Analysis Lesson 19
[MAT320/MAT640 Analysis]
with Professor Sormani
Spring 2022

Extrema,
Mean Value Theorem,
Increasing/Decreasing
and Continuity

Your work for today's lesson will go in a googledoc you create entitled MAT320S22-Lesson19-Lastname-Firstname with your last name and your first name. The googledoc will be shared with the professor sormanic@gmail.com as an editor. Put any questions you have inside your doc and email me to let me know it is there. Be sure to complete one page of HW on paper and take a selfie holding up a few pages.

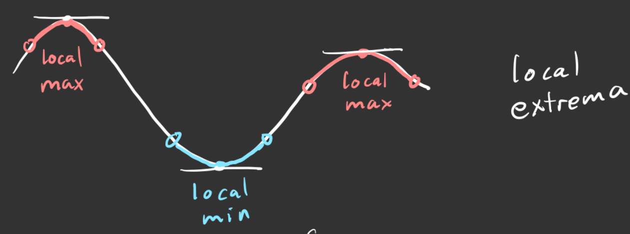
This lesson has five parts, lots of classwork and 5 HW problems.

Part 1: Extrema
Part 2: Mean Value Theorem,
Part 3: Increasing/Decreasing
Part 4: Continuity
Part 5: Second Derivatives

Part 1: Extrema

Watch [Playlist DiffExt-1to3]

Derivatives at Local Extrema



Defn f has a local max at x_0 if there is an interval (x_0-r, x_0+r) s.t. $f(x) \leq f(x_0)$ $\forall x \in (x_0-r, x_0+r)$

Defn f has a local min at x_0 if there is an interval (x_0-r, x_0+r) $f(x) \geq f(x_0)$ $\forall x \in (x_0-r, x_0+r)$

Defn f has a local extrema at x_0 if it has a local max or a local min at x_0

Thm: If f is differentiablp at a local extrema (min or max) then $f'(x_0) = 0$.

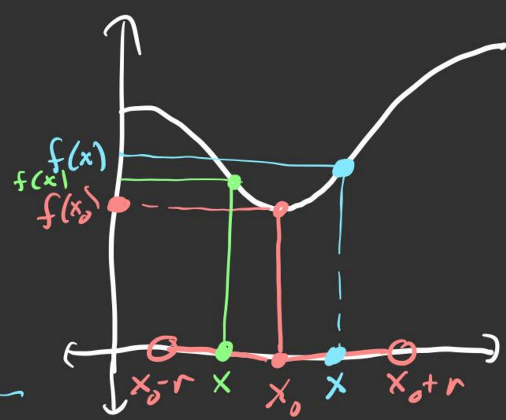
Proof of Theorem

Classwork

Given: f has a local <u>min</u> at x_0

f is differentiable at x_0

Show: $f'(x_0) = 0$.



Proof:

① $\exists r > 0$ s.t. $\forall x \in (x_0 - r, x_0 + r)$ $f(x_0) \le f(x)$

① Given & defn of local min

② $\left.\begin{array}{l} f(x) - f(x_0) \ge 0 \\ \forall x \in (x_0 - r, x_0) \end{array}\right\}$ left

$\left.\begin{array}{l} f(x) - f(x_0) \ge 0 \\ \forall x \in (x_0, x_0 + r) \end{array}\right\}$ right

② —

—

③ $\dfrac{f(x) - f(x_0)}{x - x_0} \le 0 \quad \forall x \in (x_0 - r, x_0)$

③ because $x_0 - r < x < x_0 \Rightarrow x - x_0 < 0$

$\dfrac{f(x) - f(x_0)}{x - x_0} \ge 0 \quad \forall x \in (x_0, x_0 + r)$

because $x_0 < x < x_0 + r \Rightarrow 0 < x - x_0$

(3) $$\lim_{x \to x_0^-} \frac{f(x)-f(x_0)}{x-x_0} \le 0$$

$$\lim_{x \to x_0^+} \frac{f(x)-f(x_0)}{x-x_0} \ge 0$$

(3) limit exists given because f is diffble at x_0

limit exist because f is diffble at x_0

(4) $$0 \le \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} \le 0$$

(4) left & right limits are equal because limit exists (because $f'(x_0)$ exists)

(5) $$f'(x_0) = 0$$

(5) by defn of $f'(x_0)$

QED

Classwork
do local max.

