Linear Algebra MAT313 Spring 2024
Professor Sormani
Lesson 5 Reduced Echelon Form and Homogeneous Linear Systems

Warning: do not start this lesson until you have completed Lesson 4, checked the feedback on your proof in Lesson 4, fixed it if necessary, and made a request to join your team's project document here.

Please be sure to mark down the date and time that you start this lesson. Carefully take notes on pencil and paper while watching the lesson videos. Pause the lesson to try classwork before watching the video going over that classwork. If you work with any classmates, be sure to write their names on the problems you completed together. Please wear masks when meeting with classmates even if you meet off campus.

You will cut and paste the photos of your notes and completed classwork and a selfie taken holding up the first page of your work in a googledoc entitled:

MAT313S24-lesson5-lastname-firstname

and share editing of that document with me sormanic@gmail.com. You will also include your homework and any corrections to your homework in this doc.

If you have a question, type QUESTION in your googledoc next to the point in your notes that has a question and email me with the subject MAT313 QUESTION. I will answer your question by inserting a photo into your googledoc or making an extra video.

This lesson has two parts:

Part 1: Reduced Echelon Form

There is a row reduction error in one of the classwork problems. If you catch it, mark down that you found the error. The final answer is still correct.

Watch Playlist 313S23-L5-P1.

Solving Linear Systems using Reduced Echelon Form

Given a linear system

- Convert to an Augmented Matrix
- Do Row Reduction using Reversible Row Actions following our algorithm to Echelon Form
- Convert back to a system
- Solve for leaders
- Sub up ← Very Difficult for a large system
- Write the solution set

Given a linear system

- Convert to an Augmented Matrix ✓
- Do Row Reduction using Reversible Row Actions following our algorithm to Echelon Form ✓
- Continue Row Reduction to Reduced Echelon Form
- Convert back to a system
- Solve for leaders (no sub up will be needed!)
- Write the solution set

Today we learn this new step!

Classwork:

① $\boxed{\begin{aligned} 4x+8y+4z&=16\\ x+2y+z&=4\\ x+y+z&=3 \end{aligned}}$

② $\boxed{\begin{aligned} 2x+2y&=4\\ x+3y&=1\\ 3x+5y&=6 \end{aligned}}$

③ $\boxed{\begin{aligned} x+y+4z&=12\\ x+2y+4z&=12 \end{aligned}}$

The solution of one of these is a line. Find position and direction for it.

First practice → Augmented Matrix

① Linear System

$\boxed{\begin{aligned} 4x+8y+4z&=16\\ 1x+2y+1z&=4\\ 1x+1y+1z&=3 \end{aligned}}$

Notice we put 1 in front of variables with no coefficient:

$x = 1x$ and so on

Augmented Matrix Form

$$\left[\begin{array}{ccc|c} 4 & 8 & 4 & 16\\ 1 & 2 & 1 & 4\\ 1 & 1 & 1 & 3 \end{array}\right]$$

open bracket for a matrix

represents the equals signs

close bracket

② Linear System

$\boxed{\begin{aligned} 2x+2y&=4\\ 1x+3y&=1\\ 3x+5y&=6 \end{aligned}}$

Augmented Matrix Form

$$\left[\begin{array}{cc|c} 2 & 2 & 4\\ 1 & 3 & 1\\ 3 & 5 & 6 \end{array}\right]$$

③ Linear System

$\boxed{\begin{aligned} 1x+1y+4z&=12\\ 1x+2y+4z&=12 \end{aligned}}$

Augmented Matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 4 & 12\\ 1 & 2 & 4 & 12 \end{array}\right]$$

don't forget $x = 1x$

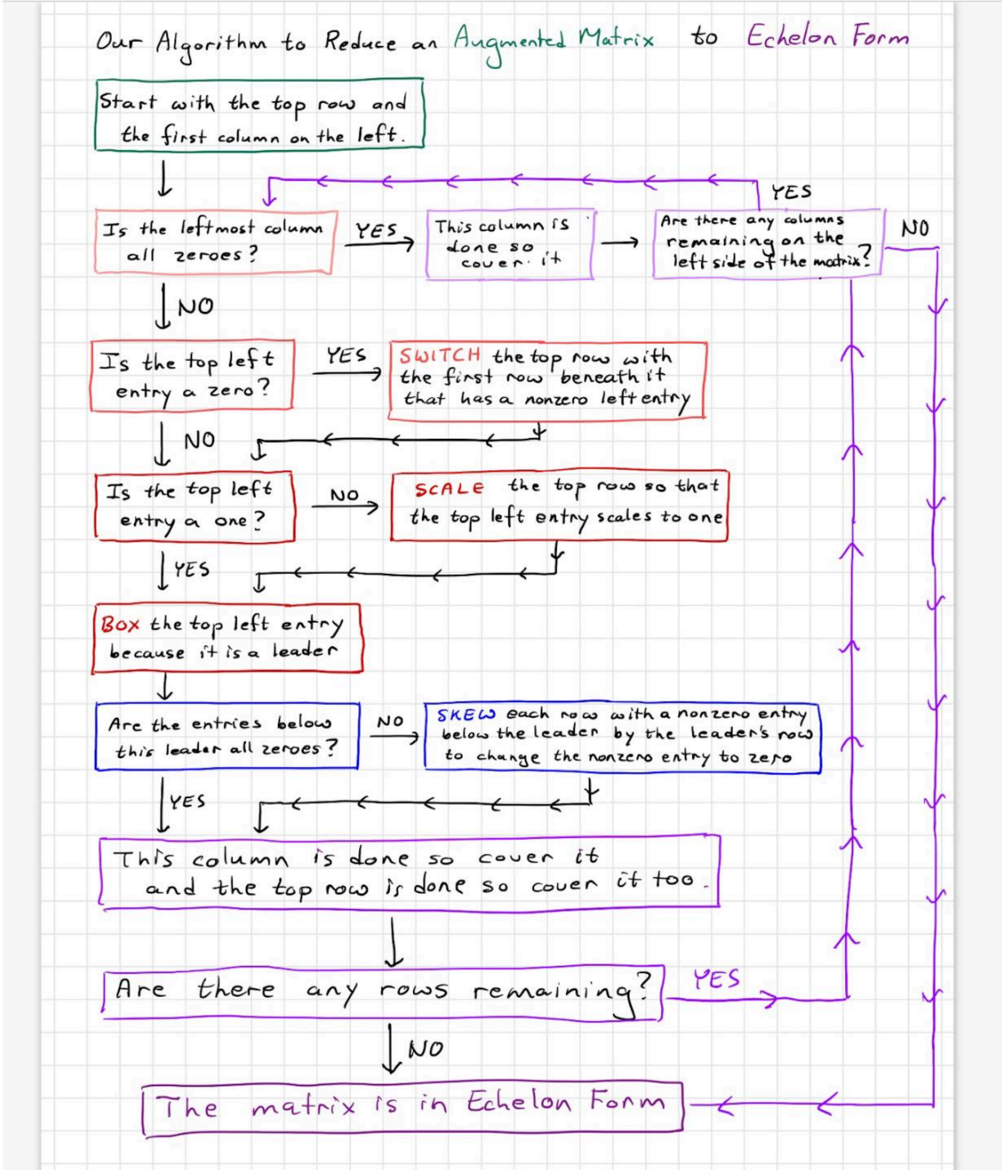
④ $\boxed{\begin{aligned} x+z&=10\\ y+z&=8 \end{aligned}}$

