Basic LP formulations

Linear programming formulations are typically composed of a number of standard problem types.

In these notes we review **four basic problems examining their:**

a. Basic Structure

b. Formulation

c. Example application

d. Answer interpretation

The problems examined are the:

Resource Allocation Problem

Basic Concept

The classical LP problem involves the allocation of an endowment of scarce resources among a number of competing products so as to maximize profits.

Objective: Maximize Profits

Indexes: Let the competing products index be j
Let the scarce resources index be i

Variables: Let us define our primary decision variable X_j, as the number of units of the j^{th} product made

Restrictions: Non negative production
Resource usage across all production possibilities is less than or equal to the resource endowment

Resource Allocation Problem

Algebraic Setup:

$$\text{Max} \quad \sum_j c_j X_j$$

$$\text{s.t.} \quad \sum_j a_{ij} X_j \leq b_i \quad \text{for all } i$$

$$X_j \geq 0 \quad \text{for all } j$$

where

- c_j is the profit contribution per unit of the j^{th} product

- a_{ij} depicts the number of units of the i^{th} resource used when producing one unit of the j^{th} product

- b_i depicts the endowment of the i^{th} resource

Resource Allocation Problem

Example: E-Z Chair Makers

Objective: determine the number of two types of chairs to produce that will maximize profits.

Chair Types: Functional and Fancy

Resources: Large &, Small Lathe, Chair Bottom Carver, and Labor

Profit Contributions: (revenue – material cost - cost increase due to lathe shifts)

Functional $82 - $15 = $67

Fancy $105 - $25 = $80

Resource Requirements When Using the Normal Pattern

Wh the Normal Pattern

	Hours of Use per Chair Type	
	Functional	Fancy
Small Lathe	0.8	1.2
Large Lathe	0.5	0.7
Chair Bottom Carver	0.4	1.0
Labor	1.0	0.8

Resource Requirements and Increased Costs for Alternative Methods of Production in Hours of Use per Chair and Dollars

	Maximum Use of Small Lathe		Maximum Use of Large Lathe	
	Functional	Fancy	Functional	Fancy
Small Lathe	1.30	1.70	0.20	0.50
Large Lathe	0.20	0.30	1.30	1.50
Chair Bottom Carver	0.40	1.00	0.40	1.00
Labor	1.05	0.82	1.10	0.84
Cost Increase	$1.00	$1.50	$0.70	$1.60

Resource Allocation Problem

Example: E-Z Chair Makers

Algebraic Setup:

$$\begin{aligned} \text{Max} \quad & \sum_j c_j X_j \\ \text{s.t.} \quad & \sum_j a_{ij} X_j \le b_i \quad \text{for all } i \\ & X_j \ge 0 \quad \text{for all } j \end{aligned}$$

Empirical Setup:

Max	$67X_1$	+	$66X_2$	+	$66.3X_3$	+	$80X_4$	+	$78.5X_5$	+	$78.4X_6$		
s.t.	$0.8X_1$	+	$1.3X_2$	+	$0.2X_3$	+	$1.2X_4$	+	$1.7X_5$	+	$0.5X_6$	$\le$	140
	$0.5X_1$	+	$0.2X_2$	+	$1.3X_3$	+	$0.7X_4$	+	$0.3X_5$	+	$1.5X_6$	$\le$	90
	$0.4X_1$	+	$0.4X_2$	+	$0.4X_3$	+	X_4	+	X_5	+	X_6	$\le$	120
	X_1	+	$1.05X_2$	+	$1.1X_3$	+	$0.8X_4$	+	$0.82X_5$	+	$0.84X_6$	$\le$	125
	X_1	,	X_2	,	X_3	,	X_4	,	X_5	,	X_6	$\ge$	0

Where:

X_1 = the # of functional chairs made with the normal pattern;
X_2 = the # of functional chairs made with maximum use of the small lathe;
X_3 = the # of functional chairs made with maximum use of the large lathe;
X_4 = the # of fancy chairs made with the normal pattern;
X_5 = the # of fancy chairs made with maximum use of the small lathe;
X_6 = the # of fancy chairs made with maximum use of the large lathe.

Resource Allocation Problem

GAMS Formulation:

```
SET     PROCESS    TYPES OF PRODUCTION PROCESSES
        /FUNCTNORM , FUNCTMXSML , FUNCTMXLRG
        ,FANCYNORM , FANCYMXSML , FANCYMXLRG/
        RESOURCE      TYPES OF RESOURCES
        /SMLLATHE,LRGLATHE,CARVER,LABOR/ ;

PARAMETER  PRICE(PROCESS)   PRODUCT PRICES BY PROCESS
       /FUNCTNORM  82, FUNCTMXSML  82, FUNCTMXLRG  82
       ,FANCYNORM 105, FANCYMXSML 105, FANCYMXLRG 105/
              PRODCOST(PROCESS)      COST BY PROCESS
       /FUNCTNORM 15, FUNCTMXSML 16 , FUNCTMXLRG 15.7
       ,FANCYNORM 25, FANCYMXSML 26.5,FANCYMXLRG 26.6/
           RESORAVAIL(RESOURCE) RESOURCE AVAILABLITY
                       /SMLLATHE 140, LRGLATHE  90,
                        CARVER   120, LABOR    125/;

TABLE RESOURUSE(RESOURCE,PROCESS) RESOURCE USAGE

                    FUNCTNORM   FUNCTMXSML  FUNCTMXLRG
   SMLLATHE              0.80         1.30        0.20
   LRGLATHE              0.50         0.20        1.30
   CARVER                0.40         0.40        0.40
   LABOR                 1.00         1.05        1.10
 +                  FANCYNORM   FANCYMXSML  FANCYMXLRG
   SMLLATHE              1.20         1.70        0.50
   LRGLATHE              0.70         0.30        1.50
   CARVER                1.00         1.00        1.00
   LABOR                 0.80         0.82        0.84;

POSITIVE VARIABLES
       PRODUCTION(PROCESS) ITEMS PRODUCED BY PROCESS;
VARIABLES
       PROFIT       TOTALPROFIT;
EQUATIONS
       OBJT         OBJECTIVE FUNCTION ( PROFIT )
       AVAILABLE(RESOURCE)  RESOURCES AVAILABLE ;
    OBJT.. PROFIT =E=
       SUM(PROCESS,(PRICE(PROCESS)-PRODCOST(PROCESS))
                              * PRODUCTION(PROCESS)) ;
    AVAILABLE(RESOURCE)..
       SUM(PROCESS,RESOURUSE(RESOURCE,PROCESS)
      *PRODUCTION(PROCESS)) =L= RESORAVAIL(RESOURCE);
MODEL RESALLOC /ALL/;
SOLVE RESALLOC USING LP MAXIMIZING PROFIT;
```

Resource Allocation Problem

Primal and Solution:

Max	$67X_1$	+	$66X_2$	+	$66.3X_3$	+	$80X_4$	+	$78.5X_5$	+	$78.4X_6$		
s.t.	$0.8X_1$	+	$1.3X_2$	+	$0.2X_3$	+	$1.2X_4$	+	$1.7X_5$	+	$0.5X_6$	$\leq$	140
	$0.5X_1$	+	$0.2X_2$	+	$1.3X_3$	+	$0.7X_4$	+	$0.3X_5$	+	$1.5X_6$	$\leq$	90
	$0.4X_1$	+	$0.4X_2$	+	$0.4X_3$	+	X_4	+	X_5	+	X_6	$\leq$	120
	X_1	+	$1.05X_2$	+	$1.1X_3$	+	$0.8X_4$	+	$0.82X_5$	+	$0.84X_6$	$\leq$	125
	X_1	,	X_2	,	X_3	,	X_4	,	X_5	,	X_6	$\geq$	0

Recall:
Shadow Price represents the marginal values of the resources
Reduced Cost represents the marginal costs of forcing non-basic variables into the solution

Table 5.5 Optimal Solution to the E-Z Chair Makers Problem

obj = 10417.29

Variables	Value	Reduced Cost	Constraint	Level	Shadow Price
X_1	62.23	0	1	140	33.33
X_2	0	-11.30	2	90	25.79
X_3	0	-4.08	3	103.09	0
X_4	73.02	0	4	125	27.44
X_5	0	-8.40			
X_6	5.18	0			

Resource Allocation Problem

Simple GAMS Formulation

Max	$67X_1$	+	$66X_2$	+	$66.3X_3$	+	$80X_4$	+	$78.5X_5$	+	$78.4X_6$		
s.t.	$0.8X_1$	+	$1.3X_2$	+	$0.2X_3$	+	$1.2X_4$	+	$1.7X_5$	+	$0.5X_6$	$\leq$	140
	$0.5X_1$	+	$0.2X_2$	+	$1.3X_3$	+	$0.7X_4$	+	$0.3X_5$	+	$1.5X_6$	$\leq$	90
	$0.4X_1$	+	$0.4X_2$	+	$0.4X_3$	+	X_4	+	X_5	+	X_6	$\leq$	120
	X_1	+	$1.05X_2$	+	$1.1X_3$	+	$0.8X_4$	+	$0.82X_5$	+	$0.84X_6$	$\leq$	125
	X_1	,	X_2	,	X_3	,	X_4	,	X_5	,	X_6	$\geq$	0

```
POSITIVE VARIABLES X1, X2, X3, X4, X5, X6;
VARIABLE PROFIT;
EQUATIONS OBJFUNC, CONSTR1, CONSTR2, CONSTR3, CONSTR4;
OBJFUNC..  67*X1 +   66*X2 + 66.3*X3 +  80*X4 + 78.5*X5 + 78.4*X6 =E= PROFIT;
CONSTR1.. 0.8*X1 +  1.3*X2 +  0.2*X3 + 1.2*X4 +  1.7*X5 +  0.5*X6 =L= 140;
CONSTR2.. 0.5*X1 +  0.2*X2 +  1.3*X3 + 0.7*X4 +  0.3*X5 +  1.5*X6 =L= 90;
CONSTR3.. 0.4*X1 +  0.4*X2 +  0.4*X3 +     X4 +      X5 +      X6 =L= 120;
CONSTR4..     X1 + 1.05*X2 +  1.1*X3 + 0.8*X4 + 0.82*X5 + 0.84*X6 =L= 125;
MODEL EXAMPLE1 /ALL/;
SOLVE EXAMPLE1 USING LP MAXIMIZING PROFIT;
```