Part 2: Mean Value Theorem

Watch [Playlist MVT-1to5]

Mean value theorem

From Wikipedia, the free encyclopedia

For the theorem in harmonic function theory, see Harmonic function § The mean value property.

In mathematics, the **mean value theorem** states, roughly, that for a given planar arc between two endpoints, there is at least one point at which the tangent to the arc is parallel to the secant through its endpoints. It is one of the most important results in real analysis. This theorem is used to prove statements about a function on an interval starting from local hypotheses about derivatives at points of the interval.

More precisely, the theorem states that if f is a continuous function on the closed interval $[a,b]$ and differentiable on the open interval (a,b), then there exists a point c in (a,b) such that the tangent at c is parallel to the secant line through the endpoints $(a, f(a))$ and $(b, f(b))$, that is,

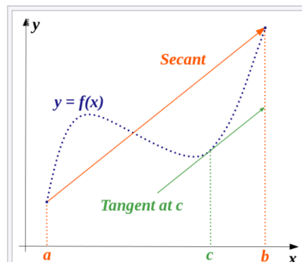
$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Part of a series of articles about
Calculus

Fundamental theorem
Leibniz integral rule
Limits of functions · Continuity
Mean value theorem · Rolle's theorem

Differential	[show]
Integral	[show]
Series	[show]
Vector	[show]
Multivariable	[show]
Specialized	[show]
Miscellaneous	[show]

V · T · E



Contents [hide]

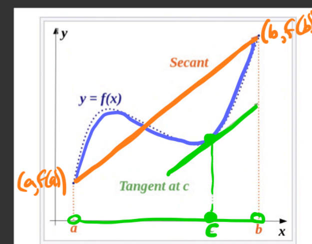
Formal statement [edit]

Let $f : [a,b] \to \mathbb{R}$ be a continuous function on the closed interval $[a,b]$, and differentiable on the open interval (a,b), where $a < b$. Then there exists some c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

The mean value theorem is a generalization of Rolle's theorem, which assumes $f(a) = f(b)$, so that the right-hand side above is zero.

Mean value theorem



Formal statement [edit]

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$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The mean value theorem is a generalization of Rolle's theorem, which assumes $f(a) = f(b)$, so that the right-hand side above is zero.

Given: $f : [a,b] \to \mathbb{R}$ Continuous on $[a,b]$ Differentiable on (a,b)

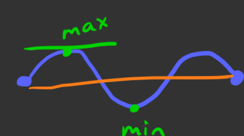
Show: $\exists c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

We will prove a special case first called "Rolles' Thm"

Assume: $f(b) = f(a)$ Show $f'(c) = 0$

Recall that if f satisfies our given then f has a max + min at some point $x_{max}, x_{min} \in [a,b]$

furthermore $f'(x_{max}) = 0$ $f'(x_{min}) = 0$



A useful fact we could use to justify a step of our proof.

Proof of Rolles' Thm by Cauchy (earliest proof by Bhaskara):

① f has a max at $x_{max} \in [a,b]$	① by f is contin on $[a,b]$ given + max existence
Case I $x_{max} \in (a,b)$ **Case II**: x_{max} is an endpoint.	
Case I $x_{max} \in (a,b)$	
② $f'(x_{max}) = 0$ So take $c = x_{max}$	② by f is diffble on (a,b) given and f' at max $= 0$.
Case II x_{max} is an endpoint	
② f has a min at $x_{min} \in [a,b]$	② by f contin on $[a,b]$ given + min existence.
Case II i: $x_{min} \in (a,b)$ Case II ii: x_{min} is an endpoint	
③ $f'(x_{min}) = 0$ take $c = x_{min}$	③ by f diffble on (a,b) given + f' at min $= 0$
Case II ii: x_{min} + x_{max} are both endpoints	
③ $f(a) = f(b)$	③ Given in Rolle's Thm.
④ $f(x_{min}) = f(x_{max})$	④ endpoints are $a + b$.
⑤ $\min\limits_{x\in[a,b]} f(x) = \max\limits_{x\in[a,b]} f(x) = f(a) = f(b)$	⑤ Defn of $x_{min} + x_{max}$
⑥ f is a constant on $[a,b]$	⑥ by defn of max + min
⑦ $f'(x) = 0$ on $[a,b]$	⑦ $\frac{d}{dx} k = 0$
⑧ Choose any $c \in (a,b)$	⑧ by step 7.

