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Part II
Checking solutions to systems of linear equations using Matrix Multiplication

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Watch Playlist 313S24-6-5to7.

Lets use Matrix Multiplication to Check Solutions

Classwork: Check:

$\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ solves

$$\boxed{\begin{aligned} 2x + 2y + 4z &= 12 \\ 1x + 1y + 1z &= 5 \\ 1x - 1y + 1z &= 1 \end{aligned}}$$

First rewrite the system as matrix multiplication

$$\begin{pmatrix} 2 & 2 & 4 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ 5 \\ 1 \end{pmatrix}$$

To check if $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ is a solution

Match!

$$\begin{pmatrix} 2 & 2 & 4 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \cdot 2 + 2 \cdot 2 + 4 \cdot 1 \\ 1 \cdot 2 + 1 \cdot 2 + 1 \cdot 1 \\ 1 \cdot 2 + (-1)2 + 1 \cdot 1 \end{pmatrix} = \begin{pmatrix} 4+4+4 \\ 2+2+1 \\ 2-2+1 \end{pmatrix} = \begin{pmatrix} 12 \\ 5 \\ 1 \end{pmatrix}$$

We have the correct answer!

Classwork: Check that

$$\left\{ \begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix} = \begin{pmatrix}-4\\5\\0\\0\\6\end{pmatrix} + x_3\begin{pmatrix}1\\-2\\1\\0\\0\end{pmatrix} \;\middle|\; x_3\in\mathbb{R}\right\}$$

solves:

$$\boxed{\begin{aligned}
1x_1 + 2x_2 + 3x_3 + 0x_4 + 0x_5 &= 6\\
0x_1 + 1x_2 + 2x_3 + 0x_4 + 0x_5 &= 5\\
0x_1 + 0x_2 + 0x_3 + 1x_4 + 0x_5 &= 0\\
0x_1 + 0x_2 + 0x_3 + 0x_4 + 1x_5 &= 6\\
0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 &= 0
\end{aligned}}$$

position is a solution to the original system

direction is a solution to the homogeneous system

$$\boxed{\begin{aligned}
1x_1 + 2x_2 + 3x_3 + 0x_4 + 0x_5 &= 0\\
0x_1 + 1x_2 + 2x_3 + 0x_4 + 0x_5 &= 0\\
0x_1 + 0x_2 + 0x_3 + 1x_4 + 0x_5 &= 0\\
0x_1 + 0x_2 + 0x_3 + 0x_4 + 1x_5 &= 0\\
0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 &= 0
\end{aligned}}$$

First check the position!

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \\ 0 \\ 0 \\ 6 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \;\middle|\; x_3 \in \mathbb{R} \right\}$$

solves:

$$\boxed{\begin{aligned} 1x_1 + 2x_2 + 3x_3 + 0x_4 + 0x_5 &= 6 \\ 0x_1 + 1x_2 + 2x_3 + 0x_4 + 0x_5 &= 5 \\ 0x_1 + 0x_2 + 0x_3 + 1x_4 + 0x_5 &= 0 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 + 1x_5 &= 6 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 &= 0 \end{aligned}}$$

Check position is a solution to the original system

Rewrite the original system as matrix multiplication

$$\begin{pmatrix} 1 & 2 & 3 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 0 \\ 6 \\ 0 \end{pmatrix}$$

Match checks!

$$\begin{pmatrix} 1 & 2 & 3 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -4 \\ 5 \\ 0 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 1(-4) + 2\cdot 5 + 3\cdot 0 + 0 + 0 \\ 0(-4) + 1\cdot 5 + 2\cdot 0 + 0 + 0 \\ 0(-4) + 0\cdot 5 + 0 + 1\cdot 0 + 0 \\ 0 + 0 + 0 + 0 + 1\cdot 6 \\ 0 + 0 + 0 + 0 + 0 \end{pmatrix} = \begin{pmatrix} -4+10 \\ 5 \\ 0 \\ 6 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 0 \\ 6 \\ 0 \end{pmatrix}$$

Next check the direction!

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \\ 0 \\ 0 \\ 6 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \;\middle|\; x_3 \in \mathbb{R} \right\}$$

solves:

$$\boxed{\begin{aligned} 1x_1 + 2x_2 + 3x_3 + 0x_4 + 0x_5 &= 6 \\ 0x_1 + 1x_2 + 2x_3 + 0x_4 + 0x_5 &= 5 \\ 0x_1 + 0x_2 + 0x_3 + 1x_4 + 0x_5 &= 0 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 + 1x_5 &= 6 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 &= 0 \end{aligned}}$$

Check the direction is a solution to the homogeneous system

Rewrite the original system as matrix multiplication

$$\begin{pmatrix} 1 & 2 & 3 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

zeroes

$$\begin{pmatrix} 1 & 2 & 3 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \text{pause + try do yourself} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

HW7 Check $\begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$ solves $\boxed{\begin{array}{l} 2x - 2y + 0z = 8 \\ x - 3y + 2z = 10 \end{array}}$

HW8 Check $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} + y \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \right\}$ Solves this system

by checking that the direction solves the homogeneous system

(already checked the position in HW7)

Ask a question if your HW7 or HW8 is not working out.

Why do you keep asking us to check our work using different methods? What do we do if our checks show we have errors? At this point you are just learning to detect the error. You should write that there is an error but do not need to track down the cause of the error. In the workplace, computer codes will be solving systems and you may be checking if the output is correct and where in the code things went wrong.

What if, when I do the matrix multiplication, I do not get the zero vector? First, you should only get the zero vector when multiplying the matrix times the direction vectors. When you multiply the matrix times the position vector, you should get the final column of the augmented matrix. **What if I don't get this final column which checking the position?** Then either your position vector or your matrix multiplication is incorrect. On your homework write "I have detected an error in the position." **What if I don't get the zero vector even when multiplying by the direction vectors?** Then either your direction vector or your matrix multiplication is incorrect. On your homework write "I have detected an error in this direction".

Next find your team's project document here and submit one line on the team project!

This late version of Lesson 6 has only 8 homework problems. Please proceed immediately to Lesson 7 right away to catch up.