The Law of Information Symmetric Compressibility

A Unified Principle of Information, Symmetry, and Compression

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Abstract

We derive and formalize a single conservation law governing the relationship between information content, symmetry, and compressibility across mathematics, computation, physics, and dynamics. We show that information is the logarithm of the orbit size under any group action, symmetry is the logarithm of the stabilizer size, and the two are complementary and conserved: their sum equals the logarithm of the group order. We prove this relationship is the **unique** information measure satisfying five natural axioms (orbit-dependence, null symmetry, monotonicity, additivity, and normalization), establish that compressibility scales monotonically with symmetry group size, and demonstrate that Boltzmann entropy, Noether's theorem, and Landauer's principle are all exact instantiations of this single law.

The law additionally provides a Shannon entropy recovery theorem (Shannon entropy equals I(d) exactly when the source distribution is uniform over the orbit) and exhibits structural analogy with Kolmogorov complexity (positive rank correlation, computable upper bound, but no exact equivalence owing to uncomputability). The extension to countably infinite discrete groups requires the Kolmogorov-Sinai entropy framework from ergodic theory.

Keywords: symmetry, information theory, compressibility, group actions, orbit-stabilizer theorem, Noether's theorem, Boltzmann entropy, Landauer principle, conservation laws, Shannon entropy, Kolmogorov-Sinai entropy

1. Introduction

1.1 Motivation

Across disparate scientific domains — information theory, statistical mechanics, quantum field theory, nonlinear dynamics, and machine learning — a single structural pattern appears: **information content is related inversely to symmetry; compressibility scales with the size of the symmetry group**. Yet this principle has been stated piecemeal in domain-specific languages, never consolidated as a unified law.

- In Kolmogorov complexity, random objects have trivial symmetry and are incompressible; symmetric objects have minimal description length. (Structural analogy — see §6.7 for precise scope.)
- In Shannon information theory, the capacity of a symmetric channel depends only on one row of its transition matrix; symmetry reduces the number of independent parameters.
- In statistical mechanics, equilibrium states have maximal microstate degeneracy (orbit size) under permutation symmetry, yet minimal macroscopic complexity.
- In Noether's theorem, every continuous symmetry of the action yields a conserved quantity — a reduction in active degrees of freedom.
- In nonlinear dynamics, transient chaotic saddles possess maximal Lyapunov entropy and minimal symmetry; attractors regain symmetry and achieve compression.

These are not analogies in all cases. This paper shows that Boltzmann entropy, Noether's theorem, and Landauer's principle are exact instances of a single mathematical object: **the orbit-stabilizer theorem of group theory, expressed as a conservation law for information and symmetry**.

1.2 Main Contributions

We formalize the **Law of Information Symmetric Compressibility (LISC)** and prove:

1. **Uniqueness theorem:** The function $I(d) = \log|Orb\_G(d)|$ is the unique information measure satisfying five natural axioms on a description system.
2. **Conservation law:** For any description d under group G, $I(d) + \Sigma(d) = \log|G|$, where $\Sigma(d) = \log|Stab\_G(d)|$.
3. **Compressibility theorem:** Compressibility $C(d) = \Sigma(d)/\log|G| \in [0,1]$ is the unique normalized measure of description compression, monotonic in symmetry group size.
4. **Shannon recovery theorem:** $H(X) \leq I(d)$ for any distribution X over Orb(d), with equality if and only if X is uniform over the orbit.
5. **Physical instantiation:** Boltzmann entropy, Landauer's erasure principle, and Noether conserved charges are three exact instances of the same conservation law at different physical scales.

1.3 Scope, Limitations, and Relation to Prior Work

Scope

- Applies to any finite group action on a finite set; extends to Lie groups via dimension in place of cardinality (§5).
- Does not propose an algorithm for finding orbit representatives or computing stabilizers (these remain domain-specific).
- Does not introduce new mathematics — the orbit-stabilizer theorem is 19th-century group theory — but identifies a previously unrecognized unified structure.
- Does not require adoption of the physics layer; the law is a pure mathematical theorem.

Infinite Discrete Groups

The framework as stated applies to finite groups and Lie groups. For countably infinite discrete groups (e.g., $\mathbb{Z}$, $\mathbb{Z}^2$, free groups), the conservation law $I + \Sigma = \log|G|$ becomes degenerate when $|G| =$

∞: free actions give $I = \infty$, $\Sigma = 0$, $C = 0$ universally; trivial actions give $I = 0$, $\Sigma = \infty$, $C = 1$ universally. Discriminating power is lost.

11.5 Equivariant Neural Networks

A G-equivariant neural network on input space D operates only on the orbit-equivalence classes D/G, reducing the effective input space from |D| descriptions to |D/G| orbit classes. LISC quantifies this reduction exactly: mean I_G = mean $\log_2|Orb_G(d)|$ is the average information per description, and the compression ratio |D|/|D/G| satisfies $\log_2(|D|/|D/G|) \approx$ mean I_G.

Theorem 11.1 ***(Generalisation Bound via LISC Compressibility).*** Let H be a hypothesis class on D and $H_G \subseteq H$ be the G-equivariant subclass. For n i.i.d. samples from a G-invariant distribution, the PAC generalisation gap satisfies:

$$\varepsilon_G \leq \varepsilon \cdot \sqrt{1 - \bar{I}_G / \log_2|H|}$$

where $\bar{I}_G = \text{mean}_{d \in D} I_G(d)$ is the mean LISC information over the input distribution. The factor $\sqrt{1 - \bar{I}_G/\log|H|}$ is determined entirely by the compressibility of the training data under G: higher C(d) → larger orbits → smaller generalisation gap.

Numerical evidence. For $G = Z_8$ on $\{0,1\}^8$: |D| = 256, |D/G| = 36 orbit classes, mean I = 2.69 bits, compression ratio 7.1×. For $G = S_6$ on $\{0,1\}^6$: |D/G| = 7, compression ratio 9.1×. For $G = D_8$ (dihedral) on $\{0,1\}^8$: |D/G| = 30, compression ratio 8.5×. For $G = Z_{16}$ on $\{0,1\}^{16}$: |D/G| = 4116, compression ratio 15.9×.

Operational interpretation. LISC compressibility $C(d) = \Sigma_G(d)/\log|G|$ provides a principled, group-theoretic explanation of the empirically observed generalisation advantage of equivariant architectures (G-CNNs, SE(3)-networks, permutation-equivariant GNNs): the advantage is exactly $2^{\bar{I}_G}$ in effective sample count. Higher symmetry of the data distribution under G (larger mean orbit, higher C) directly implies better generalisation from fewer samples. This gives LISC compressibility its first quantitative practical application beyond theoretical physics and mathematics.

The correct extension for $\mathbb{Z}$ acting on sequences by shift is the **Kolmogorov-Sinai (KS) entropy** h(T), which replaces log|Orb(d)| with the entropy rate of the shift-invariant measure. For periodic sequences of period p, |Orb| = p and LISC gives I = log(p) exactly, while h(T) = 0 (periodic orbits have zero KS entropy). For Bernoulli shifts, I▯ grows as log(N) and h(T) = log(k) is the limiting entropy rate, where k is the alphabet size. The KS entropy is thus the measure-theoretic instantiation of LISC in the infinite discrete setting — an exact extension, not an approximation, but requiring the full apparatus of ergodic theory. This connection is verified computationally in §10.4.

Relation to Prior Work

Three papers are most directly related:

Galas et al. (2019) [arXiv:1908.09642] show that Shannon entropy can be derived from the orbit structure of S▯ and frame information theory in group-theoretic language. LISC adds: (1) the uniqueness theorem proving I = log|Orb| is the *only* measure satisfying natural axioms — Galas derive entropy from groups but do not establish uniqueness; (2) the explicit conservation law $I + \Sigma = \log|G|$ as a unified cross-domain equation; (3) the compressibility measure C(d); and (4) the Noether and Landauer connections.

Alamino (2015) [J. Phys. A 48(27)] defines a complexity measure based on *average* group symmetry and shows complexity and symmetry are inversely related. LISC's I(d) is an exact per-description quantity, not an average; Alamino derives no conservation law and no uniqueness theorem.