Rolle's Thm QED

Still must prove the Mean Value Theorem.

Examples [edit]

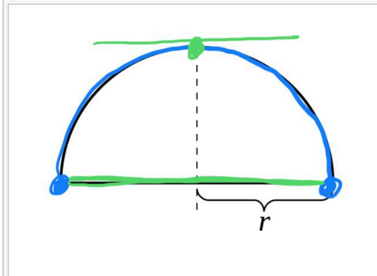
Rolle's Theorem

First example [edit]

For a radius $r > 0$, consider the function

$$f(x) = \sqrt{r^2 - x^2}, \quad x \in [-r, r].$$

Its graph is the upper semicircle centered at the origin. This function is continuous on the closed interval $[-r, r]$ and differentiable in the open interval $(-r, r)$, but not differentiable at the endpoints $-r$ and r. Since $f(-r) = f(r)$, Rolle's theorem applies, and indeed, there is a point where the derivative of f is zero. Note that the theorem applies even when the function cannot be differentiated at the endpoints because it only requires the function to be differentiable in the open interval.



A **semicircle** of radius r.

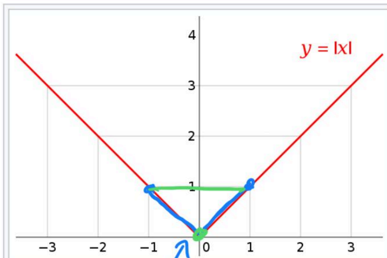
Second example [edit]

If differentiability fails at an interior point of the interval, the conclusion of Rolle's theorem may not hold. Consider the absolute value function

$$f(x) = |x|, \qquad x \in [-1, 1].$$

Then $f(-1) = f(1)$, but there is no c between –1 and 1 for which the $f'(c)$ is zero. This is because that function, although continuous, is not differentiable at $x = 0$. Note that the derivative of f changes its sign at $x = 0$, but without attaining the value 0. The theorem cannot be applied to this function because it does not satisfy the condition that the function must be differentiable for every x in the open interval. However, when the differentiability requirement is dropped from Rolle's theorem, f will still have a critical number in the open interval (a, b), but it may not yield a horizontal tangent (as in the case of the absolute value represented in the graph).

fails Rolle's Thm.



The graph of the absolute value function.

not diffble at 0

Generalization [edit]

Mean value theorem

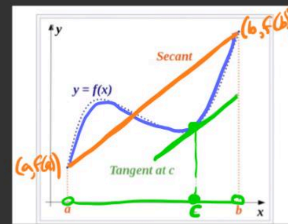
Can now prove this!

Formal statement [edit]

Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the closed interval $[a, b]$, and differentiable on the open interval (a, b), where $a < b$. Then there exists some c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The mean value theorem is a generalization of Rolle's theorem, which assumes $f(a) = f(b)$, so that the right-hand side above is zero.



Given: $f : [a,b] \to \mathbb{R}$ Continuous on $[a,b]$ Differentiable on (a,b)

Show: $\exists c \in (a,b)$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

We've already proven the special case:
Assume: $f(b) = f(a)$ Show $f'(c) = 0$ ← Rolle's Thm

Proof: ① Let $g(x) = f(x) - mx$
then $\boxed{g'(x) = f'(x) - m}$
so g is contin on $[a,b]$ and diffble on (a,b)

① by diff laws and cont. laws and given info about f.

② Let $m = \dfrac{f(b)-f(a)}{b-a}$

③ $b \neq a$ so $b - a \neq 0$

Goal: show $g'(c) = 0$ so $f'(c) = m = \dfrac{f(b)-f(a)}{b-a}$
Show g satisfies hypotheses of Rolle's Thm

③ $g(a) = f(a) - ma = f(a) - \left(\dfrac{f(b)-f(a)}{b-a}\right)a$

$g(b) = f(b) - mb$
$= f(b) - \left(\dfrac{f(b)-f(a)}{b-a}\right)b$

Check these are equal

④ $g(a) = g(b)$

Classwork!