fill in missing variables with coefficient zero

$\boxed{\begin{aligned} 1x+0y+1z&=10\\ 0x+1y+1z&=8 \end{aligned}}$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 10\\ 0 & 1 & 1 & 8 \end{array}\right]$$

A quick glance at our algorithm:



Continuing with Classwork 1:

$$\begin{aligned} \boxed{1x} + 2y + 1z &= 4 \\ 0x + \boxed{1y} + 0z &= 1 \\ 0x + 0y + 0z &= 0 \end{aligned} \longrightarrow \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 4 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

In Echelon Form!

If we do sub up from Echelon Form: Solve for leaders

$\boxed{1x} + 2y + 1z = 4$ Solve $x = 4 - 2y - 1z = 4 - 2\cdot(1) - z = 2 - z$ ← free

$\boxed{1y} + 0z = 1$ $y = 1$

$0 = 0$ (free $z = z$)

this sub up can take awhile if we have many rows and complicated formulas may need to be sub in.

Why do we have to sub up?

because some rows in Echelon Form have more than one leader.

Can we continue row actions, past Echelon Form, to avoid this?

the reason leader y is in the first row is **because of the 2 above the boxed $\boxed{1}$ for y.**

So remove entries above boxed leaders!

It would be nice to avoid the subup of the previous leaders

We would prefer not to have any leaders in our formulas for other leaders.

Continue Past Echelon Form to **Reduced Echelon Form**

using more row actions (which can be coded)

Linear Algebra 0 | Vector Calculus 226F21

* Reduced Echelon Form *

$\boxed{1x} + 2y + 1z = 4$
$0x + \boxed{1y} + 0z = 1$
$0x + 0y + 0z = 0$

Our Echelon Form
$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 4 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

avoid previous leaders in the other leader's row

continue row reduction so that we have zeroes above each leader starting with the bottom leader

use skew actions to get zeroes above it.

to remove $2y$ from row 1

$\rho_1 \to \rho_1 - 2\rho_2$

$$\rho_1 \to \rho_1 - 2\rho_2$$

$\boxed{1x} + 0y + 1z = 2$
$0x + \boxed{1y} + 0z = 1$
$0x + 0y + 0z = 0$

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 2 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Zeroes above all leaders so this is reduced echelon form!

Now we solve for leaders & we are done

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2-z \\ 1 \\ z \end{pmatrix} : z \in \mathbb{R} \right\}$$

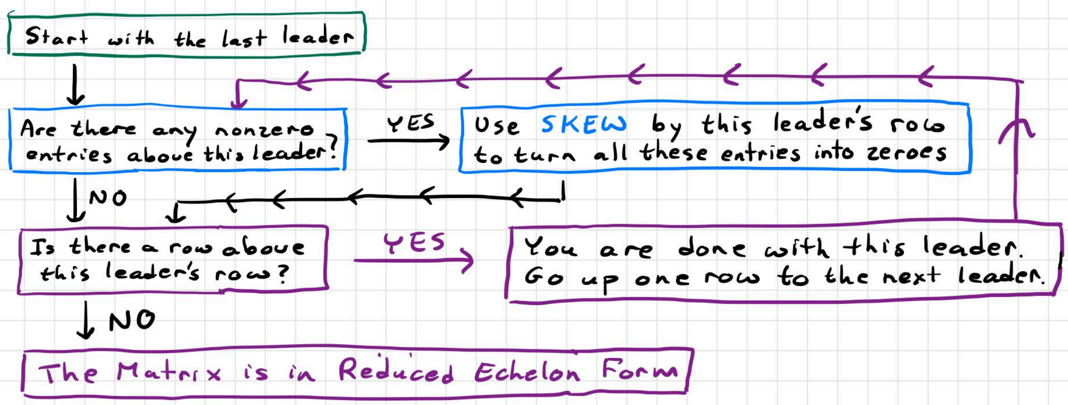
because solve each leader

$x = 2 - z$ (no y to sub)
$y = 1$
$0 = 0$

free: $z = z$

Defn: A matrix is in Reduced Echelon Form if it is in Echelon Form and only has zeroes above the leaders.

An algorithm to go from an Echelon Form to Reduced Echelon Form



Always go to Echelon Form first in this course.