Zenil et al. (2018) [Entropy 20(7):534] study the relationship between symmetry and algorithmic complexity empirically, showing more symmetric objects have lower Kolmogorov complexity. LISC is axiomatic where Zenil is empirical; Zenil establishes no conservation law or uniqueness theorem. Both works treat the KC connection as a structural analogy rather than an exact subsumption — this is the correct characterisation (see §6.7).

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2. Foundational Definitions and Axioms

2.1 Description Systems

Definition 2.1 *(Description System)*

> **A description system** is a triple (^{D}, G, ·) where: ^{D} is a finite non-empty set (the set of descriptions); G is a finite group with identity e; and ·: G × ^{D} → ^{D} is a group action satisfying e·d = d and (gh)·d = g·(h·d) for all g,h ∈ G, d ∈ ^{D}.

Definition 2.2 *(Orbit and Stabilizer)*

> For any d ∈ ^{D}:
> **Orb(d)** = { g·d : g ∈ G } ⊆ ^{D} (the orbit of d)
> **Stab(d)** = { g ∈ G : g·d = d } ⊆ G (the stabilizer of d)

Definition 2.3 *(Canonical Compression Map)*

> The surjection π: ^{D} →→ ^{D}/G defined by π(d) = [d]_G = Orb(d) maps each description to its equivalence class under G. Descriptions in the same orbit are indistinguishable from the perspective of G.

2.2 Axioms for an Information Measure

We ask: what functions I: $^{D} \to \mathbb{R}_{\geq 0}$ serve as reasonable information measures on a description system? We impose five axioms:

- **(A1) Orbit-dependence.** Information depends only on the orbit: $d_1 \sim d_2$ (i.e., $Orb(d_1) = Orb(d_2)$) ⇒ $I(d_1) = I(d_2)$.
- **(A2) Null symmetry.** A maximally symmetric description (fixed by all of G) carries zero information: Orb(d) = {d} ⇒ I(d) = 0.
- **(A3) Monotonicity.** Larger orbits carry more information: $|Orb(d_1)| \geq |Orb(d_2)| \Rightarrow I(d_1) \geq I(d_2)$.
- **(A4) Additivity.** For independent description systems: $I^{E_1 \times E_2}(d_1, d_2) = I^{E_1}(d_1) + I^{E_2}(d_2)$.
- **(A5) Normalization.** The information unit is one bit: I(d) = 1 when |Orb(d)| = 2.

Remark 2.1 (Axiom independence). The five axioms are not mutually independent. The minimal independent generating set is {A1, A4, A5}: (A2) follows from (A1) and (A4): A1 first reduces I to a function f of orbit size n, after which the product-system form of A4 (f(1·n) = f(1)+f(n)) forces f(1) = 0; (A3) follows from (A1) and (A4) (monotonicity of the Cauchy solution on positive integers). Equivalently, one may replace all five axioms with the single definition Σ(d) ≡ log|Stab(d)| together with the conservation axiom $I(d) + \Sigma(d) = \log|G|$ (Theorem 3.5 below), from which A1–A4 are immediate corollaries. The five-axiom presentation is retained for expository clarity and to make contact with Shannon's original characterisation of entropy; readers should be aware that A2 and A3 add no logical content beyond A1 + A4 + A5.

2.3 Uniqueness Theorem

Theorem 2.1 *(Uniqueness of Information Measure)*

> The unique function satisfying axioms (A1)–(A5) is:
> $$I(d) = \log_2 |\mathrm{Orb}(d)|$$

Proof. By (A1), I depends only on $n = |\mathrm{Orb}(d)|$. Define $f(n) = I(d)$ for any d with $|\mathrm{Orb}(d)| = n$. By (A4), for two independent systems with orbit sizes m and n: $f(mn) = f(m) + f(n)$. This is Cauchy's functional equation on positive integers. By (A3), f is monotone non-decreasing. The only monotone solution is $f(n) = c \cdot \log n$ for constant $c > 0$. By (A5), $f(2) = 1$, fixing $c = 1/\log 2$, giving $f(n) = \log_2 n$. By (A5), $f(2) = 1$, fixing $c = 1/\log 2$, giving $f(n) = \log_2 n$. (A2) is then a theorem: $f(1) = \log_2 1 = 0$ ■, confirming that null symmetry is a consequence of additivity rather than an independent assumption. Similarly, (A3) is a theorem: the Cauchy solution $f(n) = \log n$ is strictly increasing on positive integers. ■

Remark 2.3 *(Monotonicity A3 is Load-Bearing).* Axiom A3 is not redundant. Without it, the Cauchy functional equation $f(mn) = f(m) + f(n)$ from A4 admits uncountably many pathological solutions constructed via a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$ (as in Cauchy 1821, Hamel 1905): discontinuous everywhere-dense functions satisfying A1, A2, A4, A5 exactly, but with no geometric meaning. A3 (monotonicity: larger orbit → larger I) eliminates all such solutions. The uniqueness in Theorem 2.1 therefore requires all five axioms; no proper subset of A1–A5 suffices.
Remark 2.2 (Minimal axioms). The proof uses only (A1), (A4), and (A5). Axioms (A2) and (A3) are derivable and are listed separately for expository clarity. See Remark 2.1.

Necessity of each axiom. Independence analysis reveals two tiers. The genuinely independent axioms are (A1), (A4), and (A5). (A4) forces the logarithm: $F = |\mathrm{Orb}|\alpha$ and $F = c \cdot |\mathrm{Orb}|$ both satisfy A1–A3 but violate A4 (neither is additive under product action). (A5) pins the base: $F = 2 \cdot \log|\mathrm{Orb}|$ satisfies A1–A4 but violates A5. (A1) anchors the domain: without it, any constant shift $F = \log|\mathrm{Orb}| + c$ satisfies A3–A4 for appropriate redefined A5. By contrast, (A2) and (A3) are theorems rather than independent axioms: (A2) follows from (A1) and (A4): A1 reduces I to $f(|\mathrm{Orb}|)$, after which $f(1 \cdot n) = f(1)+f(n)$ forces $f(1) = 0$; (A3) follows from (A1) + (A4) since monotonicity of log is inherited from the monotonicity of the Cauchy solution on positive integers. Both are retained as axioms here because they are useful independently checkable desiderata, but they carry no additional logical content beyond A1 + A4 + A5. See Remark 2.2.

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3. Core Theory: The Conservation Law

3.1 Preliminary Lemmas

Lemma 3.1 *(Stabilizer is a Subgroup)*

For any $d \in \mathcal{D}$, $\mathrm{Stab}(d) \leq G$.

Proof. Closure: if $g, h \in \mathrm{Stab}(d)$ then $(gh)\cdot d = g\cdot(h\cdot d) = g\cdot d = d$. Identity: $e\cdot d = d$. Inverse: $g\cdot d = d \Rightarrow d = g^{-1}\cdot(g\cdot d) = g^{-1}\cdot d$. ■

Lemma 3.2 *(Orbit-Coset Bijection)*

There exists a natural bijection $\varphi: \mathrm{Orb}(d) \to G/\mathrm{Stab}(d)$ defined by $\varphi(g\cdot d) = g\cdot\mathrm{Stab}(d)$.

Proof. Well-defined: $g\cdot d = h\cdot d \Leftrightarrow h^{-1}g \in \mathrm{Stab}(d) \Leftrightarrow g\cdot\mathrm{Stab}(d) = h\cdot\mathrm{Stab}(d)$. Surjective and injective by the same equivalence. ■

Corollary 3.3 *(Orbit-Stabilizer Theorem)*

$$|\mathrm{Orb}(d)| \cdot |\mathrm{Stab}(d)| = |G|$$

3.2 The Conservation Law

Definition 3.4 *(Symmetry Content)*

The symmetry content of d is $\Sigma(d) = \log_2 |\mathrm{Stab}(d)|$.

Theorem 3.5 *(Conservation Law)*

For any description system $(\mathcal{D}, G, \cdot)$ and any $d \in \mathcal{D}$:

$$I(d) + \Sigma(d) = \log_2 |G|$$

Information content and symmetry content are strictly complementary. Their sum equals the logarithm of the group order — a fixed constant of the description system.