⑤ $\exists c \in (a,b)$ s.t $g'(c) = 0$ ⑤ Rolle's Thm

⑥ $f'(c) = m = \dfrac{f(b)-f(a)}{b-a}$ ⑥ $0 = g'(c) = f'(c) - m$ and choice of m in step 2.

QED

Part 3: Increasing/Decreasing

Watch [Playlist MVT-6to7]

Increasing and Decreasing

<u>Defn</u> f is increasing on $[c,d]$ if $\forall a,b \in [c,d]$ s.t $a<b$ we have $f(a)<f(b)$.

Example: $f(x)=mx$ ~~is~~ is inc if $m>0$ ~~is inc~~.

Show: $f(b)=mb>ma=f(a)$

<u>Proof</u>: ① $a<b$ ① given

② $ma<mb$ ② $m>0$ $\quad a<b$ and $c=m>0 \Rightarrow ac<bc$

③ $f(a)<f(b)$ ③ defn of f.

<u>Defn</u> f is decreasing on $[c,d]$. if $\forall a,b \in [c,d]$ s.t $a<b$ we have $f(a)>f(b)$

HW1 $f(x)=mx$ is dec if $m<0$.

Prove this!

Theorem: If $^{(i)}$ f is continuous on (c,d)
and $^{(ii)}$ f is differentiable on (c,d)
and $^{(iii)}$ $f'(x) > 0$ on (c,d) } Given

then f is strictly increasing on (c,d) } Show
$\forall a,b \in (c,d)^{*}\ a<b \Rightarrow f(a) < f(b)$