Classwork 2:

(2) Linear System

Solve it using Augmented Matrices and Reduced Echelon Form

$$\boxed{\begin{aligned} 2x + 2y &= 4 \\ 1x + 3y &= 1 \\ 3x + 5y &= 6 \end{aligned}}$$

convert to an Augmented Matrix

$$\left[\begin{array}{cc|c} \boxed{2} & 2 & 4 \\ 1 & 3 & 1 \\ 3 & 5 & 6 \end{array}\right]$$

next do row reduction to Echelon Form using only this

change 1st leader into a 1 using scale (or switch if zero)

$$\xrightarrow{\rho_1 \to \frac{1}{2}\rho_1} \left[\begin{array}{cc|c} \boxed{1} & 1 & 2 \\ 1 & 3 & 1 \\ 3 & 5 & 6 \end{array}\right]$$

all entries in row 1 are divided by 2

rest are copies

box the leader

make all entries below the leader into zeroes using skew $\rho_i \to \rho_i - k\rho_1$ because leader in row 1

$$\xrightarrow[\rho_3 \to \rho_3 - 3\rho_1]{\rho_2 \to \rho_2 - \rho_1} \left[\begin{array}{cc|c} \boxed{1} & 1 & 2 \\ 0 & \boxed{2} & -1 \\ 0 & 2 & 0 \end{array}\right]$$

copy row 1

$3-3(1)$ $\quad 5-3(1)$ $\quad 6-3(2)$

box the next leader

Change the second leader into a 1 by scaling (or switch if needed)

$$\xrightarrow{\rho_2 \to \frac{1}{2}\rho_2} \left[\begin{array}{cc|c} \boxed{1} & 1 & 2 \\ 0 & \boxed{1} & -1/2 \\ 0 & 2 & 0 \end{array}\right]$$

Make all entries below the leader into zeroes using skew $\rho_i \to \rho_i - k\rho_2$ because leader is in row 2.

$$\xrightarrow{\rho_3 \to \rho_3 - 2\rho_2} \left[\begin{array}{cc|c} \boxed{1} & 1 & 2 \\ 0 & \boxed{1} & -1/2 \\ 0 & 0 & 1 \end{array}\right]$$

$0 - 2\left(\frac{-1}{2}\right) = 1$

Echelon Form

Continue to Reduced Echelon Form

$2 - \left(\frac{-1}{2}\right) = \frac{4}{2} + \frac{1}{2} = \frac{5}{2}$

Make all entries above the last leader into zeroes using skew $\rho_i \to \rho_i - k\rho_2$ because last leader is in row 2

$$\xrightarrow{\rho_1 \to \rho_1 - \rho_2} \left[\begin{array}{cc|c} \boxed{1} & 0 & 5/2 \\ 0 & \boxed{1} & -1/2 \\ 0 & 0 & 1 \end{array}\right]$$

Reduced Echelon Form

Rewrite as a linear system

$$1x + 0y = 5/2$$
$$0x + 1y = -1/2$$
$$0x + 0y = 1$$

Solve for leaders $x = \frac{5}{2}$

$y = -1/2$

that's all our variables (no free variables)

But wait! Final Line is $\boxed{0=1}$

No solution

$\varnothing$

can notice this earlier and so no solution sooner if you wish.

See $\left[\begin{array}{cc|c} \sim & \sim & - \\ \sim & \sim & - \\ 0 & 0 & 1 \end{array}\right]$

at any time during row reduction and there is no solution!

③ Linear System

$$\boxed{\begin{aligned} x + y + 4z &= 12 \\ x + 2y + 4z &= 12 \end{aligned}}$$

Solve using Augmented Matrices and Reduced Echelon Form

Convert to Augmented Matrix:

$$\boxed{\begin{aligned} 1x + 1y + 4z &= 12 \\ 1x + 2y + 4z &= 12 \end{aligned}} \qquad \left[\begin{array}{ccc|c} \boxed{1} & 1 & 4 & 12 \\ 1 & 2 & 4 & 12 \end{array}\right]$$

Row reduction to Echelon Form

1^{st} leader to a 1 ✓

zeroes under first leader using skew $\rho_2 \to \rho_2 + k\rho_1$

$$\xrightarrow{\rho_2 \to \rho_2 - \rho_1} \left[\begin{array}{ccc|c} \boxed{1} & 1 & 4 & 12 \\ 0 & \boxed{1} & 0 & 0 \end{array}\right]$$

2^{nd} leader to a 1 ✓

nothing under 2^{nd} leader

already Echelon Form

Row Reduction to Reduced Echelon Form

zeroes above last leader $\boxed{1}$ using skew $\rho_i \to \rho_i + k\rho_2$

$$\xrightarrow{\rho_1 \to \rho_1 - \rho_2} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 4 & 12 \\ 0 & \boxed{1} & 0 & 0 \end{array}\right]$$

This is reduced Echelon Form

Change back into a linear system

$$\boxed{1x} + 0y + 4z = 12$$
$$0x + \boxed{1y} + 0z = 0$$

leaders: x, y
free $z = z$

Solve for leaders $x = 12 - 4z$
$y = 0$
$z = z$ (free)

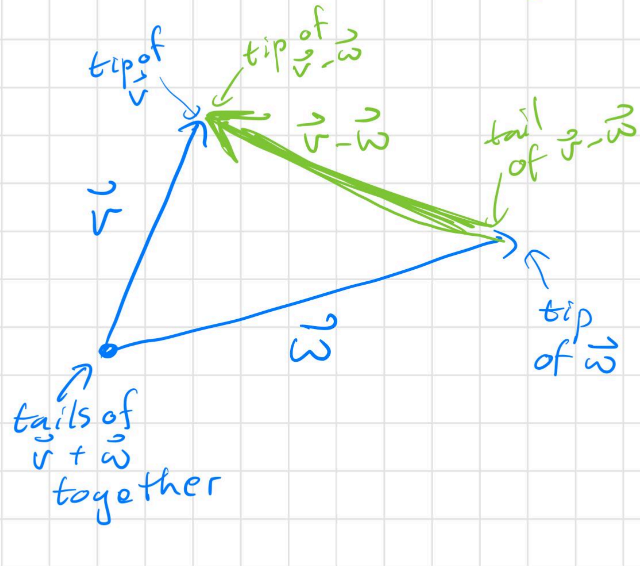
$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12-4z \\ 0 \\ z \end{pmatrix} : z \in \mathbb{R} \right\}$$