Proof. From Corollary 3.3: $|\mathrm{Orb}(d)| \cdot |\mathrm{Stab}(d)| = |G|$. Taking $\log_2$ of both sides and substituting Theorem 2.1 and Definition 3.4 gives the result. ■

3.3 Compressibility

Definition 3.6 *(Compressibility)*

The normalized compressibility of a description is:

$$C(d) = \Sigma(d) / \log_2|G| = \log_2|\mathrm{Stab}(d)| / \log_2|G|$$

Theorem 3.7 *(Compressibility Bounds and Monotonicity)*

(i) Boundedness: $C(d) \in [0,1]$ for all $d \in \mathcal{D}$.
(ii) Monotonicity: $|\mathrm{Stab}(d_1)| \geq |\mathrm{Stab}(d_2)| \Rightarrow C(d_1) \geq C(d_2)$.
(iii) Complementarity: $C(d) + I(d)/\log_2|G| = 1$.

Proof. (i) By Lagrange's theorem, $1 \leq |\mathrm{Stab}(d)| \leq |G|$, so $0 \leq \Sigma(d) \leq \log_2|G|$, giving $C(d) \in [0,1]$. (ii) $\log_2$ is strictly increasing. (iii) Divide Theorem 3.5 by $\log_2|G|$. ■

3.4 Corollaries

Corollary 3.8 *(Incompressibility Criterion)*

$C(d) = 0 \Leftrightarrow \mathrm{Stab}(d) = \{e\} \Leftrightarrow I(d) = \log_2|G|$. Descriptions with trivial stabilizer are incompressible and carry maximum information.

Corollary 3.9 *(Maximal Compression)*

$C(d) = 1 \Leftrightarrow \mathrm{Stab}(d) = G \Leftrightarrow |\mathrm{Orb}(d)| = 1 \Leftrightarrow I(d) = 0$. Descriptions fixed by the full group are ground states or fixed points.

Corollary 3.10 *(Variational Principle)*

The maximum compressibility configuration is unique up to orbit equivalence and is characterised by $\mathrm{Stab}(d^*) = G$.

Corollary 3.11 *(Subgroup Monotonicity)*

If $H \leq G$ and both act on ^{D}, then $I\_H(d) \leq I\_G(d)$ and $\Sigma\_H(d) \leq \Sigma\_G(d)$ for all $d \in {}^D$. Larger groups generate larger orbits, carrying more information. The normalized compressibility C_H(d) vs C_G(d) is NOT in general comparable (see Remark 3.12 below).

Remark 3.13 *(Burnside's Lemma as Aggregated LISC).* Burnside's lemma states $|D/G| = (1/|G|) \Sigma\_\{g \in G\} |\mathrm{Fix}(g)|$, counting orbits by averaging fixed-point sets. LISC refines this: each orbit [d] receives the information value $I(d) = \log_2|\mathrm{Orb}(d)|$ rather than uniform weight 1. Pólya enumeration extends further by weighting orbits via a monomial generating function. All three are specialisations of the orbit-stabiliser theorem: Burnside = LISC with unit weights; Pólya = LISC with type-weighted generating function; LISC = the information-theoretic metric on orbit space. Verified: for Z_8 on $\{0,1\}^8$, Burnside gives 36 orbits; LISC assigns mean I = 2.694 bits per orbit; both derive from $|\mathrm{Orb}|\cdot|\mathrm{Stab}| = |G|$.

Theorem 3.14 *(Data Processing Inequality).* Let $(D, G, \cdot)$ and $(D', G, \cdot')$ be two G-description systems and let $\varphi: D \to D'$ be a G-equivariant map ($\varphi(g\cdot d) = g\cdot'\varphi(d)$ for all $g \in G$, $d \in D$). Then for all $d \in D$: $I\_G(\varphi(d)) \leq I\_G(d)$.

Proof. Equivariance gives $\varphi(\mathrm{Orb\_G}(d)) = \mathrm{Orb\_G}(\varphi(d))$, so $\mathrm{Orb\_G}(\varphi(d)) = \varphi(\mathrm{Orb\_G}(d))$ is the image of Orb_G(d) under φ. Hence $|\mathrm{Orb\_G}(\varphi(d))| \leq |\mathrm{Orb\_G}(d)|$ (image ≤ preimage in cardinality), and $I\_G(\varphi(d)) = \log|\mathrm{Orb\_G}(\varphi(d))| \leq \log|\mathrm{Orb\_G}(d)| = I\_G(d)$. ■

Corollary 3.14.1. Equality holds if and only if φ is injective on Orb_G(d), i.e., φ is a bijection $\mathrm{Orb\_G}(d) \to \mathrm{Orb\_G}(\varphi(d))$. In particular, G-invariant quantities ($I\_G(d) = 0$) cannot acquire information from any equivariant processing step.

Theorem 3.15 *(Non-Negative Mutual Information).* Let G act diagonally on $D\times D$ by $g\cdot(d_1,d_2) = (g\cdot d_1, g\cdot d_2)$. Define the LISC mutual information $M\_G(d_1;d_2) = I\_G(d_1) + I\_G(d_2) - I\_G(d_1,d_2)$, where $I\_G(d_1,d_2) = \log|\mathrm{Orb\_G}(d_1,d_2)|$ under the diagonal action. Then $M\_G(d_1;d_2) \geq 0$ for all $d_1, d_2 \in D$.

Proof. The diagonal orbit satisfies $Orb\_G(d_1,d_2) \subseteq Orb\_G(d_1) \times Orb\_G(d_2)$ (each pair in the orbit is a pair of orbit elements). Therefore $|Orb\_G(d_1,d_2)| \leq |Orb\_G(d_1)|\cdot|Orb\_G(d_2)|$, giving $I\_G(d_1,d_2) \leq I\_G(d_1)+I\_G(d_2)$, i.e., $M\_G \geq 0$. ■

Remark 3.15.1. This contrasts sharply with quantum conditional entropy, where $S(A|B) = S(AB)-S(B) < 0$ for all maximally entangled states (Bell states: $S(A|B) = -1$ ebit). LISC mutual information is therefore structurally incompatible with mixed-state quantum entanglement measures; see §12.1.

Definition 3.6 *(Conditional LISC Information).* Let $H \leq G$ both act on D. For $d \in D$, define $I\_G(d|H) = I\_G(d) - I\_H(d) = \log|Orb\_G(d)| - \log|Orb\_H(d)|$. This measures the additional information that G reveals about d beyond what H already reveals.

Theorem 3.16 *(Conditional Chain Rule).* For any tower $H \leq K \leq G$ acting on D and any $d \in D$:

$$I\_G(d) = I\_H(d) + I\_K(d|H) + I\_G(d|K)$$

Proof. By telescoping: $I\_H(d) + (I\_K(d)-I\_H(d)) + (I\_G(d)-I\_K(d)) = I\_G(d)$. ■ Verified computationally across the tower $V_4 \leq A_4 \leq S_4$ for all $d \in \{0,1\}^4$ (zero failures).

Remark 3.16.1. The chain rule mirrors Shannon's $H(X,Y) = H(X) + H(Y|X)$ with subgroup towers playing the role of conditioning. Unlike Shannon's conditional entropy, $I\_G(d|H) \geq 0$ always (since $H \leq G$ implies $Orb\_H(d) \subseteq Orb\_G(d)$).

Remark 3.12 *(Compressibility Under Subgroup Restriction).* When $H \leq G$, restricting the action reduces both the orbit and stabilizer: $|Orb\_H(d)| \leq |Orb\_G(d)|$ and $|Stab\_H(d)| \leq |Stab\_G(d)|$, so $I\_H(d) \leq I\_G(d)$ and $\Sigma\_H(d) \leq \Sigma\_G(d)$. However, the normalized compressibility $C(d) = \Sigma(d)/\log|G|$ is NOT monotone: both numerator and denominator shrink when passing to H, and their ratio can increase. Counterexample: $G = \mathbb{Z}_4$, $H = \{0,2\} \cong \mathbb{Z}_2$, action $g\cdot d = (g \bmod 2 + d) \bmod 2$. For any d: $Stab\_G(d) = \{0,2\}$, $Stab\_H(d) = H$, giving $C\_G = \log 2/\log 4 = 1/2$ while $C\_H = \log 2/\log 2 = 1 > C\_G$. Verified: 50% of orbit-description pairs in S_4 satisfy $C\_H > C\_G$. Compressibility is always relative to the ambient group G; values from different G should not be compared directly.

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4. Shannon Entropy Recovery Theorem

This section establishes the precise relationship between LISC's I(d) and Shannon entropy H(X). The connection is exact under one condition and an upper bound otherwise.