Proof ① Given $a,b \in (c,d)$ where $a<b$ — ① Given*

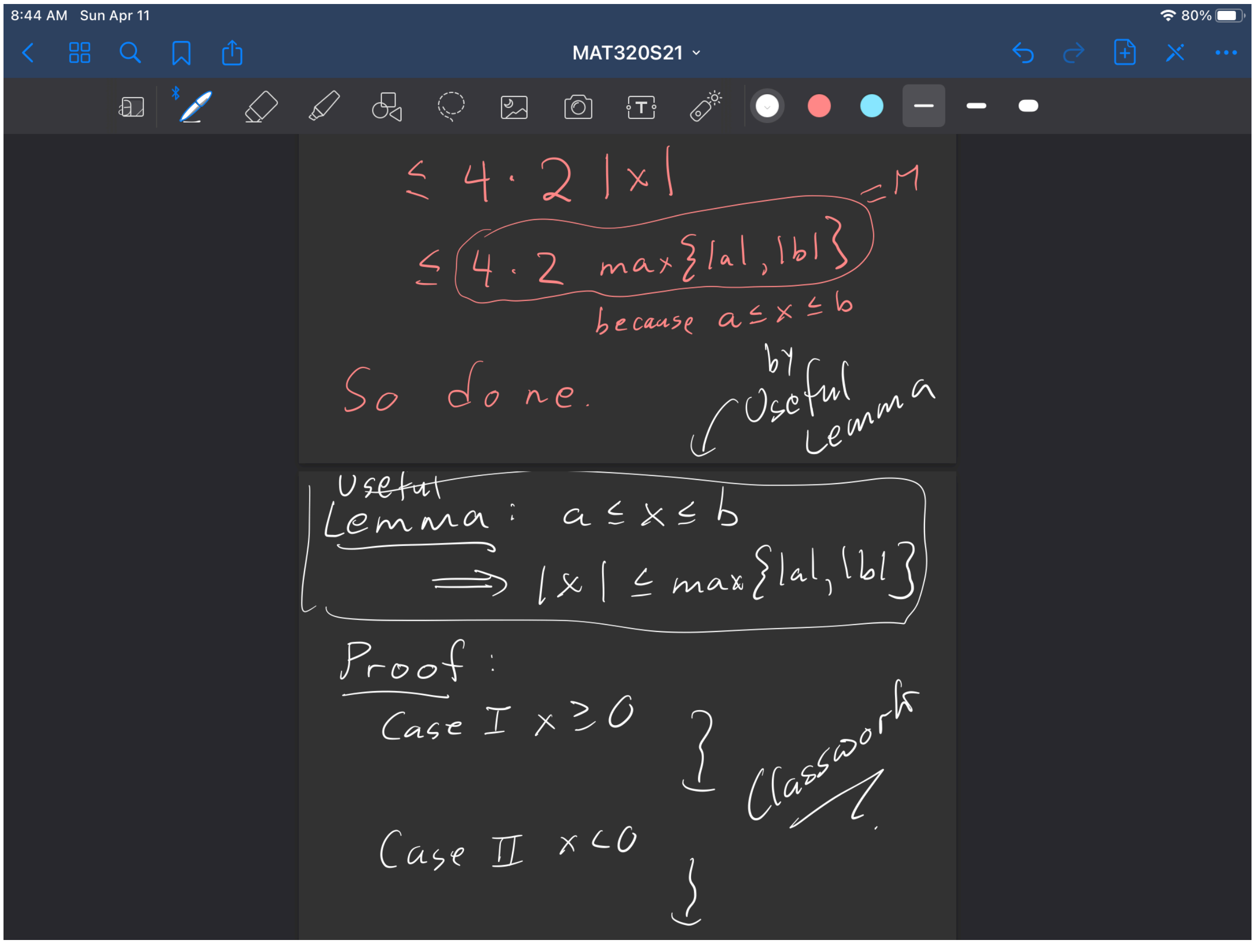
Proof Structure from Show

② f is contin on $[a,b] \subset (c,d)$	② Given (i)
③ f is diffble on $(a,b) \subset (c,d)$	③ Given (ii)
④ $\exists c \in (a,b) \in (c,d)$ s.t $f'(c) = \frac{f(b)-f(a)}{b-a}$	④ Mean Value Theorem
⑤ $\frac{f(b)-f(a)}{b-a} = f'(c) > 0$	⑤ Given (iii)
⑥ $f(b) - f(a) > 0(b-a) = 0$	⑥ $A<B$ and $C = b-a > 0$ by Given* $\Rightarrow AC < BC$
⑦ $f(b) > f(a)$	⑦ add $f(a)$ to both sides
final $f(a) < f(b)$	⑧ $>$ to $<$ QED

$\boxed{\text{HW 2}}$ Prove:

Theorem: If $^{(i)}$ f is continuous on (c,d)
and $^{(ii)}$ f is differentiable on (c,d)
and $^{(iii)}$ $f'(x) < 0$ on (c,d)
then f is strictly decreasing on (c,d)

$\boxed{\text{HW 3}}$ Prove: If $^{(i)}$ f is continuous on (c,d)
and $^{(ii)}$ f is differentiable on (c,d)
and $^{(iii)}$ $f'(x) \geq 0$ on (c,d)
then $\forall a,b \in (c,d)\ a<b \Rightarrow f(a) \leq f(b)$.



Part 4: Continuity

Watch [Playlist MVT-8to10]

Continuity

Theorem: If $f:[a,b] \rightarrow \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b) and $\forall x \in (a,b)$ $|f'(x)| \le M$

then $\forall \varepsilon > 0$ $\exists \delta_\varepsilon = \frac{\varepsilon}{M} > 0$ s.t $\begin{matrix} |x-x_0| < \delta_\varepsilon \\ x, x_0 \in [a,b] \end{matrix} \Rightarrow |f(x) - f(x_0)| < \varepsilon$

Proven using the mean value theorem:

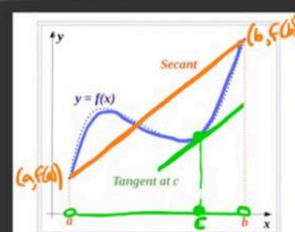
Mean value theorem

Formal statement [edit]

Let $f : [a,b] \to \mathbb{R}$ be a continuous function on the closed interval $[a,b]$, and differentiable on the open interval (a,b), where $a < b$. Then there exists some c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The mean value theorem is a generalization of Rolle's theorem, which assumes $f(a) = f(b)$, so that the right-hand side above is zero.



Given: $f:[a,b] \rightarrow \mathbb{R}$ Continuous on $[a,b]$ Differentiable on (a,b)

Show: $\exists c \in (a,b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

Continuity IMPORTANT!