Resource Allocation Problem

Primal and **Dual** Algebra:

Primal

$$\begin{aligned} Max \quad & \sum_j c_j X_j \\ s.t. \quad & \sum_j a_{ij} X_j \le b_i \quad \text{for all i} \\ & X_j \ge 0 \quad \text{for all j} \end{aligned}$$

Dual

$$\begin{aligned} \text{Min} \quad & \sum_i U_i b_i \\ \text{s.t.} \quad & \sum_i U_i a_{ij} \ge c_j \quad \text{for all j} \\ & U_i \ge 0 \quad \text{for all i} \end{aligned}$$

Resource Allocation Problem

Primal and Dual Empirical:

Primal

Max	$67X_1$	+	$66X_2$	+	$66.3X_3$	+	$80X_4$	+	$78.5X_5$	+	$78.4X_6$		
s.t.	$0.8X_1$	+	$1.3X_2$	+	$0.2X_3$	+	$1.2X_4$	+	$1.7X_5$	+	$0.5X_6$	$\leq$	140
	$0.5X_1$	+	$0.2X_2$	+	$1.3X_3$	+	$0.7X_4$	+	$0.3X_5$	+	$1.5X_6$	$\leq$	90
	$0.4X_1$	+	$0.4X_2$	+	$0.4X_3$	+	X_4	+	X_5	+	X_6	$\leq$	120
	X_1	+	$1.05X_2$	+	$1.1X_3$	+	$0.8X_4$	+	$0.82X_5$	+	$0.84X_6$	$\leq$	125
	X_1	,	X_2	,	X_3	,	X_4	,	X_5	,	X_6	$\geq$	0

Dual

Min	$140U_1$	+	$90U_2$	+	$120U_3$	+	$125U_4$		
s.t.	$0.8U_1$	+	$0.5U_2$	+	$0.4U_3$	+	U_4	$\geq$	67
	$1.3U_1$	+	$0.2U_2$	+	$0.4U_3$	+	$1.05U_4$	$\geq$	66
	$0.2U_1$	+	$1.3U_2$	+	$0.4U_3$	+	$1.1U_4$	$\geq$	66.3
	$1.2U_1$	+	$0.7U_2$	+	U_3	+	$0.8U_4$	$\geq$	80
	$1.7U_1$	+	$0.3U_2$	+	U_3	+	$0.82U_4$	$\geq$	78.5
	$0.5U_1$	+	$1.5U_2$	+	U_3	+	$0.84U_4$	$\geq$	78.4
	U_1	,	U_2	,	U_3	,	U_4	$\geq$	0

Resource Allocation Problem

Dual:

$$\begin{aligned} \text{Min} \quad & \sum_i U_i b_i \\ \text{s.t.} \quad & \sum_i U_i a_{ij} \geq c_j \quad \text{for all } j \\ & U_i \geq 0 \quad \text{for all } i \end{aligned}$$

Dual Solution:

obj = 10417.29					
Variables	Value	Reduced Cost	Row	Level	Shadow Price
U_1	33.33	0	1	67.00	62.23
U_2	25.79	0	2	77.30	0
U_3	0	16.91	3	70.38	0
U_4	27.44	0	4	80.00	73.02
			5	86.90	0
			6	78.40	5.18

Resource Allocation

Primal and Dual Solution Comparison:

Table 5.5 Optimal Solution to the E-Z Chair Makers Problem

obj = 10417.29

Variables	Value	Reduced Cost	Constraint	Level	Shadow Price
X_1	62.23	0	1	140	33.33
X_2	0	-11.30	2	90	25.79
X_3	0	-4.08	3	103.09	0
X_4	73.02	0	4	125	27.44
X_5	0	-8.40			
X_6	5.18	0			

Table 5.5a Optimal Solution to the Dual E-Z Chair Makers Problem

obj = 10417.29

Variables	Value	Reduced Cost	Constraint	Level	Shadow Price
U_1	33.33	0	1	67.00	62.23
U_2	25.79	0	2	77.30	0
U_3	0	16.91	3	70.38	0
U_4	27.44	0	4	80.00	73.02
			5	86.90	0
			6	78.40	5.18

Resource Allocation

Evaluate Dual Constraints: For X1 and X2

$$0.8U_1 + 0.5U_2 + 0.4U_3 + U_4 \geq 67$$
$$1.3U_1 + 0.2U_2 + 0.4U_3 + 1.05U_4 \geq 66$$

Dual Solution

U1 33.33

U2 25.79

U3 0

U4 27.44

Evaluation

X1

$0.8(33.33) + 0.5(25.79) + 0.4(0) + 27.44$

$= 26.6664 + 12.895 + 0 + 27.44$

$= 67 \geq 67$

Reduced cost=0

X2

$1.3U_1 + 0.2U_2 + 0.4U_3 + 1.05U_4$

$= 1.3(33.33) + 0.2(25.79) + 0.4(0) + 1.05(27.44)$

$= 43.33 + 5.16 + 0 + 28.812$

$= 77.3$

$= 66 + 11.3 \geq 66$

Reduced cost=11.3

Table 5.3. First GAMS Formulation of Resource Allocation Example

```
SET   PROCESS       TYPES OF PRODUCTION PROCESSES
                            /FUNCTNORM , FUNCTMXSML , FUNCTMXLRG
                            ,FANCYNORM , FANCYMXSML ,FANCYMXLRG/
        RESOURCE      TYPES OF RESOURCES
                            /SMLLATHE,LRGLATHE,CARVER,LABOR/ ;
 PARAMETER PRICE(PROCESS)      PRODUCT PRICES BY PROCESS
                  /FUNCTNORM  82, FUNCTMXSML  82, FUNCTMXLRG  82
                  ,FANCYNORM 105, FANCYMXSML 105, FANCYMXLRG 105/
             PRODCOST(PROCESS)     COST BY PROCESS
                 /FUNCTNORM 15, FUNCTMXSML 16,  FUNCTMXLRG  15.7
                 ,FANCYNORM 25, FANCYMXSML 26.5,FANCYMXLRG  26.6/
             RESORAVAIL(RESOURCE) RESOURCE AVAILABLITY
                 /SMLLATHE 140, LRGLATHE  90,
                  CARVER   120, LABOR    125/;

 TABLE RESOURUSE(RESOURCE,PROCESS) RESOURCE USAGE

                 FUNCTNORM   FUNCTMXSML  FUNCTMXLRG
SMLLATHE            0.80       1.30        0.20
LRGLATHE            0.50       0.20        1.30
CARVER              0.40       0.40        0.40
LABOR               1.00       1.05        1.10
 +               FANCYNORM   FANCYMXSML  FANCYMXLRG
SMLLATHE            1.20       1.70        0.50
LRGLATHE            0.70       0.30        1.50
CARVER              1.00       1.00        1.00
LABOR               0.80       0.82        0.84;
 POSITIVE VARIABLES
          PRODUCTION(PROCESS) ITEMS PRODUCED BY PROCESS;
 VARIABLES
          PROFIT              TOTALPROFIT;
 EQUATIONS
          OBJT                   OBJECTIVE FUNCTION ( PROFIT )
          AVAILABLE(RESOURCE)    RESOURCES AVAILABLE ;

  OBJT..  PROFIT =E=
             SUM(PROCESS,(PRICE(PROCESS)-PRODCOST(PROCESS))
                           * PRODUCTION(PROCESS)) ;
  AVAILABLE(RESOURCE)..
 SUM(PROCESS,RESOURUSE(RESOURCE,PROCESS)*PRODUCTION(PROCESS))
                       =L= RESORAVAIL(RESOURCE);
 MODEL RESALLOC /ALL/;
 SOLVE RESALLOC USING LP MAXIMIZING PROFIT;
```