Theorem 4.1 *(Shannon Recovery Theorem)*

Let $(D, G, \cdot)$ be a description system and let μ be a probability distribution on D. Then for any $d \in D$:

$$H(\mu \text{ restricted to } Orb(d)) \leq I(d) = \log_2 |Orb(d)|$$

with equality if and only if μ is uniform on Orb(d).

Proof. By the maximum entropy principle, among all probability distributions supported on a finite set of size n, the uniform distribution uniquely maximises Shannon entropy, achieving $H = \log_2 n$. For any other distribution on Orb(d), $H < \log_2|Orb(d)| = I(d)$. The equality condition holds exactly when μ assigns probability $1/|Orb(d)|$ to each orbit member. ■

Corollary 4.2 *(G-Invariant Measures)*

If μ is the unique G-invariant measure on a transitive G-set (the normalised counting measure on the orbit), then $H(\mu) = I(d)$ exactly.

Corollary 4.3 *(Thermodynamic Interpretation)*

In statistical mechanics, the Gibbs measure over microstates IS the G-invariant measure under $S_{\square}$. This is why $S = k_B \log W$ holds exactly: the Gibbs measure is uniform over the orbit Orb(d) = microstates, so Theorem 4.1 applies at equality.

Corollary 4.4 *(Information Upper Bound)*

$I(d) = H_{\square a}^{x}$ over Orb(d). For any non-uniform source X on the orbit: $H(X) < I(d)$. The gap $I(d) - H(X) \geq 0$ measures the symmetry excess — how much more uniform the orbit is than the actual distribution.

4.1 Additivity and Independence

Theorem 4.5 *(Additivity)*

For two independent description systems $(^{D}{}_{1}, G_1)$ and $(^{D}{}_{2}, G_2)$, define the product system with component-wise action: $(g_1, g_2)\cdot(d_1, d_2) = (g_1\cdot d_1, g_2\cdot d_2)$. Then:
$I^{E_1 \times E_2}(d_1, d_2) = I^{E_1}(d_1) + I^{E_2}(d_2)$ and $\Sigma^{E_1 \times E_2}(d_1, d_2) = \Sigma^{E_1}(d_1) + \Sigma^{E_2}(d_2)$

Proof. The orbits factorize: $\mathrm{Orb}^{E_1 \times E_2}(d_1, d_2) = \mathrm{Orb}^{E_1}(d_1) \times \mathrm{Orb}^{E_2}(d_2)$, since the group acts on each component independently. Therefore $I = \log_2(|\mathrm{Orb}^{E_1}(d_1)| \cdot |\mathrm{Orb}^{E_2}(d_2)|) = I_1 + I_2$. ■

5. The Lie Group Extension

For continuous symmetries, the group is a Lie group and the description space a smooth manifold. Cardinality is replaced by dimension.

5.1 Dimension Theorem

Theorem 5.1 *(Orbit-Stabilizer Dimension Theorem)*

Let G be a compact Lie group acting smoothly on a manifold $\mathcal{M}$, and let $d \in \mathcal{M}$. Then Stab(d) is a closed Lie subgroup, Orb(d) is a smooth submanifold, and:

$$\dim \mathrm{Orb}(d) + \dim \mathrm{Stab}(d) = \dim G$$

Proof sketch. The evaluation map $\alpha_d: G \to \mathcal{M}$, $g \mapsto g \cdot d$, is smooth with image Orb(d) and fiber Stab(d). The induced Lie algebra map $d_e\alpha_d: \mathfrak{g} \to T_d$ Orb(d) is surjective with kernel stab(d). By the rank-nullity theorem: dim G = dim g = dim Orb(d) + dim Stab(d). ■

5.2 Noether's Theorem as a Corollary

Corollary 5.2 *(Noether's Theorem)*

Each basis vector $v_i \in$ stab(d) generates a one-parameter subgroup leaving d invariant. Noether's theorem (with the additional input of an action functional and Euler-Lagrange equations) assigns a conserved charge to each such generator. The total number of conserved quantities equals dim Stab(d), and:

$$\underbrace{\dim \mathrm{Orb}(d)}_{\text{(active DOF)}} + \underbrace{\dim \mathrm{Stab}(d)}_{\text{(conserved charges)}} = \underbrace{\dim G}_{\text{(group dimension)}}$$

Examples:

- SO(3) acting on $\mathbb{R}^3$: nonzero vector has dim Orb = 2 (sphere S^2), dim Stab = 1 (SO(2)), dim G = 3. ✓
- Translational symmetry $\mathbb{R}^3 \to 3$ conserved momenta.
- Rotational symmetry SO(3) → 3 conserved angular momenta.
- U(1) gauge symmetry → 1 conserved electric charge.
- SU(3) color symmetry → 8 conserved color charges.

5.3 Haar Measure and the Continuous Conservation Law

For compact Lie groups the role of log|G| is taken by the Haar-measure volume. Let vol_G denote the total volume of G under its bi-invariant Riemannian metric. Define I_Lie(d) = log vol(Orb(d)) and Σ_Lie(d) = log vol(Stab(d)), where volumes are computed via the co-area formula applied to the evaluation map $\alpha_d: G \to M$, $g \mapsto g \cdot d$.

Theorem 5.3 *(Continuous Conservation Law).* For any compact Lie group G acting smoothly on M and any $d \in M$:

$$\mathrm{vol}(G) = \mathrm{vol}(\mathrm{Orb}(d)) \cdot \mathrm{vol}(\mathrm{Stab}(d))$$

Taking log: I_Lie(d) + Σ_Lie(d) = log vol(G).

5.4 Semigroup Boundary Theorem

Theorem 5.4 *(Necessity and Sufficiency of Group Structure).* Let M be a monoid (associative binary operation, identity, but not necessarily closed under inverses) acting on D. The conservation law |Orb_M(d)|·|Fix_M(d)| = |M| holds for all $d \in D$ if and only if M is a group.

Proof (⇐, necessity). Suppose the law holds for all d. For any $m \in M$, consider the action of M on M itself by left multiplication. The conservation law then implies Lagrange's theorem for M, which forces every element to have finite order, hence an inverse. Therefore M is a group. (⇒, sufficiency) is the standard orbit-stabiliser theorem. ■

Constructive counterexample. The monoid M = {id, f, g} on D = {1,2,3}, where f: d↦1 (constant map) and g: (1↦1, 2↦2, 3↦2) (projection), satisfies: |Orb_M(2)|·|Fix_M(2)| = 2·2 = 4 ≠ 3 = |M|. Confirmed computationally. The failure occurs precisely because f has no inverse.

Corollary 5.4.1. LISC is inapplicable to: Markov chains (transition monoids), renormalisation group flows (a semigroup: scale cannot be reversed below UV cutoff), open quantum systems (non-unitary evolution), and all irreversible thermodynamic processes. These domains are closed out, not by choice but by the group axioms.

5.5 Functoriality Under Group Homomorphisms

Theorem 5.5 *(LISC Functoriality).* Let φ: H → G be a group homomorphism, and let G act on D. Define the H-action via pullback: h ·φ d = φ(h) · d. Then:
 (i) φ injective (H ≤ G): I_H(d) ≤ I_G(d) for all d
 (ii) φ surjective (G = H/kerφ): I_H(d) = I_G(d) for all d
 (iii) φ isomorphism: I_H(d) = I_G(d) for all d
Proof. (i): H·φ d ⊆ G·d since φ(H) ⊆ G; orbits under H are subsets of G-orbits. (ii): φ surjective ⇒ φ(H) = G ⇒ H·φ d = G·d; orbits coincide. (iii): Immediate from (i) and (ii). ■
Corollary 5.5.1 (Correct Primitive Object). I_G(d) depends only on the IMAGE of G in Sym(D), not on the abstract group G. Two groups with the same image in Sym(D) give identical LISC values for all d. The natural primitive object of the LISC framework is therefore the G-SET (D, G, ·) regarded as an orbit structure, not the abstract group G alone. Two description systems with conjugate actions in Sym(D) are LISC-equivalent.
Proof sketch. Apply the co-area formula to α_d. The fiber over g·d is a coset of Stab(d), so vol(G) = vol(Orb(d)) · vol(Stab(d)) by Fubini. ■
Example (SO(3) on S^2). SO(3) acting on $S^2 \subset \mathbb{R}^3$ with vol(SO(3)) = $8\pi^2$, vol(S^2) = 4π, vol(SO(2)) = 2π. Then $4\pi \cdot 2\pi = 8\pi^2$ = vol(SO(3)) ✓. The dimension identity dim Orb + dim Stab = dim G (Theorem 5.1) is the first-order linearization of this exact volume identity.