Theorem: If $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b) and

$\forall x\in(a,b)\ |f'(x)|\le M$

then $\underbrace{\forall\varepsilon>0\ \exists\delta_\varepsilon=\frac{\varepsilon}{M}>0 \text{ s.t } \begin{matrix}|x-x_0|<\delta_\varepsilon\\ x,x_0\in[a,b]\end{matrix} \Rightarrow |f(x)-f(x_0)|<\varepsilon}$

So f is uniformly continuous.

So if we have f_j satisfying the above hypothesis with the same M for the f_j then the f_j are equicontinuous

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Theorem: If $f:[a,b]\to\mathbb{R}$ is (i) continuous on $[a,b]$ and (ii) differentiable on (a,b) and

$\forall x\in(a,b)\ |f'(x)|\le M$ where $M>0$

then $\forall\varepsilon>0\ \exists\delta_\varepsilon=\frac{\varepsilon}{M}>0$ s.t $\begin{matrix}|x-x_0|<\delta_\varepsilon\\ x,x_0\in[a,b]\end{matrix} \Rightarrow |f(x)-f(x_0)|<\varepsilon$

Proof: (1) Given $\varepsilon>0$ Choose $\delta_\varepsilon=\frac{\varepsilon}{M}>0$ (1) by $\begin{matrix}M>0\\ \varepsilon>0\end{matrix}$

(2) Whenever $|x-x_0|<\delta_\varepsilon$ and $x,x_0\in[a,b]$ (2) by given (i) and (ii)

Case I $x<x_0$ Case II $x>x_0$

Case I: we have f contin on $[x,x_0]$ and diffble on (x,x_0)

(3) $\exists c\in(x_0,x)$

Classwork

(Final) $|f(x)-f(x_0)|<\varepsilon$.

Proof: ① Given $\varepsilon > 0$ choose $\delta_\varepsilon = \frac{\varepsilon}{M} > 0$ — ① by $M > 0$, $\varepsilon > 0$

② Whenever $|x - x_0| < \delta_\varepsilon$ and $x, x_0 \in [a,b]$ — ② by given (i) and (ii)

Case I $x < x_0$
Case II $x > x_0$
classwork

Case I: we have f cont'n on $[x, x_0]$ and diff'ble on (x, x_0)

③ $\exists c \in (x, x_0)$ s.t. $f'(c) = \frac{f(x_0) - f(x)}{x_0 - x}$ — ③ Mean Value Thm

④ $|f'(c)| = \left|\frac{f(x_0) - f(x)}{x_0 - x}\right| = \frac{|f(x_0) - f(x)|}{|x_0 - x|}$ — ④ $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$

⑤ $|x_0 - x| \cdot |f'(c)| = |f(x_0) - f(x)|$ — ⑤ mult both sides by $|x_0 - x|$

⑥ $|f(x_0) - f(x)| = |x_0 - x||f'(c)| \leq |x_0 - x| \cdot M$ — ⑥ Given (iii)

⑦ $|f(x_0) - f(x)| < \delta_\varepsilon \cdot M$ — ⑦ by step 2 mult by $c = M > 0$ on both sides

⑧ $|f(x_0) - f(x)| < \frac{\varepsilon}{\cancel{M}} \cdot \cancel{M} = \varepsilon$ — ⑧ by choice of δ_ε in step 1.

(final) $|f(x) - f(x_0)| < \varepsilon$. — (final) $|a - b| = |b - a|$

Case II is classwork
$x > x_0$ interval (x_0, x)

Application:

Very Important!

Theorem: If $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b) and

$\forall x\in(a,b)\quad |f'(x)|\le M$

then $\forall \varepsilon>0\ \exists\, \delta_\varepsilon=\frac{\varepsilon}{M}>0$ s.t. $|x-x_0|<\delta_\varepsilon$, $x,x_0\in[a,b]$ $\Rightarrow |f(x)-f(x_0)|<\varepsilon$

Classwork:

Prove $f_j(x)=\frac{5jx}{j+1}$ are equicontinuous on $[a,b]$

$f_j'(x)=\frac{5j}{j+1}$ $\qquad |f_j'(x)|=\underbrace{\frac{5j}{j+1}\le 5}_{\text{jusify with an upper bound proof}}\quad \forall j\in\mathbb{N}$