Table 5.4. Second GAMS Formulation of Resource Allocation Example

```
  SET    CHAIRS        TYPES OF CHAIRS  /FUNCTIONAL,FANCY/
         PROCESS       TYPES OF PRODUCTION PROCESSES
                           /NORMAL , MXSMLLATHE , MXLRGLATHE/
         RESOURCE      TYPES OF RESOURCES
                           /SMLLATHE,LRGLATHE,CARVER,LABOR/ ;

  PARAMETER PRICE(CHAIRS)      PRODUCT PRICES BY PROCESS
                             /FUNCTIONAL 82, FANCY 105/
            COST(CHAIRS)     BASE COST BY CHAIR
                             /FUNCTIONAL 15, FANCY 25/
            EXTRACOST(CHAIRS,PROCESS) EXTRA COST BY PROCESS
                           / FUNCTIONAL.MXSMLLATHE  1.0  ,
                             FUNCTIONAL.MXLRGLATHE  0.7
                            ,FANCY.     MXSMLLATHE  1.5,
                             FANCY.     MXLRGLATHE  1.6/
            RESORAVAIL(RESOURCE) RESOURCE AVAILABLITY
                                 /SMLLATHE 140,  LRGLATHE  90,
                                  CARVER   120,  LABOR    125/;
  TABLE RESOURUSE(RESOURCE,CHAIRS,PROCESS) RESOURCE USAGE
     FUNCTIONAL.NORMAL FUNCTIONAL.MXSMLLATHEFUNCTIONAL.MXLRGLATHE
 SMLLATHE        0.80              1.30                 0.20
 LRGLATHE        0.50              0.20                 1.30
 CARVER          0.40              0.40                 0.40
 LABOR           1.00              1.05                 1.10
+           FANCY.NORMAL     FANCY.MXSMLLATHE    FANCY.MXLRGLATHE
 SMLLATHE        1.20              1.70                 0.50
 LRGLATHE        0.70              0.30                 1.50
 CARVER          1.00              1.00                 1.00
 LABOR           0.80              0.82                 0.84  ;

  POSITIVE VARIABLES
           PRODUCTION(CHAIRS,PROCESS) ITEMS PRODUCED BY PROCESS;
  VARIABLES
           PROFIT                TOTAL PROFIT;
  EQUATIONS
           OBJT                   OBJECTIVE FUNCTION ( PROFIT )
           AVAILABLE(RESOURCE)    RESOURCES AVAILABLE ;

  OBJT..  PROFIT =E= SUM((CHAIRS,PROCESS),
           PRICE(CHAIRS)-COST(CHAIRS)-EXTRACOST(CHAIRS,PROCESS))
                     * PRODUCTION(CHAIRS,PROCESS)) ;
   AVAILABLE(RESOURCE)..
       SUM((CHAIRS,PROCESS),
   RESOURUSE(RESOURCE,CHAIRS,PROCESS)*PRODUCTION(CHAIRS,PROCESS))
                        =L= RESORAVAIL(RESOURCE);

  MODEL RESALLOC /ALL/;
  SOLVE RESALLOC USING LP MAXIMIZING PROFIT;
```

Transportation Problem

Basic Concept This problem involves the shipment of a homogeneous product from a number of supply locations to a number of demand locations.

Supply Locations	Demand Locations
1	**A**
2	**B**
…	**…**
m	**n**

Problem: given needs at the demand locations how should I take limited supply at supply locations and move the goods to meet needs. Further suppose we wish to minimize cost.

Objective: Minimize cost

Variables Quantity of goods shipped from each supply point to each demand point

Restrictions: Non negative shipments

Supply availability at the supply point

Demand needs at a demand point

Transportation Problem
Formulating the Problem

Basic notation and the decision variable

Let us denote the supply locations as $supply_i$

Let us denote the demand locations as $demand_j$

Let us define our fundamental decision variable as the set of individual shipment quantities from each supply location to each demand location and denote this variable algebraically as

$$Move_{supply_i, demand_j}$$

Transportation Problem

Formulating the Problem The objective function:

We want to minimize total shipping cost so we need an expression for shipping cost

Let us define a data item giving the per unit cost of shipments from each supply location to each demand location as $cost_{supplyi,demandj}$ Our objective then becomes to minimize the sum of the shipment costs over all supplyi, demandj pairs or

$$\text{Minimize} \sum_{supplyi} \sum_{demandj} cost_{supplyi,demandj} Move_{supplyi,demandj}$$

which is the per unit cost of moving from each supply location to each demand location times the amount shipped summed over all possible shipment routes

Transportation Problem

Formulating the Problem

There are three types of constraints:

1) Supply availability: limiting shipments from each supply point to existing supply so that the sum of outgoing shipments from the supplyith supply point to all possible destinations (demandj) to not exceed $supply_{supplyi}$

$$\sum_{demandj} Move_{supplyi,\ demandj} \le supply_{supplyi}$$

2) Minimum demand: requiring shipments into the demandjth demand point be greater than or equal to demand at that point. Incoming shipments include shipments from all possible supply points supplyi to the demandjth demand point.

$$\sum_{supplyi} Move_{supplyi,\ demandj} \ge demand_{demandj}$$

3) Nonnegative shipments:

$$Move_{supplyi,\ demandj} \ge 0$$

Transportation Problem

Formulating the Problem

Formulation and Example:

$$\text{Min} \quad \sum_i \sum_j c_{ij} X_{ij}$$

$$\begin{aligned} \text{s.t.} \quad & \sum_j X_{ij} \le s_i && \text{for all } i \\ & \sum_i X_{ij} \ge d_j && \text{for all } j \\ & X_{ij} \ge 0 && \text{for all } i, j \end{aligned}$$

Explicitly:

$$\text{Min} \sum_{\text{supplyi}} \sum_{\text{demandj}} \text{cost}_{\text{supplyi, demandj}} \times \text{Move}_{\text{supplyi, demandj}}$$

$$\begin{aligned} s.t. \quad & \sum_{\text{demandj}} \text{Move}_{\text{supplyi, demandj}} \le \text{supply}_{\text{supplyi}} \quad \forall \text{supplyi} \\ & \sum_{\text{supplyi}} \text{Move}_{\text{supplyi, demandj}} \ge \text{demand}_{\text{demandj}} \quad \forall \text{demandj} \\ & \text{Move}_{\text{supplyi, demandj}} \ge 0 \quad \forall \text{supplyi, demandj} \end{aligned}$$

Transportation Problem

Example: Shipping Goods

Three plants: New York, Chicago, Los Angeles

Four demand markets: Miami, Houston, Minneapolis, Portland

Supply Available		**Demand Required**	
New York	100	Miami	30
Chicago	75	Houston	75
Los Angeles	90	Minneapolis	90
		Portland	50

Distances:

	Miami	Houston	Minneapolis	Portland
New York	3	7	6	23
Chicago	9	11	3	13
Los Angeles	17	6	13	7

Transportation costs = 5 +5*Distance:

	Miami	Houston	Minneapolis	Portland
New York	20	40	35	120
Chicago	50	60	20	70
Los Angeles	90	35	70	40

Transportation Problem

Example: Shipping Goods

$$\text{Min} \quad \sum_i \sum_j c_{ij} X_{ij}$$

$$\text{s.t.} \quad \sum_j X_{ij} \leq s_i \quad \text{for all } i$$

$$\sum_i X_{ij} \geq d_j \quad \text{for all } j$$

$$X_{ij} \geq 0 \quad \text{for all } i, j$$

Min	$20X_{11}$	+	$40X_{12}$	+	$35X_{13}$	+	$120X_{14}$	+	$50X_{21}$	+	$60X_{22}$	+	$20X_{23}$	+	$70X_{24}$	+	$90X_{31}$	+	$35X_{32}$	+	$70X_{33}$	+	$40X_{34}$		
s.t.	X_{11}	+	X_{12}	+	X_{13}	+	X_{14}																	$\leq$	100
									X_{21}	+	X_{22}	+	X_{23}	+	X_{24}									$\leq$	75
																	X_{31}	+	X_{32}	+	X_{33}	+	X_{34}	$\leq$	90
	X_{11}							+	X_{21}							+	X_{31}							$\geq$	30
			X_{12}							+	X_{22}							+	X_{32}					$\geq$	75
					X_{13}							+	X_{23}							+	X_{33}			$\geq$	90

							X_{14}							+	X_{24}							+	X_{34}	$\geq$	50
									$X_{ij} \geq 0$																

Transport Problem

Alternative Formulation Formats

$$\text{Min} \sum_{supplyi} \sum_{demandj} cost_{supplyi,\, demandj} \times Move_{supplyi,\, demandj}$$

$$s.t. \quad \sum_{demandj} Move_{supplyi,\, demandj} \le supply_{supplyi} \quad \forall supplyi$$

$$\sum_{supplyi} Move_{supplyi,\, demandj} \ge demand_{demandj} \quad \forall demandj$$

$$Move_{supplyi,\, demandj} \ge 0 \quad \forall supplyi,\ demandj$$

MOVE E_1 1	MOVE E_1 2	MOVE E_1 3	MOVE E_1 4	MOVE E_2 1	MOVE E_2 2	MOVE E_2 3	MOVE E_2 4	MOVE E_3 1	MOVE E_3 2	MOVE E_3 3	MOVE E_3 4		
20	40	35	120	50	60	20	70	90	35	70	40	Minimize	
1	1	1	1									≤	100
				1	1	1	1					≤	75
								1	1	1	1	≤	90
+1				+1				+1				≥	+30
	+1				+1				+1			≥	+75

		+1				+1				+1		≥	+90
			+1				+1				+1	≥	+50
1,	1,	1,	1,	1,	1,	1,	1,	1,	1,	1,	1,	≥	0

Transportation Problem

Example: Shipping Goods – Solution

- shadow price represents marginal values of the resources
  i.e. marginal value of additional units in Chicago = \$15
- reduced cost represents marginal costs of forcing non-basic variable into the solution
  i.e. shipments from New York to Portland costs \$75
- twenty units are left in New York

Optimal Solution: Objective value \$7,425

Table 5.8. Optimal Solution to the ABC Company Problem

Variable	Value	Reduced Cost	Equation	Slack	Shadow Price
$Move_{11}$	30	0	1	20	0
$Move_{12}$	35	0	2	0	-15
$Move_{13}$	15	0	3	0	-5
$Move_{14}$	0	75	4	0	20
$Move_{21}$	0	45	5	0	40
$Move_{22}$	0	35	6	0	35
$Move_{23}$	75	0	7	0	45
$Move_{24}$	0	40			
$Move_{31}$	0	75			
$Move_{32}$	40	0			
$Move_{33}$	0	40			
$Move_{34}$	50	0			

Optimal Shipping Patterns

Origin	Destination							
	Miami		Houston		Minneapolis		Portland	
	Units	Variable	Units	Variable	Units	Variable	Units	Variable
New York	30	$Move_{11}$	35	$Move_{12}$	15	$Move_{13}$		
Chicago					75	$Move_{23}$		

Los Angeles			40	$Move_{32}$			50	$Move_{34}$

Transport Problem

Primal and Dual Algebra:

Primal

$$\text{Min} \quad \sum_i \sum_j c_{ij} X_{ij}$$

$$\text{s.t.} \quad \sum_j X_{ij} \leq s_i \quad \text{for all i}$$

$$\sum_i X_{ij} \geq d_j \quad \text{for all j}$$

$$X_{ij} \geq 0 \quad \text{for all i, j}$$

Dual

$$\text{Max} \quad - \sum_i s_i U_i + \sum_j d_j V_j$$

$$\text{s.t.} \quad - U_i + V_j \leq c_{ij} \quad \text{for all i,j}$$

$$U_i \ , \ V_j \geq 0 \quad \text{for all i,j}$$

Transport Problem

Empirical Primal and Dual

Primal

20	40	35	120	50	60	20	70	90	35	70	40	Minimize	
1	1	1	1									≤	100
				1	1	1	1					≤	75
								1	1	1	1	≤	90
+1				+1				+1				≥	+30
	+1				+1				+1			≥	+75
		+1				+1				+1		≥	+90
			+1				+1				+1	≥	+50

Dual

Max	$-100U_1$	-	$75U_2$	-	$90U_3$	+	$30V_1$	+	$75V_2$	+	$90V_3$	+	$50V_4$		
s.t.	$-U_1$					+	V_1							≤	20
	$-U_1$							+	V_2					≤	40
	$-U_1$									+	V_3			≤	35
	$-U_1$											+	V_4	≤	120
			$-U_2$			+	V_1							≤	50
			$-U_2$					+	V_2					≤	60
			$-U_2$							+	V_3			≤	20
			$-U_2$									+	V_4	≤	70
					$-U_3$	+	V_1							≤	90
					$-U_3$			+	V_2					≤	35
					$-U_3$					+	V_3			≤	70
					$-U_3$							+	V_4	≤	40

Transport Problem

Empirical Dual and Solution

	Max	$-100U_1$	-	$75U_2$	-	$90U_3$	+	$30V_1$	+	$75V_2$	+	$90V_3$	+	$50V_4$		
	s.t.	$-U_1$					+	V_1							$\leq$	20
		$-U_1$							+	V_2					$\leq$	40
		$-U_1$									+	V_3			$\leq$	35
		$-U_1$											+	V_4	$\leq$	120
				$-U_2$			+	V_1							$\leq$	50
				$-U_2$					+	V_2					$\leq$	60
				$-U_2$							+	V_3			$\leq$	20
				$-U_2$									+	V_4	$\leq$	70
						$-U_3$	+	V_1							$\leq$	90
						$-U_3$			+	V_2					$\leq$	35
						$-U_3$					+	V_3			$\leq$	70
						$-U_3$							+	V_4	$\leq$	40

Table 5.10. Optimal Dual Solution to the ABC Company Problem

Variable	Value	Reduced Cost	Constraint	Level	Shadow Price
U_1	0	20	1	-20	30
U_2	15	0	2	40	35
U_3	5	0	3	35	15
V_1	20	0	4	45	0
V_2	40	0	5	5	0
V_3	35	0	6	25	0
V_4	45	0	7	20	75
			8	30	0
			9	15	0
			10	35	40
			11	30	0
			12	40	50

Transport Problem

Primal/Dual Solution Comparison:

Primal

Variable	Value	Reduced Cost	Constraint	Level	Shadow Price
$Move_{11}$	30	0	1	80	0
$Move_{12}$	35	0	2	75	15
$Move_{13}$	15	0	3	90	5
$Move_{14}$	0	75	4	30	20
$Move_{21}$	0	45	5	75	40
$Move_{22}$	0	35	6	90	35
$Move_{23}$	75	0	7	50	45
$Move_{24}$	0	40			
$Move_{31}$	0	75			
$Move_{32}$	40	0			
$Move_{33}$	0	40			
$Move_{34}$	50	0			

Dual

Variable	Value	Reduced Cost	Constraint	Level	Shadow Price
U_1	0	-20	1	-20	30
U_2	15	0	2	40	35
U_3	5	0	3	35	15
V_1	20	0	4	45	0
V_2	40	0	5	5	0
V_3	35	0	6	25	0
V_4	45	0	7	20	75
			8	30	0
			9	15	0
			10	35	40

	11	30	0
	12	40	50

Transport Problem

Table 5.7. GAMS Statement of Transportation Example

```
SETS  PLANT    PLANT LOCATIONS
            /NEWYORK, CHICAGO, LOSANGLS/
     MARKET   DEMAND MARKETS
            /MIAMI, HOUSTON, MINEPLIS, PORTLAND/

PARAMETERS SUPPLY(PLANT) QUANT AVAILABLE AT EACH PLANT
            /NEWYORK 100, CHICAGO 75, LOSANGLS 90/
         DEMAND(MARKET) QUANT REQUIRED BY DEMAND MARKET
            /MIAMI    30, HOUSTON  75,
             MINEPLIS 90, PORTLAND 50/;

TABLE   DISTANCE(PLANT,MARKET) DIST FROM PLANT TO MARKET

             MIAMI    HOUSTON   MINEPLIS   PORTLAND
   NEWYORK      3        7         6         23
   CHICAGO      9       11         3         13
   LOSANGLS    17        6        13          7;


PARAMETER COST(PLANT,MARKET) CALC COST OF MOVING GOODS ;
    COST(PLANT,MARKET) = 5 + 5 * DISTANCE(PLANT,MARKET) ;

POSITIVE VARIABLES
  SHIPMENTS(PLANT,MARKET) AMOUNT SHIPPED OVER A ROUTE;
VARIABLES
  TCOST       TOTAL COST OF SHIPPING OVER ALL ROUTES;
EQUATIONS
  TCOSTEQ           TOTAL COST ACCOUNTING EQUATION
  SUPPLYEQ(PLANT)    LIMIT ON SUPPLY AVAILABLE AT A PLANT
  DEMANDEQ(MARKET)   MIN REQUIREMENT AT A DEMAND MARKET;

TCOSTEQ..TCOST =E=
        SUM((PLANT,MARKET),SHIPMENTS(PLANT,MARKET)
          * COST(PLANT,MARKET)) ;
 SUPPLYEQ(PLANT)..SUM(MARKET,SHIPMENTS(PLANT,MARKET))
               =L=   SUPPLY(PLANT);
 DEMANDEQ(MARKET)..SUM(PLANT,SHIPMENTS(PLANT,MARKET))
               =G=   DEMAND(MARKET);
 MODEL TRANSPORT /ALL/;
 SOLVE TRANSPORT USING LP MINIMIZING TCOST;
```

Feed Mix Problem

Basic ConceptThis problem involves composing a minimum cost diet from a set of available ingredients while maintaining nutritional characteristics within certain bounds.

Objective: Minimize total diet costs

Variables: how much of each feedstuff is used in the diet

Restrictions: Non negative feedstuff

Minimum requirements by nutrient

Maximum requirements by nutrient

Total volume of the diet

This problem requires two types of indices

Type of feed ingredients available from which the diet can be composed

ingredient_j = {corn, soybeans, salt, etc.}Type of nutritional characteristics which must fall within certain limits nutrient = {protein, calories, etc.}

Feed Mix Problem

Basic Concept

Variable -- Feed $_{ingredientj}$ amount of feedstuff ingredient$_j$ fed to animalObjective – total costWe want to minimize total diet costs across all the feedstuffs so we need an expression for feedstuff costs.

Let us define a data item giving the per unit cost of ingredients as cost$_{ingredientj}$. Our objective then becomes to minimize the sum of the diet costs over all feed ingredients

Minimize $\sum_{ingredientj} cost_{ingredientj}\ Feed_{ingredientj}$ **which is the per unit cost of ingredients summed over the feed ingredients.**

Feed Mix Problem Basic Concept Additional parameters representing how much of each nutrient is present in each feedstuff as well as the dietary minimum and maximum requirements for that nutrient are needed.

Let

1). $a_{nutrient,ingredientj}$ be the amount of the nutrientth nutrient present in one unit of the ingredientjth feed ingredient

2). $UL_{nutrient}$ and $LL_{nutrient}$ be the maximum and minimum amount of the nutrientth nutrient

in the diet Then the nutrient constraints are formed by summing the nutrients generated from each feedstuff ($a_{nutrient,ingredientj}F_{ingredientj}$) and requiring these to exceed the dietary minimum and/or be less than the maximum. Problem then focuses on how much of each feedstuff is used in the diet to maintain nutritional characteristics within certain bounds.

Feed Mix Problem

Formulating the Problem

There are four general types of constraints:

1) minimum nutrient requirements restricting the sum of the nutrients generated from each feedstuff ($a_{nutrient,ingredientj}F_{ingredientj}$) to meet the dietary minimum

$$\sum_{ingredientj} a_{nutrient,ingredientj} \; Feed_{ingredientj} \geq minimum_{nutrient}$$

2) maximum nutrient requirements restricting the sum of the nutrients generated from each feedstuff ($a_{nutrient,ingredientj}F_{ingredientj}$) to not exceed the dietary maximum

$$\sum_{ingredientj} a_{nutrient,ingredientj} \; Feed_{ingredientj} \leq maximum_{nutrient}$$

3) total volume of the diet constraint requiring the ingredients in the diet equal the required weight of the diet. Suppose the weight of the formulated diet and the feedstuffs are the same, then

$$\sum_{\text{ingredientj}} Feed_{ingredientj} = 1$$

4) nonnegative feedstuff

$$Feed_{ingredientj} \geq 0$$

Feed Mix Problem

Example: cattle feeding

Seven nutritional characteristics:

energy, digestible protein, fat, vitamin A, calcium, salt, phosphorus

Seven feed ingredient availability:

corn, hay, soybeans, urea, dical phosphate, salt, vitamin A

New product: potato slurry

Ingredient costs per kilogram ($c_{ingredient j}$)

Ingredient Costs for Diet Problem Example per kg

Corn	\$0.133
Dical	\$0.498
Alfalfa hay	\$0.077
Salt	\$0.110
Soybeans	\$0.300
Vitamin A	\$0.286
Urea	\$0.332