Check: $x + y + 4z = 12$
$(12-4z) + 0 + 4z = 12$ ✓

original system

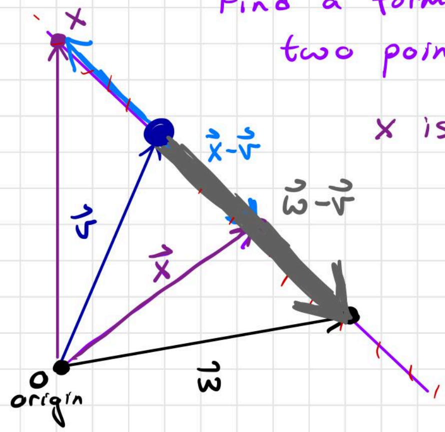
$x + 2y + 4z = 12$
$(12-4z) + 2 \cdot 0 + 4z = 12$ ✓

So next we want to write this in line form.



Lines written in Vector Notation

Find a formula for a line through two points $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix}$



x is a typical point on the line

$$\vec{x} - \vec{v} = t\,(\vec{w} - \vec{v})$$

$$\left\{ \vec{x} = \vec{v} + t\,\underbrace{(\vec{w} - \vec{v})} \;\middle|\; t \in \mathbb{R} \right\}$$

position vector $\vec{v}$ on the line

direction vector $(\vec{w} - \vec{v})$ where $\vec{w}$ is also on the line

as we change the value of t we get different points $\vec{x}$ lying on the line

$$\left\{ \begin{pmatrix} x\\ y\\ z\end{pmatrix} = \begin{pmatrix} 12-4z\\ 0\\ z\end{pmatrix} \,\middle|\, z\in\mathbb{R} \right\}$$

So next we want to write this in line form $\left\{ \vec{x} = \vec{v} + t\,\underbrace{(\vec{w}-\vec{v})} \,\middle|\, t\in\mathbb{R} \right\}$

t is the free variable $t\in\mathbb{R}$

position vector $\vec{v}$ on the line

direction vector $(\vec{w}-\vec{v})$ where $\vec{w}$ is also on the line

Our solution set $z\in\mathbb{R}$ so $z=t\in\mathbb{R}$

position is the constant terms without the free variable $t=z$

$$\left\{ \begin{pmatrix} x\\ y\\ z\end{pmatrix} = \begin{pmatrix} 12-4z\\ 0+0z\\ 0+1z\end{pmatrix} \,\middle|\, z\in\mathbb{R} \right\}$$

$\vec{x}$ $\qquad \vec{v} = \begin{pmatrix} 12\\ 0\\ 0\end{pmatrix}$ $\qquad$ direction $(\vec{w}-\vec{v}) = \begin{pmatrix} -4\\ 0\\ 1\end{pmatrix}$

The solution in line form:

$$\left\{ \vec{x} = \begin{pmatrix} 12\\ 0\\ 0\end{pmatrix} + z\begin{pmatrix} -4\\ 0\\ 1\end{pmatrix} \,\middle|\, z\in\mathbb{R} \right\}$$

Try classwork 1 solution in line form.

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2-z \\ 1 \\ z \end{pmatrix} : \underline{z \in \mathbb{R}} \right\}$$

Classwork ①

Convert to line form

$$\left\{ \vec{x} = \vec{v} + t(\vec{w} - \vec{v}) : t \in \mathbb{R} \right\}$$

What is our free variable?

In this case it is $z \in \mathbb{R}$

(we have only one free variable) ✓

(Only a line if we have a single free variable)

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2-z \\ 1 \\ z \end{pmatrix} = \begin{pmatrix} 2-1z \\ 1+0z \\ 0+1z \end{pmatrix} : \underline{z \in \mathbb{R}} \right\}$$

position $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ direction $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} : z \in \mathbb{R} \right\}$$ ← line form

student who took vector calc can graph this.

(4) $\boxed{\begin{aligned} x + z &= 10 \\ y + z &= 8 \end{aligned}}$ → to convert to an augmented matrix add in missing variables with zero coefficients

$$\boxed{\begin{aligned} 1x + 0y + z &= 10 \\ 0x + 1y + 1z &= 8 \end{aligned}}$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 10 \\ 0 & \boxed{1} & 1 & 8 \end{array}\right]$$

Already in Reduced Echelon Form

two leaders with coeff = 1 already and zeroes below and above.

Convert back to a system:

$$\boxed{\begin{aligned} \boxed{1x} + 0y + z &= 10 \\ 0x + \boxed{1y} + 1z &= 8 \end{aligned}}$$

leaders: x and y

free: z

Solve for leaders

$x = 10 - z$

$y = 8 - z$

$z = z$ (free)

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 - z \\ 8 - z \\ z \end{pmatrix} = \begin{pmatrix} 10 - 1z \\ 8 - 1z \\ 0 + 1z \end{pmatrix} \;\middle|\; z \in \mathbb{R} \right\}$$

only one free so we can write it as a line

position $\begin{pmatrix} 10 \\ 8 \\ 0 \end{pmatrix}$ direction $\begin{pmatrix} -1 \\ -1 \\ +1 \end{pmatrix}$

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 8 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ -1 \\ +1 \end{pmatrix} \;\middle|\; z \in \mathbb{R} \right\}$$

This next classwork has a deliberate error in the row reduction but still has the correct Echelon Form. Find the error and explain it in your classwork, redo it correctly, and email me for Extra Credit.