$M=5$

f_j contin on $[a,b]$
diffble on (a,b)

So $\quad \delta_\varepsilon=\frac{\varepsilon}{M}>0$ So equicontinuous

Classwork

Prove $f_j(x) = \dfrac{(j+3)\,x^2}{j} + j$ is equicontinuous on $[a,b]$

Pause + Try

f_j is continuous on $[a,b]$

diffble on (a,b)

$$f_j'(x) = \left(\frac{j+3}{j}\right)2x + 0$$

$$|f_j'(x)| = \left|\left(\frac{j+3}{j}\right)2x\right|$$

$$= \left|\left(\frac{j}{j} + \frac{3}{j}\right)\right| \cdot |2x|$$

$$\leq |(1+3)| \cdot |2| \cdot |x|$$

$\leq 4 \cdot 2|x|$

$\leq 4 \cdot 2 \max\{|a|,|b|\}$ $=M$

because $a \leq x \leq b$ by Useful Lemma

So done.

~~Useful~~

Lemma: $a \leq x \leq b$

$\Longrightarrow |x| \leq \max\{|a|,|b|\}$

Proof:

Case I $x \geq 0$

Classwork

Case II $x < 0$

$\boxed{\text{HW4}}$ Prove $f_j(x) = \dfrac{(jx+1)}{(jx-1)}$

are equicontinuous on $[4,8]$

using the method above where $M = \max_{x \in [4,8]} |f_j'(x)|$

and so $|f_j(x) - f_j(a)| \leq M|x-a|$ so $\delta_\varepsilon = \frac{\varepsilon}{M}$.

Part 5: Concavity

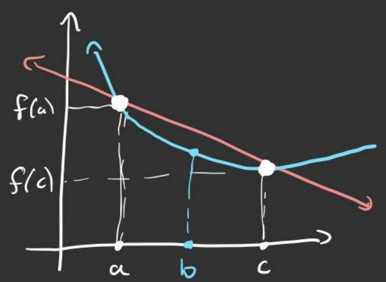
Watch Video MVT11

Concavity

concave up

Defn: A function is convex

if $a < b < c \implies f(b) \le \underbrace{\left(\frac{f(c)-f(a)}{c-a}\right)(b-a) + f(a)}_{\text{a line through } (a, f(a)) \text{ and } (c, f(c))}$



Defn: f is "concave down" if $-f$ is concave up

Thm: If f is twice differentiable on an interval and $f''(x) \ge 0$ on the interval then f is convex on the interval.

Proof: (Fill in Justifications for HW)

1. f is continuous on $[a,c]$ and f is diffble on (a,c)
2. $\exists p \in (a,b)$ s.t. $\frac{f(b)-f(a)}{b-a} = f'(p)$

1. g diffble $\Rightarrow$ g contin applied to $g = f'$
2.

(3) $\exists\, q \in (b, c)$ s.t. $\dfrac{f(c) - f(b)}{c - b} = f'(q)$ (3)

(4) $g(x) = f'(x)$ is increasing (4)

(5) $a < p < b < q < c$ (5)

(6) $f'(p) < f'(q)$ (6)

(7) $\dfrac{f(b) - f(a)}{b - a} < \dfrac{f(c) - f(b)}{c - b}$ (7)

(8) $f(c) - f(a) = f(c) - f(b) + f(b) - f(a)$ (8)

(9) $= \dfrac{f(c) - f(b)}{c - b}(c - b) + \dfrac{f(b) - f(a)}{b - a}(b - a)$ (9)

(10) $> \dfrac{f(b) - f(a)}{b - a}(c - b) + \dfrac{f(b) - f(a)}{b - a}(b - a)$ (10)

(11) $= \dfrac{f(b) - f(a)}{b - a}(c - b + b - a)$ (11)

(12) $= \dfrac{f(b) - f(a)}{b - a}(c - a)$ (12)

(13) $\dfrac{f(c) - f(a)}{c - a} > \dfrac{f(b) - f(a)}{b - a}$ (13)

(14) $\dfrac{f(c) - f(a)}{c - a}(b - a) > f(b) - f(a)$ (14)

(15) $\dfrac{f(c) - f(a)}{c - a}(b - a) + f(a) > f(b)$ (15)

QED

HW5 is to fill in the justifications above.