Required Nutrient Characteristics per Kilogram

Nutrient	Unit	Minimum Amount	Maximum Amount
Net energy	Mega calories	1.34351	--
Digestible protein	Kilograms	0.071	0.13

Fat	Kilograms	--	0.05
Vitamin A	International Units	2200	--
Salt	Kilograms	0.015	0.02
Calcium	Kilograms	0.0025	0.01
Phosphorus	Kilograms	0.0035	0.012
Weight	Kilograms	1	1

Feed Mix Problem
Example: cattle feeding

Table 5.13. Nutrient Content per Kilogram of Feeds

Characteristic	Corn	Hay	Soybean	Urea	Dical Phosphate	Salt	Vitamin A Concentrate	Potato Slurry
Net energy	1.48	0.49	1.29					1.39
Digestible protein	0.075	0.127	0.438	2.62				0.032
Fat	0.0357	0.022	0.013					0.009
Vitamin A	600	50880	80				2204600	
Salt						1		
Calcium	0.0002	0.0125	0.0036		0.2313			0.002
Phosphorus	0.0035	0.0023	0.0075	0.68	0.1865			0.0024

Feed Mix Problem
Example: cattle feeding

Table 5.14. Primal Formulation of Feed Problem

	Corn		Hay		Soybean		Urea		Dical		Salt		Vitamin A		Slurry		
Min	$.133X_{C}$	+	$.077X_{H}$	+	$.3X_{SB}$	+	$.332X_{Ur}$	+	$.498X_{d}$	+	$.110X_{SLT}$	+	$.286X_{VA}$	+	PX_{SL}		
Protein	$.075X_{C}$	+	$.127X_{H}$	+	$.438X_{SB}$	+	$2.62X_{Ur}$							+	$.032X_{SL}$	$\leq$	.13
Fat	$.0357X_{C}$	+	$.022X_{H}$	+	$.013X_{SB}$									+	$.009X_{SL}$	$\leq$	.05
Salt											X_{SLT}					$\leq$	.02
Calcium	$.0002X_{C}$	+	$.0125X_{H}$	+	$.0036X_{SB}$			+	$.2313X_{d}$					+	$.002X_{SL}$	$\leq$	.01
Phosphorus	$.0035X_{C}$	+	$.0023X_{H}$	+	$.0075X_{SB}$	+	$.68X_{Ur}$	+	$.1865X_{d}$					+	$.0024X_{SL}$	$\leq$	.012
Energy	$1.48X_{C}$	+	$.49X_{H}$	+	$1.29X_{SB}$									+	$1.39X_{SL}$	$\geq$	1.34351
Protein	$.075X_{C}$	+	$.127X_{H}$	+	$.438X_{SB}$	+	$2.62X_{Ur}$							+	$.032X_{SL}$	$\geq$	.071
Vit A	$600X_{C}$	+	$50880X_{H}$	+	$80X_{SB}$							+	$2204600X_{VA}$			$\geq$	2200
Salt											X_{SLT}					$\geq$	.015
Calcium	$.0002X_{C}$	+	$.0125X_{H}$	+	$.0036X_{SB}$			+	$.2313X_{d}$					+	$.002X_{SL}$	$\geq$	.0025
Phosphorus	$.0035X_{C}$	+	$.0023X_{H}$	+	$.0075X_{SB}$	+	$.68X_{Ur}$	+	$.1865X_{d}$					+	$.0024X_{SL}$	$\geq$	.0035
Volume	X_{C}	+	X_{H}	+	X_{SB}	+	X_{Ur}	+	X_{d}	+	X_{SLT}	+	X_{VA}	+	X_{SL}	=	1

Feed Mix Problem
Example: cattle feeding – Solution

-least cost feed ration is 95.6% slurry, 0.1% vitamin A, 1.5% salt, 0.2% dicalcium phosphate, 1.4%urea, 1.1% soybeans, and 0.1% hay

- reduced costs of feeding corn is 0.95 cents.
- shadow prices: nonzero values indicate the binding constraints (the phosphorous maximum constraint along with the net energy, protein, salt, and calcium minimums and the weight constraint).
- If relaxing the energy minimum, we save $0.065.

Optimal Solution: Objective value = $0.021

Table 5.16. Optimal Primal Solution to the Diet Example Problem

Variable	Value	Reduced Cost	Equation	Slack	Shadow Price
X_C	0	0.095	Protein L Max	0.059	0
X_H	0.001	0	Fat Max	0.041	0
X_{SB}	0.011	0	Salt Max	0.005	0
X_{Ur}	0.014	0	Calcium Max	0.007	0
X_d	0.002	0	Phosphrs	0.000	-2.207
X_{SLT}	0.015	0	Net Engy Min	0.000	0.065
X_{VA}	0.001	0	Protein Min	0.000	0.741
X_{SL}	0.956	0	Vita Lim Min	0.000	0
			Salt Lim Min	0.000	0.218
			Calcium Min	.000	4.400
			Phosphrs	0.008	0
			Weight	0.000	-0.108

Feed Mix Problem

Primal and Dual Algebra:

Primal

$$
\begin{array}{llcll}
\text{Min} & \sum_j c_j F_j & & & \\
\text{s.t.} & \sum_j a_{ij} F_j & \leq & UL_i & \text{for all i} \\
 & \sum_j a_{ij} F_j & \geq & LL_i & \text{for all i} \\
 & \sum_j F_j & = & 1 & \\
 & F_j & \geq & 0 & \text{for all j}
\end{array}
$$

Dual

$$
\begin{array}{lcccccccl}
\text{Max} & -\sum_i \gamma_i UL_i & + & \sum_i \beta_i LL_i & + & \alpha & & & \\
\text{s.t.} & -\sum_i \gamma_i a_{ij} & + & \sum_i \beta_i a_{ij} & + & \alpha & \leq & c_j & \text{for all j} \\
 & \gamma_i, & & \beta_i & & & \geq & 0 & \text{for all i} \\
 & & & & & & & \alpha & \text{unrestricted}
\end{array}
$$

Feed Mix Problem

Notes on Dual Algebra:

Here we have an equality constraint. In fact a general LP is

$$\begin{array}{lll} \text{Max} & cX & \\ \text{s.t.} & AX & \leq b \\ & DX & \geq e \\ & FX & = g \\ & X & \geq 0 \end{array}$$

To write the dual we need to get in standard form and I will use max subject to less thans

First get rid of equality

$$\begin{array}{lll} \text{Max} & cX & \\ \text{s.t.} & AX & \leq b \\ & DX & \geq e \\ & FX & \leq g \\ & FX & \geq g \\ & X & \geq 0 \end{array}$$

Then covert all to less thans

$$\begin{array}{lll} \text{Max} & cX & \\ \text{s.t.} & AX & \leq b \\ & -DX & \leq -e \\ & FX & \leq g \\ & -FX & \leq -g \\ & X & \geq 0 \end{array}$$

Feed Mix Problem

Notes on Dual Algebra:

$$
\begin{array}{llll}
\text{Max} & cX & & \\
\text{s.t.} & AX & \leq & b \\
 & -DX & \leq & -e \\
 & FX & \leq & g \\
 & -FX & \leq & -g \\
 & X & \geq & 0
\end{array}
$$

writing dual

$$
\begin{array}{llll}
\text{Min} & ub \;\; -ve \;\; +wg - yg & & \\
\text{s.t.} & uA \;\; -vD +wF - yF & \geq & c \\
 & u, \;\; v \;\;, w \;\;, y & \geq & 0
\end{array}
$$

Then note one could make substitutions

$v' = v$ and $z = w - y$

yielding

$$
\begin{array}{llll}
\text{Min} & ub \;\; +v'e \;\; +zg & & \\
\text{s.t.} & uA \;\; +v'D \;\; +zF & \geq & c \\
 & u & \geq & 0 \\
 & v' & \leq & 0 \\
 & z & \overset{\leq}{\underset{>}{}} & 0
\end{array}
$$

so when primal has $\geq$ dual variable is negative

and when primal has = dual variable is unrestricted in sign

Feed Mix Problem

Dual and Example

$$
\begin{array}{llllllll}
\text{Max} & -\sum_i \gamma_i UL_i & + & \sum_i \beta_i LL_i & + \ \alpha & & & \\
\text{s.t.} & -\sum_i \gamma_i a_{ij} & + & \sum_i \beta_i a_{ij} & + \ \alpha & \le & c_j & \text{for all } j \\
 & \gamma_i, & & \beta_i & & \ge & 0 & \text{for all } i \\
 & & & & & & \alpha & \text{unrestricted}
\end{array}
$$

Table 5.17. Dual Formulation of Feed Mix Example Problem

	γ_1	γ_2	γ_3	γ_4	γ_5	β_1	β_2	β_3	β_4	β_5	β_6	α		
Max	- .13	- .05	- .02	- .01	- .12	+ 1.34351	+ .071	+ 2200	+ .015	+ .0025	+ .0035	+ 1		
	- .075	- .0357		- .0002	- .0035	+ 1.48	+ .075	+ 600		+ .0002	+ .0035	+ 1	$\le$	.133
	- .127	- .022		- .0125	- .0023	+ .49	+ .127	+ 50880		+ .0125	+ .0023	+ 1	$\le$	.077
	- .438	- .013		- .0036	- .0075	+ 1.29	+ .438	+ 80		+ .0036	+ .0075	+ 1	$\le$	.3
	-2.62				- .68		+ 2.62				+ .68	+ 1	$\le$	.332
				- .2313	- .1865					+ .2313	+ .1865	+ 1	$\le$	.498
			- 1						+ 1			+ 1	$\le$	.110
								+ 2204600				+ 1	$\le$	.286
	- .032	- .009		- .002	- .0024	+ 1.39	+ .032			+ .002	+ .0024	+ 1	$\le$	P