Classwork: Convert to an Augmented Matrix
Do Row Reduction to Reduced Echelon Form
Solve for leader and write Solution Set
If there is one free variable write the solution set as a line with position and direction

Six Variables! $x_1\ x_2\ x_3\ x_4\ x_5\ x_6$

$$\boxed{\begin{aligned} 2x_1 + 4x_2 + 6x_6 &= 0 \\ 1x_1 + 2x_2 + 1x_4 &= 0 \\ 4x_1 + 8x_2 + 10x_6 &= 0 \\ 2x_1 + 4x_2 + 2x_5 &= 0 \\ 2x_4 + x_5 &= 0 \\ x_4 + x_5 + 2x_6 &= 0 \end{aligned}}$$

$2x_1 + 4x_2 + 0x_3 + 0x_4 + 0x_5 + 6x_6 = 0$

rewrite those as well with the zeroes
Then write the Augmented Matrix

pause
& do this.

$$\begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\ \end{array}$$

← Our variables

$$\left[\begin{array}{cccccc|c} \boxed{2} & 4 & 0 & 0 & 0 & 6 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 4 & 8 & 0 & 0 & 0 & 10 & 0 \\ 2 & 4 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 & 0 \end{array}\right]$$

pause + try

← final column is all 0 in a homogeneous system

Box the first leader
turn it into a 1

$$\xrightarrow{\rho_1 \to \frac{1}{2}\rho_1}$$

$$\left[\begin{array}{cccccc|c} \boxed{1} & 2 & 0 & 0 & 0 & 3 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 4 & 8 & 0 & 0 & 0 & 10 & 0 \\ 2 & 4 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 & 0 \end{array}\right]$$

$10-4\cdot 3=10-12=-2$

$10-2\cdot 3=10-6=4$

must put zeroes under the leader
using skew by leader's row $\rho_i \to \rho_i + k\rho_1$

$\rho_2 \to \rho_2 - \rho_1$
$\rho_3 \to \rho_3 - 4\rho_1$
$\rho_4 \to \rho_4 - 2\rho_1$
last two are 0

$$\longrightarrow \left[\begin{array}{cccccc|c} \boxed{1} & 2 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 & 0 \end{array}\right]$$

Find 2nd leader: none in the second column
(2nd variable is free!)
none in third column
(3rd variable is free!)
the leader in the fourth column is a 1
so we do not need to scale
and already in row 2 so no switch needed

Next make 0's under the leader
skew by leaders row ρ_2

$\rho_5 \to \rho_5 - 2\rho_2$
$\rho_6 \to \rho_6 - \rho_2$

$$\longrightarrow \left[\begin{array}{cccccc|c} \boxed{1} & 2 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 0 \end{array}\right]$$

$\rho_4 \to \frac{-1}{2}\rho_4$

$$\left[\begin{array}{cccccc|c} \boxed{1} & 2 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

change to 0 skew by ρ_4

$\rho_5 \to \rho_5 - 4\rho_4$

$\rho_6 \to \rho_5 + \rho_4$

$$\left[\begin{array}{cccccc|c} \boxed{1} & 2 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Echelon Form!

Next want Reduced Echelon Form with zeroes above the leaders starting with bottom leader

skew by leaders row ρ_4

$\rho_1 \to \rho_1 - 3\rho_4$

$\rho_2 \to \rho_2 + 3\rho_4$

$\rho_3 \to \rho_3 - 6\rho_4$

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \left[\begin{array}{cccccc|c} \boxed{1} & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Check third leader ✓ second leader ✓

Reduced Echelon Form.

$\boxed{1 x_1} + 2x_2 = 0$

$\boxed{1 x_4} = 0$

$\boxed{1 x_5} = 0$

$\boxed{1 x_6} = 0$

solve for leaders

$x_1 = -2x_2$

$x_4 = 0$

$x_5 = 0$

$x_6 = 0$

free

$x_2 = x_2$

$x_3 = x_3$

These columns have no leaders, so they are free

Form.

Write as a Linear System

$$\boxed{1x_1} + 2x_2 = 0 \qquad x_1 = -2x_2$$
$$\boxed{1x_4} = 0 \qquad x_4 = 0$$
$$\boxed{1x_5} = 0 \qquad x_5 = 0$$
$$\boxed{1x_6} = 0 \qquad x_6 = 0$$

solve for leaders

free variables $x_2 = x_2$, $x_3 = x_3$

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -2x_2 \\ x_2 \\ x_3 \\ 0 \\ 0 \\ 0 \end{pmatrix} : x_2, x_3 \in \mathbb{R} \right\}$$

two free variables!

cannot write in line form!

write each entry with a constant term that will be for the position and a term for each free variable color them differently

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 - 2x_2 + 0x_3 \\ 0 + 1x_2 + 0x_3 \\ 0 + 0x_2 + 1x_3 \\ 0 + 0x_2 + 0x_3 \\ 0 + 0x_2 + 0x_3 \\ 0 + 0x_2 + 0x_3 \end{pmatrix} : x_2, x_3 \in \mathbb{R} \right\}$$

position with two directions

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} ; x_2, x_3 \in \mathbb{R} \right\}$$

we get a direction for each free variable!

Students' Questions on Part I:

Why go to Echelon Form before starting to get zeroes above leaders?
Because many problems only need Echelon Form. So extra row actions are unnecessary. In particular you can stop at Echelon Form if there is no solution.
What Problems can we solve with only Echelon Form?
These kinds: Is the solution set empty?
Is the solution a single point? (all variables are leaders)
Is the solution set a line? (one variable is free)
Other applications to come later in the course.
Why go to Reduced Echelon Form?
To more quickly find the exact solution set.

Other applications to come later in the course.

