Review Day - Feb 15th
Test - Feb 20th

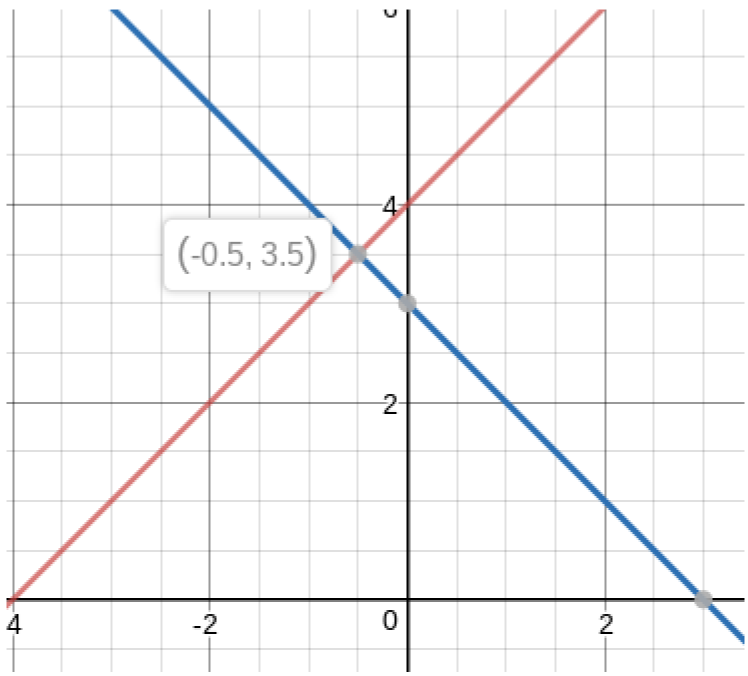
Link To Full Homework Solutions & Notes

Contents:

Lesson 0.2 Review of Solving Systems

By Graphing

- By graphing, you can get a general idea of a solution. However, it can be inaccurate and time consuming.
- If you're going to graph, using desmos is preferable than doing it by hand. Graphing by hand is imprecise and will provide an inaccurate solution.
- In Desmos, plot all equations to compare values, and click on the wanted point to see its values.
- What it might look like to solve by graphing:



- To solve a system by graphing you must find the y-intercept and the slope of each line and graph that. Extend the lines to use up the whole graph, when the two lines intersect at a point that is your solution.

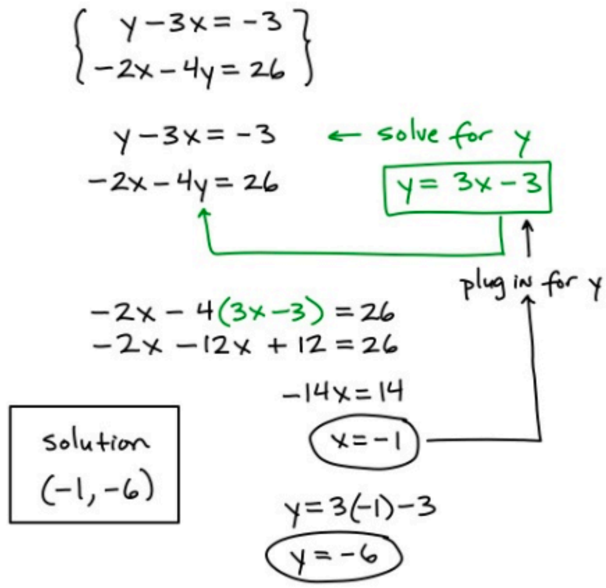
By Substitution

Khan Academy Lesson:
[Linear System: Substitution]

- Easiest when one variable is on it's own with no coefficient. If this is not the case, it can simply be isolated.
- Once isolated, the variable can then be plugged into the other equation, and be used to create an equation with only one variable.
- After isolating and solving the new one-variable equation, the found value can be plugged back into an original equation. Then, it can be used to solve for the other variable.

- Perfect for 2 missing variables with 2 given equations, but usable with 3-variable, 3-equation systems as well.