Feed Mix Problem

Primal / Dual Empirical Comparison:

Table 5.14. Primal Formulation of Feed Problem

	Corn	Hay	Soybean	Urea	Dical	Salt	Vitamin A	Slurry		
Min	$.133X_C$	$+ .077X_H$	$+ .3X_{SB}$	$+ .332X_{Ur}$	$+ .498X_d$	$+ .110X_{SLT}$	$+ .286X_{VA}$	$+ PX_{SL}$		
s.t.	$.075X_C$	$+ .127X_H$	$+ .438X_{SB}$	$+ 2.62X_{Ur}$				$+ .032X_{SL}$	$\leq$	.13
Max	$.0357X_C$	$+ .022X_H$	$+ .013X_{SB}$					$+ .009X_{SL}$	$\leq$	.05
Nut						X_{SLT}			$\leq$	.02
	$.0002X_C$	$+ .0125X_H$	$+ .0036X_{SB}$		$+ .2313X_d$			$+ .002X_{SL}$	$\leq$	.01
	$.0035X_C$	$+ .0023X_H$	$+ .0075X_{SB}$	$+ .68X_{Ur}$	$+ .1865X_d$			$+ .0024X_{SL}$	$\leq$	.012
	$1.48X_C$	$+ .49X_H$	$+ 1.29X_{SB}$					$+ 1.39X_{SL}$	$\geq$	1.34351
	$.075X_C$	$+ .127X_H$	$+ .438X_{SB}$	$+ 2.62X_{Ur}$				$+ .032X_{SL}$	$\geq$	.071
Min	$600X_C$	$+ 50880X_H$	$+ 80X_{SB}$				$+ 2204600X_{VA}$		$\geq$	2200
Nut						X_{SLT}			$\geq$	.015
	$.0002X_C$	$+ .0125X_H$	$+ .0036X_{SB}$		$+ .2313X_d$			$+ .002X_{SL}$	$\geq$	.0025
	$.0035X_C$	$+ .0023X_H$	$+ .0075X_{SB}$	$+ .68X_{Ur}$	$+ .1865X_d$			$+ .0024X_{SL}$	$\geq$	.0035
Volume	X_C	$+ X_H$	$+ X_{SB}$	$+ X_{Ur}$	$+ X_d$	$+ X_{SLT}$	$+ X_{VA}$	$+ X_{SL}$	$=$	1

Feed Mix Problem

Primal / Dual Empirical Comparison:

Dual:

Table 5.17. Dual Formulation of Feed Mix Example Problem

	γ_1	γ_2	γ_3	γ_4	γ_5	β_1	β_2	β_3	β_4	β_5	β_6	α	
Max	- .13	- .05	- .02	- .01	- .12	+ 1.34351	+ .071	+ 2200	+ .015	+ .0025	+ .0035	+ 1	
	- .075	- .0357		- .0002	- .0035	+ 1.48	+ .075	+ 600		+ .0002	+ .0035	+ 1	≤ .133
	- .127	- .022		- .0125	- .0023	+ .49	+ .127	+ 50880		+ .0125	+ .0023	+ 1	≤ .077
	- .438	- .013		- .0036	- .0075	+ 1.29	+ .438	+ 80		+ .0036	+ .0075	+ 1	≤ .3
	-2.62				- .68		+ 2.62				+ .68	+ 1	≤ .332
				- .2313	- .1865					+ .2313	+ .1865	+ 1	≤ .498
			- 1						+ 1			+ 1	≤ .110
								+ 2204600				+ 1	≤ .286
	- .032	- .009		- .002	- .0024	+ 1.39	+ .032			+ .002	+ .0024	+ 1	≤ P

Feed Mix Problem

Primal / Dual Solution Comparison

Table 5.16. Optimal Primal Solution to the Diet Example Problem

Variable	Value	Reduced Cost	Constraint	Level	Price
X_C	0	0.095	Protein L Max	0.071	0
X_H	0.001	0	Fat Max	0.009	0
X_{SB}	0.011	0	Salt Max	0.015	0
X_{Ur}	0.014	0	Calcium Max	0.002	0
X_d	0.002	0	Phosphrs	0.012	-2.207
X_{SLT}	0.015	0	Net Engy Min	1.344	0.065
X_{VA}	0.001	0	Protein Min	0.071	0.741
X_{SL}	0.956	0	Vita Lim Min	2200.0	0
			Salt Lim Min	0.015	0.218
			Calcium Min	.002	4.400
			Phosphrs	0.012	0
			Weight	1	-0.108

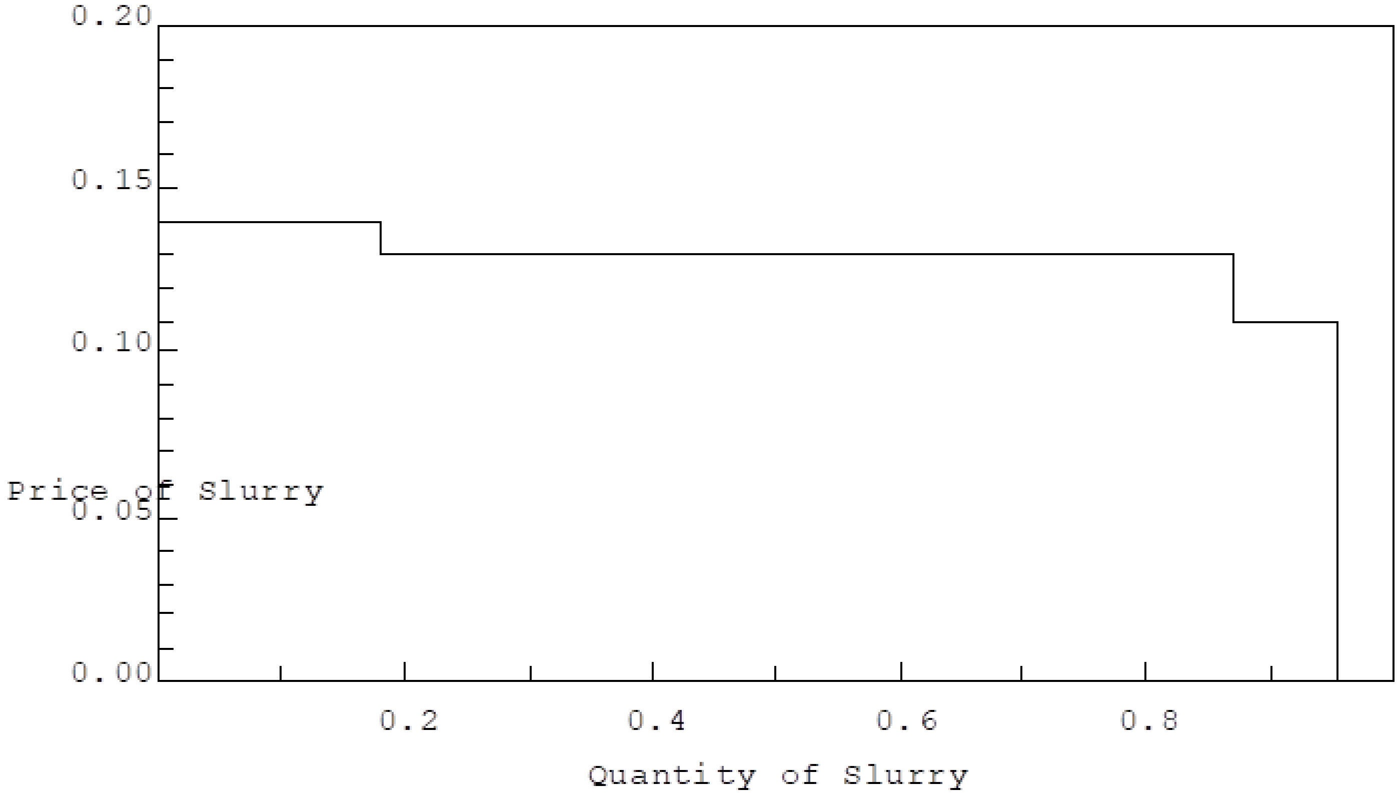
Table 5.18. Optimal Solution to the Dual of the Feed Mix Problem

Variable	Value	Reduced Cost	Constraint	Level	Shadow Price
γ_1	0	-0.059	Corn	0.038	0
γ_2	0	-0.041	Hay	0.077	0.001
γ_3	0	-0.005	Soybean	0.300	0.011
γ_4	0	-0.007	Orea	0.332	0.014
γ_5	-2.207	0	Dical	0.498	0.002
β_1	0.065	0	Salt	0.110	0.015
β_2	0.741	0	Vita	0.286	0.001
β_3	0	0	Slurry	0.01	0.956
β_4	0.218	0			
β_5	4.400	0			
β_6	0	-0.008			
A	-0.108	0			