When is it in Reduced Echelon Form?

When every leader only has zeroes above it, zeroes below it and zeroes on its left and each leader is below and to the right of the previous leader.

How do I find the position and directions?

The direction for each free variable comes from the coefficients of that free variable (see green in last example above) and the position comes from the constant terms:

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -2x_2 \\ x_2 \\ x_3 \\ 0 \\ 0 \\ 5 \end{pmatrix} : x_2, x_3 \in \mathbb{R} \right\}$$

two free variables!

cannot write in line form!

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 - 2x_2 + 0x_3 \\ 0 + 1x_2 + 0x_3 \\ 0 + 0x_2 + 1x_3 \\ 0 + 0x_2 + 0x_3 \\ 0 + 0x_2 + 0x_3 \\ 5 + 0x_2 + 0x_3 \end{pmatrix} : x_2, x_3 \in \mathbb{R} \right\}$$

directions are same

position with two directions

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 5 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} ; x_2, x_3 \in \mathbb{R} \right\}$$

new position

How do I know I found the right position and directions?

A simple check is as follows:

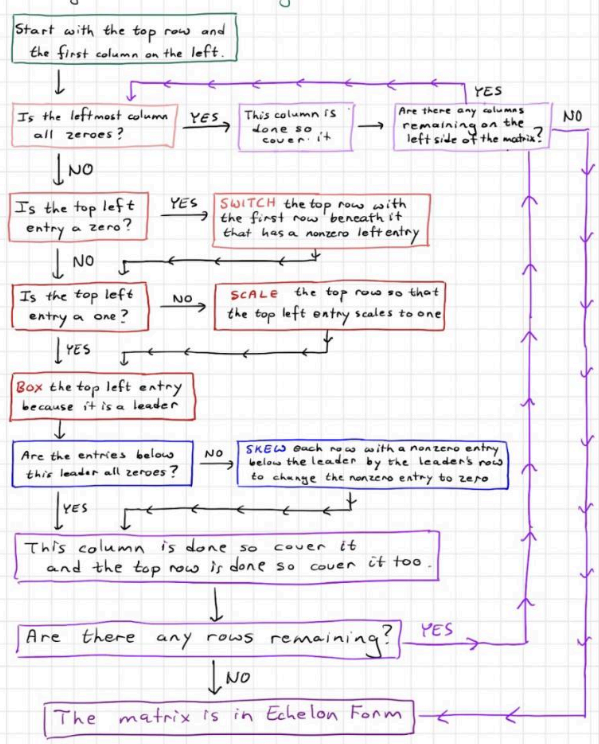
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 5 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 5 \end{pmatrix} + \begin{pmatrix} -2x_2 \\ 1x_2 \\ 0x_2 \\ 0x_2 \\ 0x_2 \\ 0x_2 \end{pmatrix} + \begin{pmatrix} 0x_3 \\ 0x_3 \\ 1x_3 \\ 0x_3 \\ 0x_3 \\ 0x_3 \end{pmatrix} = \begin{pmatrix} 0 - 2x_2 + 0x_3 \\ 0 + 1x_2 + 0x_3 \\ 0 + 0x_2 + 1x_3 \\ 0 + 0x_2 + 0x_3 \\ 0 + 0x_2 + 0x_3 \\ 5 + 0x_2 + 0x_3 \end{pmatrix} = \begin{pmatrix} -2x_2 \\ x_2 \\ x_3 \\ 0 \\ 0 \\ 5 \end{pmatrix} \leftarrow$$

which is the original solution

We will learn another way to check it in the next part.

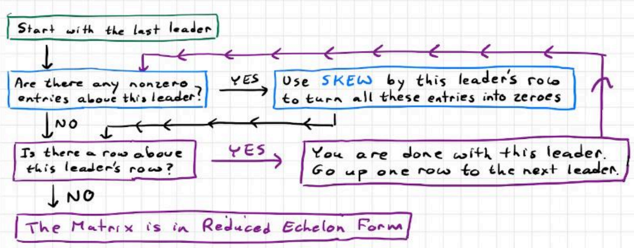
For the homework use our algorithms:

Our Algorithm to Reduce an Augmented Matrix to Echelon Form



Defn: A matrix is in Reduced Echelon Form if it is in Echelon Form and only has zeroes above the leaders.

An algorithm to go from an Echelon Form to Reduced Echelon Form



Always go to Echelon Form first in this course.

If you wish you can do the following homework before going to Part 2.

$\boxed{\text{HW1}}$ Solve

$$\boxed{\begin{aligned} x_1 + x_8 &= 2 \\ x_3 + x_5 &= 4 \\ x_2 + x_7 &= 6 \\ x_4 + x_6 &= 8 \end{aligned}}$$

$\boxed{\text{HW2}}$ Solve using work from HW1 and show it has the same direction vectors as HW1.

$$\boxed{\begin{aligned} x_1 + x_8 &= 0 \\ x_3 + x_5 &= 0 \\ x_2 + x_7 &= 0 \\ x_4 + x_6 &= 0 \end{aligned}}$$

Also be sure to find the error in the last classwork to get extra credit!

Part II: Homogeneous Linear Systems

Watch the Playlist 313S23-L5-P2.

Homogeneous Linear Systems

$$\sum_{j=1}^{m} a_{ij} x_j = 0 \quad \text{for } i=1 \text{ to } n$$

$$\boxed{\begin{aligned} a_{11}x_1 + \cdots &+ a_{1m}x_m = 0 \\ a_{21}x_1 + \cdots &+ a_{2m}x_m = 0 \\ &\vdots \\ a_{n1}x_1 + \cdots &+ a_{nm}x_m = 0 \end{aligned}}$$

$d_i = 0$ for $i = 1$ to n

$$\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1m} & 0 \\ \vdots & & \vdots & 0 \\ & & & \vdots \\ a_{n1} & \cdots & a_{nm} & 0 \end{array}\right]$$

position is always 0, the directions are solutions.