Example:



By Elimination

Khan Academy Lesson:

[Systems of equations with elimination]

- Used when the equation contains the same variable with the same coefficient to solve for another variable
- Also can be used when one equation can be easily manipulated to achieve a similar coefficient as another (ex $x - 4y = 3$ and $3x + 12y = 2$). Here, you can multiply the first equation by 3. Then, you can add or subtract one entire equation from the other, and in doing so eliminate y or x, respectively.
- Once a variable has been eliminated, you can isolate the remaining variable to solve for its value.
- When you know the value of one variable, you then substitute it back into one of the original equations to solve for the second value.
- Your final answer should be in coordinate format: (x value, y value).

Example:

$$\begin{Bmatrix} x + y = 2 \\ x - y = 14 \end{Bmatrix}$$

$$\begin{array}{r} x + y = 2 \\ x - y = 14 \\ \hline 2x \quad = 16 \end{array}$$

← eliminate the y variable by adding equations

$2x = 16$ ← solve for x

$x = 8$ ← use to find y

$$\begin{array}{r} x + y = 2 \\ 8 + y = 2 \\ -8 \qquad -8 \\ \hline y = -6 \end{array}$$

$(8, -6)$ solution

Lesson 0.3 Gauss Jordan Elimination - MATRICES! :)

The Format

- Serve to organize equation systems into arrangements that are simpler to understand and manipulate than normal equations. This is achieved by removing variables and focusing on coefficients.
- Example of equations converted into a matrix:
- Remember: These are not lines! These are PLANES!

$$2x + 3y + z = 10 \qquad x - y + 2z = 7 \qquad x + 2y - z = 1$$

$$\begin{array}{c} \\ R1 \\ R2 \\ R3 \end{array} \begin{array}{c} \begin{array}{ccc|c} x & y & z & = \end{array} \\ \left(\begin{array}{ccc|c} 2 & 3 & 1 & 10 \\ 1 & -1 & 2 & 7 \\ 1 & 2 & -1 & 1 \end{array}\right) \end{array}$$

- Here, columns each represent a variable or constant, and rows each represent one equation from the system. The vertical line between the third and fourth columns represents the "equals" signs dividing variables from constants across all equations.

Using Matrices

The rows of matrices, when used in the way we do, can be manipulated in a few ways:

- Each individual row can be multiplied or divided by a number to change its values.
- Each row can be added or subtracted from another row.
- Any combination of these two can be done, with multiples of rows being added or subtracted from other multiples of rows.
- Rows can have their locations swapped for easier interpretation.

Matrix row operation	Example
Switch any two rows	$\begin{bmatrix} 2 & 5 & 3 \\ 3 & 4 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 4 & 6 \\ 2 & 5 & 3 \end{bmatrix}$ (Interchange row 1 and row 2.)
Multiply a row by a nonzero constant	$\begin{bmatrix} 2 & 5 & 3 \\ 3 & 4 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 3 \cdot 2 & 3 \cdot 5 & 3 \cdot 3 \\ 3 & 4 & 6 \end{bmatrix}$ (Row 1 becomes 3 times itself.)
Add one row to another	$\begin{bmatrix} 2 & 5 & 3 \\ 3 & 4 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 5 & 3 \\ 3+2 & 4+5 & 6+3 \end{bmatrix}$ (Row 2 becomes the sum of rows 2 and 1.)

https://www.khanacademy.org/math/precalculus/precalc-matrices/elementary-matrix-row-operations/a/matrix-row-operations

Through these different manipulations, matrices are used to eliminate variables from equations and simplify them for finding solutions. This is done by manipulating rows until many of their variable coefficients become zero. Once a matrix is manipulated to a point known as Echelon form, it is possible to isolate and determine one variable, then substitute it into another simplified equation and find another variable, and finally sub both into the last equation and determine the third variable.

Example of Row Echelon Form:

	x	y	z		
R1	1	4	5	1	
R2	0	4	6	4	
R3	0	0	3	-1	→ Find z then use to find y

Note the equation with two zero coefficients and the equation with one zero coefficient.