Feed Mix Problem

Cost Ranging Result

Figure 5.1 Demand Schedule for Potato Slurry in Feed Mix Example



Feed Mix Problem

Table 5.15 GAMS Formulation of Diet Example

```
SET INGREDT  NAMES OF THE AVAILABLE FEED INGREDIENTS
    /CORN,HAY,SOYBEAN,UREA,DICAL,SALT,VITA,SLURRY/
    NUTRIENT NUTRIENT REQUIREMENT CATEGORIES
    /NETENGY,PROTEIN,FAT,VITALIM,SALTLIM,CALCIUM, PHOSPHRS/
    LIMITS   TYPES OF LIMITS IMPOSED ON NUTRIENTS /MIN,MAX/;
PARAMETER INGREDCOST(INGREDT) INGREDIENT COSTS PER KG BOUGHT
          /CORN   .133, HAY  .077, SOYBEAN .300, UREA    .332
         , DICAL  .498,SALT  .110, VITA    .286, SLURRY  .01/;
TABLE NUTREQUIRE(NUTRIENT, LIMITS) NUTRIENT REQUIREMENTS
                        MIN              MAX
        NETENGY       1.34351
        PROTEIN         .071            .130
        FAT               0             .05
        VITALIM         2200
        SALTLIM         .015             .02
        CALCIUM        .0025            .0100
        PHOSPHRS       .0035            .0120;
TABLE CONTENT(NUTRIENT,INGREDT) NUTR CONTENTS PER KG OF FEED
           CORN    HAY   SOYBEAN   UREA  DICAL SALT    VITA  SLURRY
 NETENGY   1.48    .49      1.29                                1.39
 PROTEIN   .075   .127      .438    2.62                       0.032
 FAT      .0357   .022      .013                               0.009
 VITALIM    600  50880        80                        2204600
 SALTLIM                                          1
 CALCIUM  .0002  .0125     .0036         .2313                  .002
 PHOSPHRS  0035  .0023     .0075     .68 .1865                 .0024;

POSITIVE VARIABLES
    FEEDUSE(INGREDT) AMOUNT OF EACH INGREDIENT USED IN MIXING FEED;
VARIABLES
      COST   PER KG COST OF THE MIXED FEED;
EQUATIONS
  OBJT              OBJECTIVE FUNCTION ( TOTAL COST OF THE FEED )
  MAXBD(NUTRIENT) MAX LIMITS ON EACH NUTRIENT IN THE BLENDED FEED
  MINBD(NUTRIENT) MIN LIMITS ON EACH NUTRIENT IN THE BLENDED FEED
  WEIGHT          REQUIRED THAT EXACTLY ONE KG OF FEED BE PRODUCED;

  OBJT..COST =E= SUM(INGREDT,INGREDCOST(INGREDT)* FEEDUSE(INGREDT))
    MAXBD(NUTRIENT)$NUTREQUIRE(NUTRIENT,"MAXIMUM")..
        SUM(INGREDT,CONTENT(NUTRIENT , INGREDT) * FEEDUSE(INGREDT))
                        =L= NUTREQUIRE(NUTRIENT, "MAXIMUM");
      MINBD(NUTRIENT)$NUTREQUIRE(NUTRIENT,"MINIMUM")..
          SUM(INGREDT,CONTENT(NUTRIENT, INGREDT) * FEEDUSE(INGREDT))
                        =G= NUTREQUIRE(NUTRIENT, "MINIMUM");
      WEIGHT..       SUM(INGREDT, FEEDUSE(INGREDT)) =E= 1. ;
      MODEL FEEDING  /ALL/;
      SOLVE FEEDING USING LP MINIMIZING COST;

      SET VARYPRICE PRICE SCENARIOS /1*30/
      PARAMETER SLURR(VARYPRICE,*)
      OPTION SOLPRINT = OFF;
      LOOP (VARYPRICE,
      INGREDCOST("SLURRY")= 0.01 + (ORD(VARYPRICE)-1)*0.005;
      SOLVE FEEDING USING LP MINIMIZING COST;
      SLURR(VARYPRICE,"SLURRY") = FEEDUSE.L("SLURRY");
      SLURR(VARYPRICE,"PRICE") =   INGREDCOST("SLURRY")     ) ;
      DISPLAY SLURR;
```

Joint Products Problem

Formulation

Basic notation and the decision variable

Let us denote the set of

the produced products as product

the production possibilities as process

the purchased inputs as input

the available resources as resource

Let us define three fundamental decision variables as

$Sales_{product}$:the set of produced product,

$Production_{process}$:the set of production possibilities,

$BuyInput_{input}$:the set of purchased inputs,

Joint Products Problem

Formulation

To set up the joint products problem, four additional parameter values that give the composite relationship among product, process, input, and resource are needed.

These parameters are:

$q_{product,process}$:the quantity yielded by the production possibility,

$r_{input,process}$: the amount of the inputth input used by the processth production possibility,

$s_{resource,process}$: the amount of the resourceth resource used by the processth production possibility,

$b_{resource}$:the amount of resource availability or endowment by resource,

Joint Products Problem

Formulating the Problem

The objective function:

We want to maximize total profits across all of the possible productions. To do so, three additional required parameters for sale price, input purchase cost, and other production costs associated with production are needed.
Let us define these parameters as

$SalePrice_{product}$ $InputCost_{input}$ $OtherCost_{process}$

Then the objective function becomes

Maximize

$$\sum_{product} SalePrice_{product} \; Sales_{product}$$

$$-\sum_{input} InputCost_{input} \; BuyInput_{input}$$

$$-\sum_{process} OtherCost_{process} \; Production_{process}$$

Joint Products Problem

Formulating the Problem

The constraints:

There are four general types of constraints:

1) demand and supply balance for which quantity sold of each product is less than or equal to the quantity yielded by production.

$$Sales_{product} - \sum_{process} q_{product,process} \; Production_{process} \leq 0$$

2) demand and supply balance for which quantity purchased of each fixed price input is greater than or equal to the quantity utilized by the production activities.

$$\sum_{process} r_{input,process} \; Production_{process} - BuyInput_{input} \leq 0$$

3) resource availability constraint insuring that the quantity used of each fixed quantity input does not exceed the resource endowments.

$$\sum_{process} s_{resource,process} \; Production_{process} \leq b_{resource}$$

4) nonnegativity

$$Sales_{product}, \; BuyInput_{input}, \; Production_{process} \geq 0$$

Joint Products Problem

Formulating the Problem

Maximize

$$\sum_{product} SalePrice_{product}\ Sales_{product}$$

$$-\sum_{input} InputCost_{input}\ BuyInput_{input}$$

$$-\sum_{process} OtherCost_{process}\ Production_{process}$$

s.t.

$$Sales_{product} - \sum_{process} q_{product,process}\ Production_{process} \leq 0$$

$$\sum_{process} r_{input,process}\ Production_{process} - BuyInput_{input} \leq 0$$

$$\sum_{process} s_{resource,process}\ Production_{process} \leq b_{resource}$$

$$Sales_{product},\ BuyInput_{input},\ Production_{process} \geq 0$$

Joint Products Problem

Example: wheat production

Two products produced: Wheat and wheat straw

Three inputs: Land, fertilizer, seed

Seven production processes:

Outputs and Inputs Per Acre	Process						
	1	2	3	4	5	6	7
Wheat yield in bu.	30	50	65	75	80	80	75
Wheat straw yield/bales	10	17	22	26	29	31	32
Fertilizer use in Kg.	0	5	10	15	20	25	30
Seed in pounds	10	10	10	10	10	10	10
Land	1	1	1	1	1	1	1

Wheat price = \$4/bushel, wheat straw price =\$0.5/bale

Fertilizer = \$2 per kg, Seed = \$0.2/lb.

\$5 per acre production cost for each process

Land = 500 acres

Joint Products Problem

Example: Wheat Production

Maximize

$$\sum_{product} SalePrice_{product} \; Sales_{product}$$

$$- \sum_{input} InputCost_{input} \; BuyInput_{input}$$

$$- \sum_{process} OtherCost_{process} \; Production_{process}$$

s.t.

$$Sales_{product} - \sum_{process} q_{product,process} \; Production_{process} \le 0$$

$$\sum_{process} r_{input,process} \; Production_{process} - BuyInput_{input} \le 0$$

$$\sum_{process} s_{resource,process} \; Production_{process} \le b_{resource}$$

$$Sales_{product}, \; BuyInput_{input}, \; Production_{process} \ge 0$$

Joint Products Problem

Example: Wheat Production

$$
\begin{array}{l}
\max\; 4Sale_1 + .5Sale_2 - 5Y_1 - 5Y_2 - 5Y_3 - 5Y_4 - 5Y_5 - 5Y_6 - 5Y_7 - 2Z_1 - .2Z_2 \\
\text{s.t.}\quad Sale_1 - 30Y_1 - 50Y_2 - 65Y_3 - 75Y_4 - 80Y_5 - 80Y_6 - 75Y_7 \le 0 \\
\quad Sale_2 - 10Y_1 - 17Y_2 - 22Y_3 - 26Y_4 - 29Y_5 - 31Y_6 - 32Y_7 \le 0 \\
\quad + 5Y_2 + 10Y_3 + 15Y_4 + 20Y_5 + 25Y_6 + 30Y_7 - Z_1 \le 0 \\
\quad 10Y_1 + 10Y_2 + 10Y_3 + 10Y_4 + 10Y_5 + 10Y_6 + 10Y_7 - Z_2 \le 0 \\
\quad Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6 + Y_7 \le 500 \\
\quad Sale_1, Sale_2, Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7, Z_1, Z_2 \ge 0
\end{array}
$$

Note that: Y refers to Productionprocess and Z refers to BuyInputinput

$U1 \geq 4$

$U2 \geq 0.5$

$-W1 >= -2$

Joint Products Problem

Example: Wheat Production – Solution

- 40,000 bushels of wheat and 14,500 bales of straw are produced by 500 acres of the fifth production possibility using 10,000 kilograms of fertilizer and 5,000 lbs. of seed
- reduced cost shows a $169.50 cost if the first production possibility is used.
- shadow prices are values of sales, purchase prices of the various outputs and inputs, and land values ($287.5).