Looking at an example we solved before:

Every linear system has a corresponding homogeneous system with the same directions!

(4) $\boxed{\begin{array}{l} x + z = 10 \\ y + z = 8 \end{array}}$ → to convert to an augmented matrix add in missing variables with zero coefficients

$$\boxed{\begin{array}{l} 1x + 0y + z = 10 \\ 0x + 1y + 1z = 8 \end{array}}$$ not homogeneous

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 10 \\ 0 & \boxed{1} & 1 & 8 \end{array}\right]$$ Already in Reduced Echelon Form

two leaders with coeff = 1 already and zeroes below and above.

Convert back to a system:

$$\begin{array}{l} \boxed{1x} + 0y + z = 10 \\ 0x + \boxed{1y} + 1z = 8 \end{array}$$ leaders: x and y; free: z

Solve for leaders $x = 10 - z$, $y = 8 - z$, $z = z$ (free)

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 - z \\ 8 - z \\ z \end{pmatrix} = \begin{pmatrix} 10 - 1z \\ 8 - 1z \\ 0 + 1z \end{pmatrix} \;\middle|\; z \in \mathbb{R} \right\}$$ only one free, so we can write it as a line

position $\begin{pmatrix} 10 \\ 8 \\ 0 \end{pmatrix}$ direction $\begin{pmatrix} -1 \\ -1 \\ +1 \end{pmatrix}$

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 8 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ -1 \\ +1 \end{pmatrix} \;\middle|\; z \in \mathbb{R} \right\}$$

position not 0

skew $0 + k0 = 0$
switch $0 \leftrightarrow 0$
scale $k0 = 0$

all row actions are the same on the left side of the matrix

(4) $\boxed{\begin{array}{l} x + z = 0 \\ y + z = 0 \end{array}}$ → to convert to an augmented matrix add in missing variables with zero coefficients

$$\boxed{\begin{array}{l} 1x + 0y + z = 0 \\ 0x + 1y + 1z = 0 \end{array}}$$ ← homogeneous

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 0 \\ 0 & \boxed{1} & 1 & 0 \end{array}\right]$$ Already in Reduced Echelon Form

two leaders with coeff = 1 already and zeroes below and above.

Convert back to a system:

$$\begin{array}{l} \boxed{1x} + 0y + z = 0 \\ 0x + \boxed{1y} + 1z = 0 \end{array}$$ leaders: x and y; free: z

Solve for leaders $x = 0 - z$, $y = 0 - z$, $z = z$ (free)

no constant terms

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 - z \\ 0 - z \\ z \end{pmatrix} = \begin{pmatrix} 0 - 1z \\ 0 - 1z \\ 0 + 1z \end{pmatrix} \;\middle|\; z \in \mathbb{R} \right\}$$ only one free, so we can write it as a line

position $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ direction $\begin{pmatrix} -1 \\ -1 \\ +1 \end{pmatrix}$

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ -1 \\ +1 \end{pmatrix} \;\middle|\; z \in \mathbb{R} \right\}$$

position is $\vec{0}$ ← same direction

For a homogeneous system, the position is always 0 so the directions are solutions.

Solution set
$$\left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} + x_m \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{pmatrix} \;\middle|\; x_m \in \mathbb{R} \right\}$$

Notice that if we take $x_3 = 1 \in \mathbb{R}$, the other free variables $= 0 \in \mathbb{R}$

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + 1 \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} + 0 \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{pmatrix} \in \{ \text{solution set} \}$$

$$= \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} \in \text{the solution set}$$

Thus, we can check the directions

Check $\begin{pmatrix} -1 \\ -1 \\ +1 \end{pmatrix}$ is a solution

of the corresponding homog system

$$\boxed{\begin{aligned} 1x + 0y + 1z &= 0 \\ 0x + 1y + 1z &= 0 \end{aligned}}$$

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ +1 \end{pmatrix} = 1 \cdot (-1) + 0\,(-1) + 1\,(+1) = -1 + 0 + 1 = 0 \checkmark$$

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ +1 \end{pmatrix} = 0\,(-1) + 1\,(-1) + 1\,(+1) = 0 - 1 + 1 = 0 \checkmark$$

Check directions using dot products.

To check the position

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 10 \\ 8 \\ 0 \end{pmatrix} = 1 \cdot 10 + 0 \cdot 8 + 1 \cdot 0 = 10 + 0 + 0 = 10$$

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 10 \\ 8 \\ 0 \end{pmatrix} = 0 \cdot 10 + 1 \cdot 8 + 1 \cdot 0 = 0 + 8 + 0 = 8$$

The position is a solution of the original system.

$$\boxed{\begin{aligned} 1x + 0y + 1z &= 10 \\ 0x + 1y + 1z &= 8 \end{aligned}}$$

To check positions take dot products with coeffients of rows and get d_i

To check directions take dot products with coeffients of rows and get 0.

So for now on check your solutions!

The following notes are explained in 313F21-5-9a 313F21-5-9b 313F21-5-9c:

Every Linear System: has a homogenous system:

$$\boxed{\begin{aligned} a_{11}x_1 + \dots + a_{1m}x_m &= d_1 \\ a_{21}x_1 + \dots + a_{2m}x_m &= d_2 \\ &\dots \\ a_{n1}x_1 + \dots + a_{nm}x_m &= d_n \end{aligned}} \qquad \boxed{\begin{aligned} a_{11}x_1 + \dots + a_{1m}x_m &= 0 \\ a_{21}x_1 + \dots + a_{2m}x_m &= 0 \\ &\dots \\ a_{n1}x_1 + \dots + a_{nm}x_m &= 0 \end{aligned}}$$

that has the same coefficients $a_{i,j}$.