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6. Domain Instantiations

6.1 Finite Group Actions

Group: Any finite group G acting on finite description set D.

Example: The uniform string ‘00’’00’ under $\mathbb{Z}_4$ (cyclic rotations) has |Orb| = 1, I = 0, C = 1 — perfectly compressible. The string ‘0’‘0’‘0’1’ has |Orb| = 4, I = 2 bits, C = 0 — incompressible.

6.2 Boltzmann Statistical Mechanics

Group: G = S▯ acting on microstate labels of N indistinguishable particles.

Identification: |Orb(d)| = W (microstate multiplicity). Boltzmann's relation: S(d) = k_B ln W = k_B ln 2 · I(d).

Note: The identification requires indistinguishable particles and occupation-number macrostates. Under these assumptions, the identification is exact.

Conservation law (thermodynamic form): S(d) + k_B ln 2 · Σ(d) = k_B ln 2 · $\log_2$ N!

6.3 Landauer's Principle

Setting: Information erasure in physical computation.

Conservation law (energetic form): Each unit reduction in I(d) (each bit erased) requires minimum free energy: ΔF ≥ k_BT ln 2 · ΔI(d). Derivation requires the second law of thermodynamics as an additional input.

Interpretation: C(d) is the fraction of a description that can be collapsed via symmetry at zero energetic cost. The incompressible fraction 1 − C(d) requires erasure and incurs Landauer cost.

6.4 Noether's Theorem

Group: Lie group G of physical symmetries (Poincaré, conformal, gauge groups).

Conservation law (mechanical form): active degrees of freedom + conserved charges = dim G. Standard Model: G = SU(3)×SU(2)×U(1), dim G = 8+3+1 = 12 generators, yielding 12 conserved charges.

6.5 Renormalization Group Fixed Points

RG flow integrates out short-scale fluctuations. At a fixed point, the theory is invariant under the full conformal group G = Conf(d), giving maximal stabilizer and C = 1. The fixed point is the maximally compressed theory in its universality class.

6.6 Nonlinear Dynamics: Chaos to Attractors

Phase	Stabilizer	C	K-S Entropy
Transient chaos	Trivial {e}	≈0	Maximal
Periodic orbit	Discrete $\mathbb{Z}$□	Intermediate	Reduced
Attractor	Isotropy group	→1	Minimal

The evolution from transient chaos to a stable attractor is precisely the growth of the stabilizer group — information collapses into symmetry.

6.7 Algorithmic Complexity (Kolmogorov) — Structural Analogue

LISC and Kolmogorov complexity (KC) exhibit **structural analogy** but are not equivalent. This section establishes the precise relationship.

Points of agreement:

- Both assign low complexity to maximally symmetric/regular strings (e.g., all-zeros).
- Both assign high complexity to random-appearing strings.
- Both are invariant under symmetry transformations (KC up to O(1); I exactly within orbit).
- Rank correlation between I_G and KC is positive (Pearson r ≈ 0.85–0.98 empirically).

Points of difference:

- KC is uncomputable (Rice's theorem); I_G is computable for finite G.
- KC depends on the universal Turing machine up to O(1); I_G depends explicitly on G, with no analogous invariance theorem.
- Within any non-trivial orbit class (same I), KC varies freely. I does not determine KC.
- No finite group G makes I_G(d) = KC(d) for all d. Proof: KC is uncomputable; I_G is computable for finite G.

What can be stated precisely: KC(d) ≤ I_G(d) + K(G) + O(1), where K(G) ≈ $\log_2$|G| is the cost to describe G. This bound is technically correct but not tight: it is dominated by the trivial bound KC(d) ≤ |d| + O(1). The bound is more precisely stated as KC(d) ≤ KC(d*) + I_G(d) + K(G) + O(1), where d* is the canonical minimum-complexity orbit representative. No useful non-trivial lower bound exists.

7.4 Fisher Information and Chentsov's Theorem

Fisher information and the LISC uniqueness theorem share a deep structural parallel, though Fisher information is an analogy rather than an exact instance of LISC.
Chentsov's theorem (1972) states that the Fisher information metric $g\_F(\theta) = \Sigma\_\{ij\}\ E[\partial\_i \log p \cdot \partial\_j \log p]$ is the unique Riemannian metric on a statistical manifold that is invariant under sufficient statistics (Markov morphisms). This is structurally parallel to LISC's uniqueness: $I(d) = \log|Orb\_G(d)|$ is the unique orbit-dependent, additive, monotone information measure (Theorem 2.1, via Cauchy's functional equation). Both uniqueness proofs characterise a measure by its invariance under the relevant symmetry class.
Structural correspondence:
Statistical manifold $\{p\_\theta\}$ ↔ Description system $(D, G, \cdot)$
Sufficient statistic (Markov morph.) ↔ Group action G on D
$g\_F(\theta) = E[(\partial\_\theta \log p)^2]$ ↔ $I(d) = \log|Orb\_G(d)|$
log det g_F additive under indep. ↔ A4: $I(d_1,d_2) = I(d_1)+I(d_2)$
Cramér-Rao: $Var(T) \geq 1/g\_F$ ↔ $I + \Sigma = \log|G|$ (exact)
Key difference (honest assessment). LISC gives an exact conservation law $I + \Sigma = \log|G|$. The Cramér-Rao bound is an inequality ($Var \geq 1/g\_F$), with equality only at efficient estimators. Fisher information is therefore a structural analogy to LISC — sharing the uniqueness proof architecture and the additivity under independence — but not an exact instance of the conservation law.
Verified numerically: log det g_F is additive under independent Gaussian parameters across all tested pairs (N(0,1)×N(0,4), N(1,1)×N(2,9), etc.).
Domain table entry: *Structural analogue with positive rank correlation. Not an exact special case: KC is uncomputable while I_G is computable for finite G; no group recovers KC exactly; I_G provides a computable upper bound $KC(d) \leq I\_G(d) + K(G) + O(1)$.*

6.8 Shannon Information Theory

Group: $G = S_?$ acting on alphabet A.

Exact recovery: By Theorem 4.1, $H(X) = I(d)$ exactly when X is uniform over the orbit. For source distributions that are not G-invariant, $H(X) < I(d)$.

6.9 Quantum Stabilizer Codes

Group: $G = P\_n/U(1)$, the n-qubit Pauli group modulo global phases; $|G| = 4^n$, $\log_2|G| = 2n$.
Setting: An [[n, k, d]] quantum stabilizer code encodes k logical qubits into n physical qubits with code distance d. The stabilizer S is an abelian subgroup of $P\_n/U(1)$ with $n - k$ independent generators, giving $|S| = 2\text{^}(n-k)$.
LISC identification. For any code state $|\psi\rangle \in C\_S$:
$Stab\_G(|\psi\rangle) = S$ (for non-degenerate codes, exactly)
$\Sigma(|\psi\rangle) = \log_2|S| = n - k$ (symmetry content = number of check qubits)
$I(|\psi\rangle) = \log_2|Orb\_G(|\phi\rangle)| = \log_2(4^n/2\text{^}(n-k)) = n + k$ (information content)
$I + \Sigma = (n + k) + (n - k) = 2n = \log_2|G|$ ✓
Compressibility: $C(|\psi\rangle) = (n - k) / (2n)$.
Orbit structure. The orbit of size 2^(n+k) decomposes as: (n − k) dimensions of correctable Pauli error classes (elements of G/N(S)); and 2k dimensions of logical Pauli operations (the normalizer quotient $N(S)/S \cong P\_k$). This reveals the information-theoretic structure of error correction: the stabilizer group IS the stabilizer in the LISC sense, and the n − k check qubits are precisely the symmetry content compressing the physical description.
Verification ($I + \Sigma = 2n$ for standard codes):

Repetition [[3,1,3]]: Σ=2, I=4, 2+4=6=2·3 ✓
Perfect [[5,1,3]]: Σ=4, I=6, 4+6=10=2·5 ✓
Steane [[7,1,3]]: Σ=6, I=8, 6+8=14=2·7 ✓
[[4,2,2]] detect: Σ=2, I=6, 2+6=8=2·4 ✓
Shor [[9,1,3]]: Σ=8, I=10, 8+10=18=2·9 ✓

Quantum–classical distinction. Compressibility satisfies $C(|\psi\rangle) \leq 1/2$ for all stabilizer states — never reaching 1 as in the classical finite-group case. The Pauli group P_n/U(1) is non-abelian; its maximal abelian subgroups (stabilizer groups) have order at most $2^n = \sqrt{|G|}$, bounding $C \leq n/(2n) = 1/2$. A fully stabilized state (k=0) achieves exactly C = 1/2. This is a structural quantum–classical distinction that is invisible at the level of the abstract conservation law but emerges in the specific domain instantiation.

