Q1: Show that set of integers Z is not a group under the binary operation multiplication.

Justify each of the group properties that holds or not. Marks = 5

Group: Let G be a non-empty set. The ordered pair (G,*) is said to be a group if it satisfies the following Axioms

1) $a * b \in G \quad \forall a, b \in G$ (Closure Property)
2) $a * (b * c) = (a * b) * c \quad \forall a, b, c \in G$ (Associative Law)
3) For each $a \in G$ There exist an element $e \in G$ Known as the identity element Such that $a * e = e * a = a$
4) For each $a \in G$, there exist an element $a^{-1} \in G$ Known as the inverse of a Such that $a * a^{-1} = a^{-1} * a = e$

Solution:

$G = Z \quad , \quad * = .$

Is G is a Group or not Group under multiplication:

Closure Property: Multiplication of two integers is always an integer

$\Rightarrow \; m, n \in Z \quad \forall m, n \in Z$

Z is a closed under multiplication

The closure property Hold

Associative Law: $(m.n).l = m.(n.l) \;\; \forall m, n, l \in Z$

$\Rightarrow Z$ Is Associative under multiplication

Existence of Identity: $m.1 = m = 1.m \quad \forall m \in Z$

So, 1 is the multiplicative identity in Z

Inverse:- $Let \; m \in Z$

We know that if $m \in G$

$m \in G \;\exists\; n \in G$ Such that $a * b = b * a = e$

$$m.\frac{1}{m} = 1 = \frac{1}{m}.m \quad \text{If} \quad m \in Z$$

But $\frac{1}{m}$ is not integer, $\frac{1}{m}$ does not belong to Z

$(z,*)$ Inverse does not exist

So, $(z,*)$ is not a Group under multiplication.

The reason is that (Z, *) is not a group is that most of the elements do not have inverses. Furthermore, addition is commutative, so (Z, +) is an abelian group.