When we solve them:

$$\left[\begin{array}{cccc|c} \boxed{a_{11}} & a_{12} & \cdots & a_{1m} & d_1 \\ a_{21} & a_{22} & \cdots & a_{2m} & d_2 \\ & & \cdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} & d_n \end{array}\right] \qquad \left[\begin{array}{cccc|c} \boxed{a_{11}} & a_{12} & \cdots & a_{1m} & 0 \\ a_{21} & a_{22} & \cdots & a_{2m} & 0 \\ & & \cdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} & 0 \end{array}\right]$$

Same row actions are determined by the coefficients

Reduced Echelon Form

$$\left[\begin{array}{c|c} E & \begin{matrix} \bullet \\ \bullet \\ \bullet \\ \bullet \end{matrix} \end{array}\right] \qquad \left[\begin{array}{c|c} E & \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right]$$

So when we solve for leaders

leader = position + (free)(direction) + ... + (free)(dir)

for the homogeneous system the position is zero

same free variables + directions

Sometimes the original system has no solution due to a row

$0 \quad 0 \quad 0 \quad 0 \mid \bullet$ ← nonzero entry here

but the homogeneous system always

$0 \quad 0 \quad 0 \quad 0 \mid 0$ has a solution

313F21-5-9b

Every Linear System: has a homogenous system:

$$\begin{array}{l} a_{11}x_1 + \dots + a_{1m}x_m = d_1 \\ a_{21}x_1 + \dots + a_{2m}x_m = d_2 \\ \dots \\ a_{n1}x_1 + \dots + a_{nm}x_m = d_n \end{array} \qquad \begin{array}{l} a_{11}x_1 + \dots + a_{1m}x_m = 0 \\ a_{21}x_1 + \dots + a_{2m}x_m = 0 \\ \dots \\ a_{n1}x_1 + \dots + a_{nm}x_m = 0 \end{array}$$

that has the same coefficients $a_{i,j}$.

$$\boxed{\sum_{j=1}^{m} a_{ij} x_j = d_i \text{ for } i = 1 \text{ to } n} \qquad \boxed{\sum_{j=1}^{m} a_{ij} x_j = 0 \text{ for } i = 1 \text{ to } n}$$

$$\boxed{row_i \cdot \vec{x} = d_i \text{ for } i = 1 \text{ to } n} \qquad \boxed{row_i \cdot \vec{x} = 0}$$

Remember $row_i = \rho_i = (a_{i,1}, \dots, a_{i,m})$

Theorem: If $\vec{p}$ is a particular solution to the linear system and if $\vec{y}$ is a solution to the homogeneous system with the same coefficients then $\vec{p} + \vec{y}$ is a solution to the original linear system.

Given: $\vec{p}$ is a soln to $\sum_{j=1}^{m} a_{ij} x_j = d_i$

$\vec{y}$ is a soln to $\sum_{j=1}^{m} a_{ij} x_j = 0$

Show: $\vec{x} = \vec{p} + \vec{y}$ is a soln to $\sum_{j=1}^{m} a_{ij} x_j = d_i$

313F21-5-9c

Rewrote this with formula

Given: $\sum_{j=1}^{m} a_{ij} p_j = d_i$ **and** $\sum_{j=1}^{m} a_{ij} y_j = 0$

Show: $\underbrace{\sum_{j=1}^{m} a_{ij} (p_j + y_j)}_{LHS} = \underbrace{d_i}_{\text{RHS already simplified}}$

because $x_j = p_j + y_j$ by defn of vector addition

Proof:

① $LHS = \sum_{j=1}^{m} a_{ij}(p_j + y_j) = \sum_{j=1}^{m} (a_{ij} p_j + a_{ij} y_j)$ ① by distribution of reals

② $= \sum_{j=1}^{m} a_{ij} p_j + \sum_{j=1}^{m} a_{ij} y_j$ ② by reordering sums

③ $= d_i + 0$ ③ by given $\sum_{j=1}^{m} a_{ij} p_j = d_i$; by given $\sum_{j=1}^{m} a_{ij} y_j = 0$

④ $= d_i$ ↑ RHS ④ $x + 0 = x$ for any real number x.

QED

Note there is an error to find in the classwork in Part I. If you find the error, email me for extra credit.

Homework:

Complete HW1-HW4 following the full algorithm first to Echelon form and then to reduced Echelon form below. HW5 is almost the same as the last classwork.

HW1 Solve

$$\begin{cases} x_1 + x_8 = 2 \\ x_3 + x_5 = 4 \\ x_2 + x_7 = 6 \\ x_4 + x_6 = 8 \end{cases}$$

HW2 Solve

using work from HW1 and show it has the same direction vectors as HW1.

$$\begin{cases} x_1 + x_8 = 0 \\ x_3 + x_5 = 0 \\ x_2 + x_7 = 0 \\ x_4 + x_6 = 0 \end{cases}$$

Check your answers to HW1-HW2 using dot products with every row of the Augmented matrix you found.

HW3

Solve
$$\left[\begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\ 0 & 0 & 2 & 0 & 0 & 2 & 0 & 8 \end{array}\right]$$

HW4 Solve the homogeneous system corresponding to **HW3**.

Check your answers to HW3-HW4 using dot products with every row.

$\boxed{\text{HW5}}$ Prove that if $\vec{v}$ and $\vec{w}$ solve a homogeneous system then $\vec{z} = \vec{v} + \vec{w}$ solves the same system:

Hint

Rewrote this with formula

Given: $\sum_{j=1}^{m} a_{ij} v_j = 0$ and $\sum_{j=1}^{m} a_{ij} \cdot w_j = 0$

Show: $\underbrace{\sum_{j=1}^{m} a_{ij} (v_j + w_j)}_{\text{LHS}} = \underbrace{0}_{\text{RHS already simplified}}$

because $z_j = v_j + w_j$ by defn of vector addition

Hint: imitate final classwork.

Practice your row actions here in the Row Action Self Test.