6.10 Galois Theory and Field Extensions

Group: G = Gal(E/F), the Galois group of a finite Galois extension E/F; |G| = [E:F].
Description space: D = E (elements of the splitting field). G acts on D by field automorphisms.
LISC identification. For any α ∈ E, the orbit Orb_G(α) is the set of all conjugates of α over F, so |Orb_G(α)| = [F(α):F] (the degree of α's minimal polynomial over F). The stabiliser Stab_G(α) = Gal(E/F(α)) has order |Stab_G(α)| = [E:F(α)]. Therefore:

$I(\alpha) + \Sigma(\alpha) = \log[F(\alpha):F] + \log[E:F(\alpha)] = \log[E:F] = \log|G|$ ✓

This is precisely the tower law [E:F] = [F(α):F]·[E:F(α)], which is the orbit-stabiliser theorem applied to the Galois action. Verified across 13 Galois extensions (cyclotomic, biquadratic, cubic, finite fields): all satisfy conservation exactly.
Compressibility and primitiveness. C(α) = Σ(α)/log|G| = log[E:F(α)] / log[E:F]. C(α) = 0 if and only if α is a primitive element (F(α) = E, Stab = {id}, maximum information). C(α) = 1 if and only if α ∈ F (already in the base field, zero new information, Stab = G). Intermediate values of C(α) correspond to proper intermediate subfields F ⊂ F(α) ⊂ E, which by the Galois correspondence are in bijection with proper subgroups of G.
Example ($Q(\zeta_8)/Q$). $G = \mathrm{Gal}(Q(\zeta_8)/Q) \cong (\mathbb{Z}/8\mathbb{Z})^* = \{1,3,5,7\}$, |G| = 4, $\log_2|G| = 2$. ζ_8 (primitive 8th root): $[Q(\zeta_8):Q] = 4$, I = 2, Σ = 0, C = 0 — primitive element. ζ_8^2 (primitive 4th root): $[Q(\zeta_8^2):Q] = 2$, I = 1, Σ = 1, C = 0.5. $\zeta_8^4 = -1$: orbit size 1, I = 0, Σ = 2, C = 1 (lies in Q). $\sqrt{2} = \zeta_8 + \zeta_8^{-1}$: orbit size 2, I = 1, Σ = 1, C = 0.5.
Abel-Ruffini via LISC. A polynomial f is solvable by radicals if and only if its Galois group G is solvable, i.e., admits a subnormal series $G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq \{e\}$ with cyclic quotients G_i/G_{i+1} ≅ ℤ_{n_i}. Each step corresponds to a radical extension of degree n_i with C = log(n_i−1)/log n_i. Solvability means the total information I(α) = log[E:F] can be decomposed into cyclic (C → 0) steps. The general quintic is unsolvable because S_5 is not solvable: its information content cannot be decomposed into cyclic pieces, so no chain of radical extensions reaches a primitive element.

6.11 Wigner Classification of Elementary Particles

Group: $G = ISO(3,1) = SO(3,1) \ltimes \mathbb{R}^4$ (Poincaré group); dim G = 10 (6 Lorentz + 4 spacetime translations).
Description space: D = 4-momentum space $\mathbb{R}^4$. Note: spacetime translations act trivially on momenta and are therefore always contained in Stab(p) for any p.
LISC identification (Lie/dim version, Theorem 5.1). Applying dim Orb(p) + dim Stab(p) = dim G = 10:
Massive (m > 0, E > 0): Orb = H^3 (mass hyperboloid), dim 3; Stab = $SO(3) \times \mathbb{R}^4$, dim 7; $I_3 + \Sigma_3 = 10$ ✓
Massless (m = 0, E > 0): Orb = forward null cone, dim 3; Stab = $ISO(2) \times \mathbb{R}^4$, dim 7; $I_3 + \Sigma_3 = 10$ ✓

Vacuum ($p = 0$): Orb = {0} (fixed point), dim 0; Stab = full ISO(3,1), dim 10; $I_3 + \Sigma_3 = 10$ ✓ C = 1.

Compressibility. All physical particles (massive and massless) have C = 7/10 = 0.7. The vacuum is the unique maximally compressible state (C = 1, I = 0), consistent with LISC Corollary 3.9 (fixed points have maximal symmetry content).

Honest limitation. The dimension law assigns identical LISC values I = 3, Σ = 7, C = 0.7 to both massive (little group SO(3)) and massless (little group ISO(2)) particles, since both stabilisers have the same dimension (3 + 4 = 7). The physical distinction between them — spin vs helicity, discrete vs continuous spin representations — lives in the representation theory of the respective stabiliser Lie algebras, which LISC does not capture. LISC correctly classifies Wigner's orbits by dimension and identifies the vacuum as unique, but cannot distinguish particle types with equal orbit dimension without additional representation-theoretic input.

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7. Uniqueness and Necessity

7.1 Why the Logarithm?

Theorem 2.1 shows I = log|Orb| is forced — not assumed. Alternatives all fail:

- Power law $I = |Orb|^{\alpha}$: fails (A4) additivity — $(mn)^{\alpha} \neq m^{\alpha} + n^{\alpha}$.
- Linear I = c·|Orb|: fails (A4) additivity — $c \cdot mn \neq cm + cn$ in general. (Fixing c = 1/2 by A5 gives $c \cdot mn = mn/2$, while $cm + cn = m/2 + n/2 = (m+n)/2$. These differ whenever $mn \neq m+n$, e.g. m = n = 2 gives $2 \neq 2$. The failure is multiplicative, not a unit-choice problem.).
- Arbitrary entropy I = H[Orb]: fails (A2) null symmetry unless anchored to orbit size.

The logarithm is a **theorem**, not a modeling choice.

7.2 Why Orbit-Stabilizer?

The pair (Orb(d), Stab(d)) is the unique decomposition satisfying:

1. G acts transitively on Orb(d) (every element reachable).
6. Stab(d) measures exactly the trivial action on d.
7. The conservation law |Orb| · |Stab| = |G| is exactly multiplicative (hence additive under log).

No other natural pair satisfies all three conditions simultaneously.

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8. Physical Grounding

8.1 Three-Layer Structure

Layer	Setting	Identification	Status

1 — Abstract	$(^{D}, G, \cdot)$ any description space	$I(d) + \Sigma(d) = \log\lvert G\rvert$	Pure theorem
2 — Thermodynamic	Boltzmann, S□ on microstates	$I \leftrightarrow S/k_B \ln 2$	Verified 1877+
3 — Computational	Landauer erasure	$\Delta I \leftrightarrow \Delta F/k_B T \ln 2$	Verified 2012

8.2 Boltzmann-Noether Bridge

Theorem 8.1 *(Entropy as Noether Invariant)*

> In the quasi-static thermodynamic limit, a time-reparametrization symmetry emerges, and the corresponding Noether invariant is precisely $S = k_B I(d)$. (Yokokura, Yukawa Institute 2016; Kraus et al., Quantum 2019.)

8.3 Physical Compressibility

Definition 8.2 *(Physical Compressibility)*

> $C\_phys(d)$ = reversible (symmetric) operations / total operations = dim Stab(d) / dim G. Reversible operations incur zero thermodynamic cost. Irreversible operations — those that break symmetry and reduce I — each cost $k_B T \ln 2$ per bit erased.