Optimal Solution Value 143,750

Variable	Value	Reduced Cost CCostcCost	Equation	Slack	Shadow Price
$Sale_1$	40,000	0	Wheat	0	4
$Sale_2$	14,500	0	Straw	0	0.5
$Process_1$	0	-169.50	Fertilizer	0	2
$Process_2$	0	-96.00	Seed	0	0.2
$Process_3$	0	-43.50	Land	0	287.5
$Process_4$	0	-11.50			
$Process_5$	500	0			
$Process_6$	0	-9.00			
$Process_7$	0	-38.50			
$Buyinput_{fert}$	10,000	0			
$Buyinput_{seet}$	5,000	0			

Joint Products Problem

Example: Wheat Production – Dual

$$Min \sum_{resource} U_{resource} b_{resource}$$

$$nputCost_{input} \quad \forall input \qquad V_{product}, \qquad W_{input}, \qquad U_{resource} \geq 0$$

Dual objective: minimizes total marginal resource values

$U_{resource}$ is the marginal resource value
$V_{product}$ is the marginal product value
W_{input} is the marginal input cost

1st constraint (dual implication of a sales variable) insures that V is GE the sales price. A lower bound is imposed on the shadow price for the commodity sold.
2nd constraint (dual implication of a production variable) insures that the total value of the products yielded by a process is LE the value of the inputs used in production.
3rd constraint (dual implication of a purchase variable) insures that W is no more than its purchase price. An upper bound is imposed on the shadow price of the item which can be purchased.

Joint Products Problem

Example: Wheat Production – Dual Empirical

Table 5.22. Dual Formulation of Example Joint Products Problem

Min									500U			
s.t.	V_1										$\geq$	4
			V_2								$\geq$	.5
	$-30V_1$	-	$10V_2$			+	$10W_2$	+	U		$\geq$	-5
	$-50V_1$	-	$17V_2$	+	$5W_1$	+	$10W_2$	+	U		$\geq$	-5
	$-65V_1$	-	$22V_2$	+	$10W_1$	+	$10W_2$	+	U		$\geq$	-5
	$-75V_1$	-	$26V_2$	+	$15W_1$	+	$10W_2$	+	U		$\geq$	-5
	$-80V_1$	-	$29V_2$	+	$20W_1$	+	$10W_2$	+	U		$\geq$	-5
	$-80V_1$	-	$31V_2$	+	$25W_1$	+	$10W_2$	+	U		$\geq$	-5
	$-75V_1$	-	$32V_2$	+	$30W_1$	+	$10W_2$	+	U		$\geq$	-5
					$-W_1$						$\geq$	-2
							$-W_2$				$\geq$	-0.2
	V_1	,	V_2	,	W_1	,	W_2	,	U		$\geq$	0

$$+ \quad 20W1 \quad + \quad 10W2 \quad + \quad U+5 \quad \geq \quad +80V1 \quad + \quad 29V2$$

Joint Products Problem

Example: Wheat Production – Dual

The optimal solution to the dual problem corresponds exactly to the optimal primal solution where:

Dual	**Primal**
Value	Shadow Price
Reduced Cost	Slack
Shadow Price	Value
Slacks	Reduced Cost

Objective Function Value 143,750

Variable	Value	Reduced Cost	Equation	Slacks	Shadow
V_1	4	0	Wheat	0	40,000
V_2	0.5	0	Straw	0	14,500
W_1	2	0	Prod 1	169.5	0
W_2	0.2	0	Prod 2	96	0
U	287.5	0	Prod 3	43.5	0
			Prod 4	11.5	0
			Prod 5	0	500
			Prod 6	9	0
			Prod 7	38.5	0
			Fertilizer	0	10,000
			Seed	0	5,000

Joint Products Problem

GAMS Formulation

Table 5.20. GAMS Formulation of the Joint Products Example

```
SET  PRODUCTS   LIST OF ALTERNATIVE PRODUCT    /WHEAT, STRAW/
     INPUTS     PURCHASED INPUTS               /SEED,  FERT/
     FIXED      FIXED INPUTS                   /LAND/
     PROCESS    POSSIBLE INPUT COMBINATIONS    /Y1*Y7/;

PARAMETER  PRICE(PRODUCTS) PRODUCT PRICES /WHEAT 4.00, STRAW 0.50/
           COST(INPUTS)     INPUT PRICES   /SEED 0.20,  FERT 2.00/
           PRODCOST(PROCESS) PRODUCTION COSTS BY PROCESS
           AVAILABLE(FIXED)  FIXED INPUTS AVAILABLE / LAND  500 /;

           PRODCOST(PROCESS)  = 5;

TABLE YIELDS(PRODUCTS, PROCESS) PRODUCTION POSSIBILITIES YIELDS
                  Y1     Y2     Y3     Y4     Y5     Y6     Y7
        WHEAT     30     50     65     75     80     80     75
        STRAW     10     17     22     26     29     31     32;

TABLE USAGE(INPUTS,PROCESS) PURCHASED INPUT USAGE BY PRODUCTION
POSSIBLIITIES
                  Y1     Y2     Y3     Y4     Y5     Y6     Y7
        SEED      10     10     10     10     10     10     10
        FERT       0      5     10     15     20     25     30;

TABLE FIXUSAGE(FIXED,PROCESS) FIXED INPUT USAGE BY PRODUCTION
POSSIBLIITIES
                  Y1     Y2     Y3     Y4     Y5     Y6     Y7
        LAND       1      1      1      1      1      1      1;

POSITIVE VARIABLES
           SALES(PRODUCTS)      AMOUNT OF EACH PRODUCT SOLD
           PRODUCTION(PROCESS) LAND AREA GROWN WITH EACH INPUT
PATTERN
           BUY(INPUTS)           AMOUNT OF EACH INPUT PURCHASED ;
VARIABLES
           NETINCOME  NET REVENUE  (PROFIT);
EQUATIONS
           OBJT               OBJECTIVE FUNCTION  (NET REVENUE)
           YIELDBAL(PRODUCTS) BALANCES PRODUCT SALE WITH PRODUCTION
           INPUTBAL(INPUTS)   BALANCE INPUT PURCHASES WITH USAGE
           AVAIL(FIXED)        FIXED INPUT AVAILABILITY;


OBJT..  NETINCOME =E=
           SUM(PRODUCTS , PRICE(PRODUCTS)    * SALES(PRODUCTS))
             - SUM(PROCESS   , PRODCOST(PROCESS)
                     * PRODUCTION(PROCESS))
                     - SUM(INPUTS    , COST(INPUTS)
                             * BUY(INPUTS));
YIELDBAL(PRODUCTS)..

           SUM(PROCESS, YIELDS(PRODUCTS,PROCESS)

                     * PRODUCTION(PROCESS))
                             =G=      SALES(PRODUCTS);
INPUTBAL(INPUTS)..
     SUM(PROCESS, USAGE(INPUTS,PROCESS) * PRODUCTION(PROCESS))
                               =L=      BUY(INPUTS);
AVAIL(FIXED)..
     SUM(PROCESS, FIXUSAGE(FIXED,PROCESS)*PRODUCTION(PROCESS))
                     =L= AVAILABLE(FIXED);
```

```
56  MODEL JOINT /ALL/;
57  SOLVE JOINT USING LP MAXIMIZING NETINCOME;
```

Joint Products Problem

Alternative Formulations

Table 5.24 Alternative Computer Inputs for a Model

Simple GAMS Input File

```
POSITIVE VARIABLES    X1 ,   X2,   X3
VARIABLES             Z
EQUATIONS             OBJ,  CONSTRAIN1  ,  CONSTRAIN2;
OBJ..           Z =E= 3 * X1 + 2 * X2 + 0.5* X3;
CONSTRAIN1..        X1 + X2 +X3=L= 10;
CONSTRAIN2..        X1 - X2 =L= 3;
MODEL PROBLEM /ALL/;
SOLVE PROBLEM USING LP MAXIMIZING Z;
```

Lindo Input

```
MAX 3 * X1 + 2 * X2 + 0.5* X3;
ST
X1 + X2 +X3 <  10
X1 - X2 < 3
END
GO
```

Tableau Input File

```
 5   3
 3.  2.  0.5   0.   0.
 1.  1.  1.  1.  0.  10.
 1. -1.  0.  0.  1.  3.
```

MPS Input File

```
NAME        CH2MPS
ROWS
 N          R1
 L          R2
 L          R3
COLUMNS
   X1       R0       3.         R1       1.
   X1       R3       1.
   X2       R0       2.         R1       1.
   X2       R1       -1.
   X3       R0       0.5        R1       1.
RHS
   RHS1     R1       10.        R1       3.
ENDDATA
```

More Complex GAMS input file

```
SET   PROCESS       TYPES OF PRODUCTION PROCESSES   /X1,X2,X3/
    RESOURCE      TYPES OF RESOURCES     /CONSTRAIN1,CONSTRAIN2/
PARAMETER
  PRICE(PROCESS)       PRODUCT PRICES BY PROCESS /X1 3,X2 2,X3 0.5/
  PRODCOST(PROCESS)     COST BY PROCESS          /X1 0 ,X2 0, X3 0/
  RESORAVAIL(RESOURCE)  RESOURCE AVAILABLITY
                        /CONSTRAIN1 10 ,CONSTRAIN2 3/
TABLE RESOURUSE(RESOURCE,PROCESS) RESOURCE USAGE
              X1      X2      X3
   CONSTRAIN1     1       1       1
   CONSTRAIN2     1      -1
POSITIVE VARIABLES   PRODUCTION(PROCESS) ITEMS PRODUCED BY PROCESS;
VARIABLES           PROFIT             TOTALPROFIT;
EQUATIONS           OBJT               OBJECTIVE FUNCTION ( PROFIT )
                 AVAILABLE(RESOURCE) RESOURCES AVAILABLE ;
OBJT.. PROFIT=E=   SUM(PROCESS,(PRICE(PROCESS)-PRODCOST(PROCESS))*
                PRODUCTION(PROCESS)) ;
AVAILABLE(RESOURCE).. SUM(PROCESS,RESOURUSE(RESOURCE,PROCESS)
                   *PRODUCTION(PROCESS))  =L= RESORAVAIL(RESOURCE);
MODEL RESALLOC /ALL/;
SOLVE RESALLOC USING LP MAXIMIZING PROFIT;
```