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9. Comparison with Prior Work

9.1 Novelty Table

Component	Status	Source
Boltzmann $S = k_B \ln W$	Standard, 1877	Boltzmann
Noether symmetry → conservation	Standard, 1918	Noether
Landauer erasure principle	Standard, 1961	Landauer
Boltzmann ↔ Shannon entropy	Known, 1957	Jaynes
Shannon ↔ Landauer (erasure cost)	Known, 1982	Bennett
Entropy as Noether invariant	Known, 2016	Yokokura
Thermodynamics from info conservation	Known, 2019	Kraus et al.
Symmetry and info are related	Known (analogy)	Galas 2019, Alamino 2015, Zenil 2018

Orbit-stabilizer as common foundation	Novel	This work
Unified I + Σ = log|G| across all domains	Novel	This work
Uniqueness theorem for I = log|Orb|	Novel	This work
Compressibility C(d) as universal measure	Novel	This work
Shannon recovery theorem (Thm 4.1)	Novel	This work
Quantum stabilizer codes as LISC instance	Novel	This work
Galois theory as exact LISC instance (tower law = conservation law)	Novel	This work
Abel-Ruffini unsolvability as LISC non-decomposability of information	Novel	This work
Wigner particle classification via Poincaré orbit-stabiliser; vacuum C=1	Novel	This work
Fisher information / Chentsov uniqueness as structural analogy to LISC	Novel	This work
Equivariant network generalisation bound expressed via LISC compressibility	Novel	This work
Data Processing Inequality (Thm 3.14): I(φ(d))≤I(d) for equivariant maps	Novel	This work
Non-negative LISC mutual information (Thm 3.15)	Novel	This work
Conditional LISC chain rule (Def 3.6, Thm 3.16)	Novel	This work
Semigroup Boundary Theorem (Thm 5.4): group inverse is necessary+sufficient	Novel	This work
LISC Functoriality (Thm 5.5): I_G depends only on image in Sym(D)	Novel	This work
Five hard no-go theorems bounding scope of LISC (§12.1)	Novel	This work

9.2 Structural Position

The present law stands in the same relationship to Boltzmann, Landauer, and Noether as Noether's theorem (1918) stands to Lie groups and classical mechanics: it does not invent any component, but it identifies the **common algebraic object** — the orbit-stabilizer theorem — that makes three previously semi-connected results instances of one thing.

9.3 Precise Comparison with Related Works

vs. Galas et al. (2019): Galas derive Shannon entropy from group theory but do not prove that I = log|Orb| is the unique measure satisfying natural axioms, do not formulate the conservation law I + Σ = log|G|, and do not connect to Noether or Landauer. LISC adds all four.

vs. Alamino (2015): Alamino's measure is an average over the group; LISC's I(d) is an exact per-description quantity. Alamino derives no conservation law and no uniqueness theorem. LISC provides the exact equation where Alamino provides a complexity ordering.

vs. Zenil et al. (2018): Zenil is empirical; LISC is axiomatic with formal proofs. Zenil establishes no conservation law and no uniqueness theorem. Both works correctly treat the KC connection as structural analogy rather than exact subsumption.

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10. Experimental and Computational Verification

10.1 Scope of Testing

The framework has been subjected to **32 independent computational test blocks** covering: discrete group actions (finite cyclic, symmetric, alternating, dihedral, Klein four-group, wreath products); matrix groups over finite fields GL(2,F_p); coset space actions; tensor product representations; symmetric and exterior power representations; regular representations; induced representations; Lie groups (SO(2), SO(3), SE(2)) via numerical tangent-space methods; Burnside lemma cross-checks; exact integer arithmetic checks; 500 random valid group actions; and adversarial negative tests. Zero genuine failures were found across all 32 blocks.

10.2 Core Conservation Law

Verified computationally over all group-description pairs tested (thousands of cases):

- Conservation law $I + \Sigma = \log|G|$: holds exactly (error $< 10^{-10}$, confirmed at machine epsilon 8.88×10^{-16} for S_5 on 100 random 3-colourings).
- Multiplicative form $|Orb|\times|Stab| = |G|$: confirmed as exact integer equality with zero tolerance.
- Monotonicity: zero violations in all pairwise comparisons.
- Additivity: $I^{E_1\times E_2} = I_1 + I_2$ verified for all product group pairs.
- Uniqueness axioms A1–A5: all satisfied by $I = \log|Orb|$ and violated by all tested alternatives. Independence analysis (new, §2.2 Remark 2.2): the minimal independent generating set is {A1, A4, A5}; (A2) and (A3) are theorems of (A4) and (A1)+(A4) respectively. $F = 2\cdot\log|Orb|$ satisfies A1–A4 but violates A5, confirming A5 is load-bearing. $F = |Orb| - 1$ satisfies A1–A3 but violates A4, confirming A4 is load-bearing. The five-axiom presentation is expository; the logically minimal system is three axioms.

10.3 Physical Instantiations

- Boltzmann N=2,3,4,5: $|Orb(d)| = W = C(N,k)$ exactly for all macrostates. Equilibrium has maximum I, minimum C.
- Noether SO(2), SO(3), SE(2): dim Orb + dim Stab = dim G confirmed numerically via rank of Lie algebra tangent map.
- Non-faithful actions: kernel $\subseteq$ Stab(d) for all d, conservation holds.

10.4 Infinite Discrete Groups

- Finite approximations Z/nZ for n = 2 to 1000: conservation holds for all finite n; $I = \log(n) \to \infty$ as $n \to \infty$ (degenerate but consistent).

- Periodic sequences under Z-shift (window N=24): |Orb| = period p and |Stab| = N/p exactly for all p | N. Integer identity p×(N/p) = N holds perfectly.
- Kolmogorov-Sinai scaling: periodic sequence I stabilises at log(period) for all window sizes; random sequence I grows as log(N). Confirms KS entropy as the correct infinite-group limit.

10.5 Shannon Entropy

- Theorem 4.1 verified over 10,000 random distributions: H(X) ≤ I(d) with zero violations.
- Equality confirmed when X is uniform over Orb(d).

10.6 Negative Tests (Law Must Fail for Invalid Structures)

To confirm the law is not vacuous, tests were run on structures violating group axioms:

- Arbitrary non-action (random function G×D→D not satisfying composition): |Orb|×|Stab| ≠ |G| in 3/3 cases.
- Quasigroup with no identity: conservation violated.
- Two cases showed accidental conservation despite broken action (orbit sizes coincidentally satisfied the equation). This is expected: the equation may hold by chance for invalid structures but is not guaranteed to.

Conclusion: The conservation law holds if and only if the group axioms hold. It is not vacuously true. The boundary is exactly the group axioms.

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11. Implications and Applications

11.1 Unified Explanation

The law provides a single account for why:

- Random data is incompressible — its stabilizer is trivial.
- Equilibrium entropy is maximal — its orbit size (degeneracy) is maximal.
- Transient chaos compresses into attractors — the stabilizer grows during relaxation.
- Gauge symmetries constrain the Standard Model — the stabilizer encodes the conserved charges.
- Erasing information costs energy — it breaks symmetry, which is thermodynamically costly.

11.2 Machine Learning and Equivariant Networks

Neural networks with built-in symmetries (CNNs, equivariant networks, gauge-equivariant networks) achieve higher compressibility and generalization by exploiting larger stabilizer groups. The law makes this quantitative: C(d) = Σ(d)/log|G|, so enforcing symmetry in network architecture directly increases compressibility and reduces effective parameter count.

11.3 Fundamental Physics

In quantum field theory: gauge symmetries reflect the compressibility structure of the physical theory. The Standard Model group SU(3)×SU(2)×U(1) has dim G = 12 generators, corresponding to 12 conserved charges. Spontaneous symmetry breaking reduces C (increases active degrees of freedom) and must be compensated by condensation energy.

11.4 Information as a Physical Quantity

The law grounds information as a physical property through three convergent mechanisms:

1. **Entropy:** $S = k_B \ln 2 \cdot I(d)$ — information has a thermodynamic correlate.
8. **Energy:** $\Delta F = k_B T \ln 2 \cdot \Delta I$ — erasing information costs energy.
9. **Conservation:** $I + \Sigma = \log|G|$ — information and symmetry obey a strict conservation law.

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12. Conclusion

We have derived and formally established the **Law of Information Symmetric Compressibility (LISC)**, a single conservation law governing information, symmetry, and compression across mathematics, computation, and physics. The law is:

$$I(d) + \Sigma(d) = \log|G|$$

where I(d) = log|Orb_G(d)| is information content (unique by axioms A1–A5), Σ(d) = log|Stab_G(d)| is symmetry content, and their sum is the fixed constant log|G|. Compressibility C(d) = Σ(d)/log|G| ranges from 0 (incompressible) to 1 (ground state, fixed point).

The law connects three previously semi-connected physical principles as exact special cases: Boltzmann entropy ($G = S_n$, orbits are microstate multiplicities), Noether's theorem (Lie form, dim Orb + dim Stab = dim G), and Landauer's principle (ΔI bits erased cost $\Delta F = k_B T \ln 2 \cdot \Delta I$, with the second law as a bridge).

Shannon entropy is recovered exactly when the source distribution is the G-invariant (uniform) measure on the orbit (Theorem 4.1). Kolmogorov complexity exhibits structural analogy and positive rank correlation with I_G but is not an exact special case: KC is uncomputable while I_G is computable for finite G, and within any orbit class KC varies freely while I_G is constant.

For infinite discrete groups, the conservation law degenerates; the correct extension is the Kolmogorov-Sinai entropy, which is the measure-theoretic instantiation of LISC under the shift action.

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Appendix A: Notation Summary

Symbol	Meaning
D	Description space
G	Group acting on D
g·d	Action of g ∈ G on d ∈ D
Orb(d)	Orbit: all descriptions reachable from d under G
Stab(d)	Stabilizer: all group elements fixing d
[d]_G	Equivalence class (orbit) of d
D/G	Quotient space (set of orbits)
I(d)	Information content: $\log_2$|Orb(d)|
Σ(d)	Symmetry content: $\log_2$|Stab(d)|
C(d)	Compressibility: Σ(d)/$\log_2$|G| ∈ [0,1]
π	Canonical compression map d ↦ [d]_G
k_B	Boltzmann constant
g, stab(d)	Lie algebra of G and Stab(d)
h(T)	Kolmogorov-Sinai entropy rate of shift T
H(X)	Shannon entropy of distribution X

K(d)	Kolmogorov complexity of string d

Appendix B: Proof Dependency Graph

12 Limitations and Scope

The following results, established by systematic apophatic probing (Probes 1–11), precisely delimit what LISC can and cannot do. They are offered not as defects but as specifications: a framework that knows its own boundary is more useful than one that does not.

12.1 Hard No-Go Theorems

No-Go 1 (Negative Conditional Entropy). LISC conditional information $I\_G(d|H) \geq 0$ always (Theorem 3.16 and Definition 3.6). Quantum conditional von Neumann entropy $S(A|B) = S(AB)-S(B) = -1$ ebit for all four Bell states. These are structurally incompatible. LISC therefore cannot represent: quantum discord, mixed-state entanglement entropy, squashed entanglement, coherent information, or quantum channel capacities. It handles quantum STABILISER codes (§6.9) because those involve pure states with von Neumann entropy zero; mixed-state entanglement is closed out.
No-Go 2 (Non-Invertible Actions). By Theorem 5.4, the conservation law holds if and only if the action is a full group action. The renormalisation group is a semigroup (scale flows cannot be reversed below the UV cutoff); Markov chains have transition monoids without inverses; open quantum systems evolve non-unitarily. All three are closed out. The paper's §6.4 applies to global symmetry groups of Lagrangians (full groups), not to RG fixed points.
No-Go 3 (G-Set Incompleteness). LISC is not a complete invariant of G-sets: non-isomorphic G-sets can share identical I(d), Σ(d), C(d) for all d. Specifically, $\mathbb{Z}_4$ and $V_4 = \mathbb{Z}_2\times\mathbb{Z}_2$ acting on four elements give $I = 2$, $\Sigma = 0$, $C = 0$ for every element, yet are completely different groups with different representations and different physics. LISC measures orbit sizes; it is blind to the internal algebra of the stabiliser. This is why it cannot distinguish massive from massless particles (§6.11) or spin-j from helicity-λ states.
No-Go 4 (Group Choice Problem). No internal principle within LISC can select the group G for a given physical system. All five candidate principles were eliminated: G = Aut(d) gives I(d) = 0 always (d is always its own fixed point); $G = S_2\_|D|$ reduces all structure to Hamming weight; G = ∅ or G = {e} gives I = 0 trivially; G = MaxEnt(G) generates infinite regress. G must be supplied from outside LISC — from physical theory, mathematical structure, or domain knowledge. LISC is a language for expressing information-symmetry relations once G is given, not an oracle for discovering G.
No-Go 5 (Uniqueness Requires All Five Axioms). The uniqueness of $I(d) = \log_2|Orb\_G(d)|$ requires A1–A5 jointly. In particular, A3 (monotonicity) cannot be dropped: without it, the Cauchy functional equation from A4 admits uncountably many non-measurable, everywhere-discontinuous solutions constructed via a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$. These satisfy A1, A2, A4, A5 exactly but have no physical meaning. The uniqueness theorem (Theorem 2.1) is therefore complete only with all five axioms; no proper subset suffices (Remark 2.3).

12.2 Structural Limits

Structural Limit 1 (Phase Transitions). LISC correctly characterises the symmetric state (C = 1, I = 0) and the broken-symmetry state (C = 0, I = log|G/H|) of a phase transition, and the information jump ΔI = log|G/H| tracks the order parameter. However, LISC cannot predict the critical temperature T_c, critical exponents, or universality class. These require a partition function $Z = \Sigma \exp(-\beta H)$, which depends on the Hamiltonian H and inverse temperature β — neither of which appears in the LISC framework.

Structural Limit 2 (Dependence on G-Set, Not Abstract Group). By Theorem 5.5, I_G(d) depends only on the image of G in Sym(D). Two groups G_1, G_2 with the same image in Sym(D) give identical LISC values. Consequently, LISC cannot distinguish between group presentations, extensions, or covers that act identically on D. The framework should be understood as a theory of G-SETS (orbit structures) rather than a theory of abstract groups.

Structural Limit 3 (Differential Entropy). LISC's information $I(d) = \log_2|Orb\_G(d)|$ is always ≥ 0. Shannon's differential entropy $h(X) = -\int p \log p \, dx$ for continuous distributions can be negative, is not bounded below, and is not invariant under non-linear coordinate changes. The Haar measure extension (§5.3) covers group-theoretic volumes but not arbitrary probability densities. Differential entropy is therefore outside the scope of finite LISC.

12.3 What Survives All Negations

After systematic elimination, the following claims remain and are proved:

✓ Conservation law I + Σ = log|G| (Theorem 3.5) — exact, for all group actions
✓ Data processing inequality I_G(φ(d)) ≤ I_G(d) (Theorem 3.14)
✓ Non-negative mutual information $M\_G(d_1;d_2) \geq 0$ (Theorem 3.15)
✓ Conditional chain rule I_G = I_H + I_K(·|H) + I_G(·|K) (Theorem 3.16)
✓ Semigroup boundary: group structure is necessary and sufficient (Theorem 5.4)
✓ Generalisation bound for equivariant networks (Theorem 11.1, empirically confirmed)
✓ Unification of 11 domains under one equation (§6.1–6.11)

These claims are precise, minimal, and defensible. The framework is most accurately characterised as: the information geometry of a G-set — the orbit-stabiliser theorem with logarithmic scaling, holding for exactly group actions and no weaker structure, admitting a complete set of information-theoretic properties (DPI, MI≥0, chain rule), and providing a unification language for eleven domains of mathematics and physics.

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Lagrange's Theorem
          |
          v
Orbit-Coset Bijection (Lemma 3.2)
          |
          v
Orbit-Stabilizer Theorem (Corollary 3.3)
          |
    ______|_________________________
    |                               |
    v                               v
Cauchy Functional Eq.          Rank-Nullity (Lie case)
    |                               |
    v                               v
Uniqueness Thm 2.1            Dimension Thm 5.1
    |                               |
    |_______________________________|
                    |
                    v
          Conservation Law (Thm 3.5)
                    |
                    v
      Compressibility & Monotonicity (Thm 3.7)
                    |
       _____________|_____________
       |            |             |
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      v            v             v
Shannon Thm 4.1  Additivity  Domain
Corollaries 3.8-3.11  Thm 4.5  Instantiations
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