Although Echelon form allows for easier solving than simple substitution or elimination, further matrix manipulation can create what is known as Row-reduced Echelon form.

Row-reduced Echelon Form- solves for the POI

This form simplifies the matrix to the point of displaying each variable's value without requiring any more solving after it.

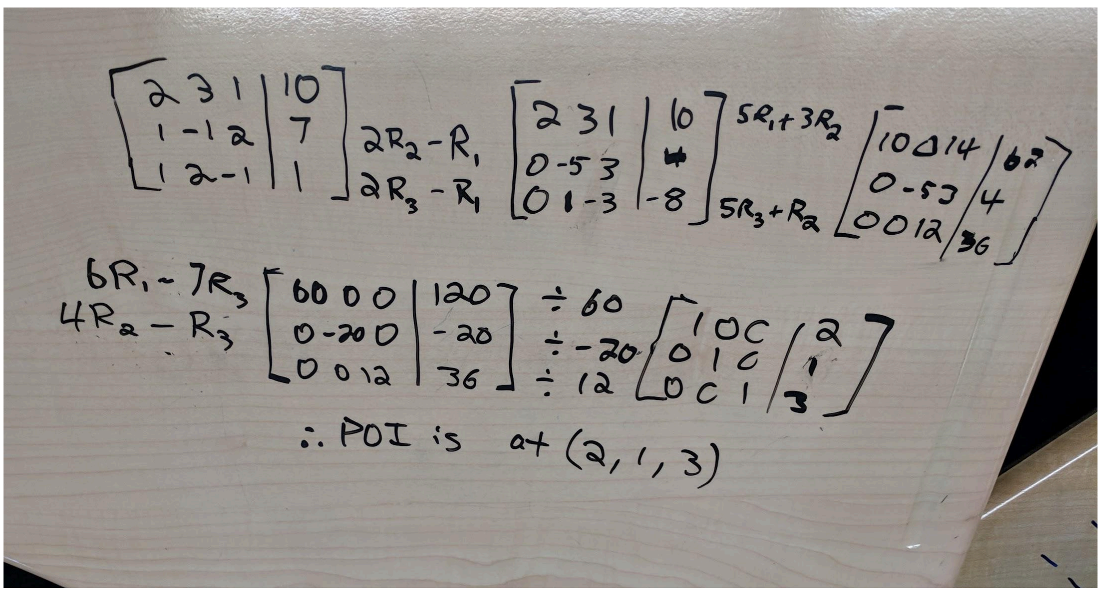
Example of Row-Reduced Echelon Form:

	x	y	z	
R1	1	0	0	1
R2	0	1	0	2
R3	0	0	1	-3

NOTICE: There is only one "1" in each row, and in each column!

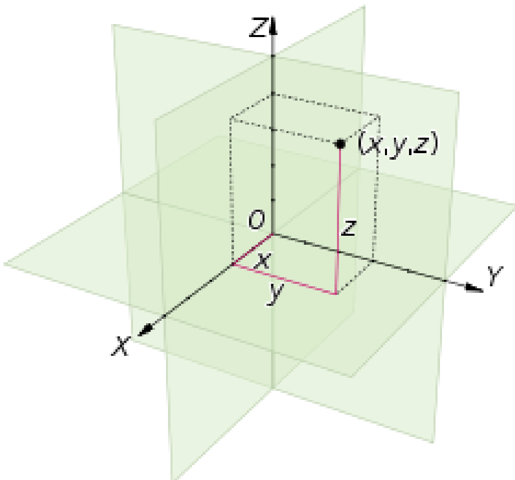
← This red box are the answers! There is a "1" which means (1)x = 1, (1)y = 2, and (1)z = -3

Example of Matrix Solved Using Reduced-Row Echelon Form:



Lesson 1.1 Plotting Points and Planes

Helpful Technology: 3D Graphing Calculator



(x, y, z) → ordered triples

Coordinate Planes

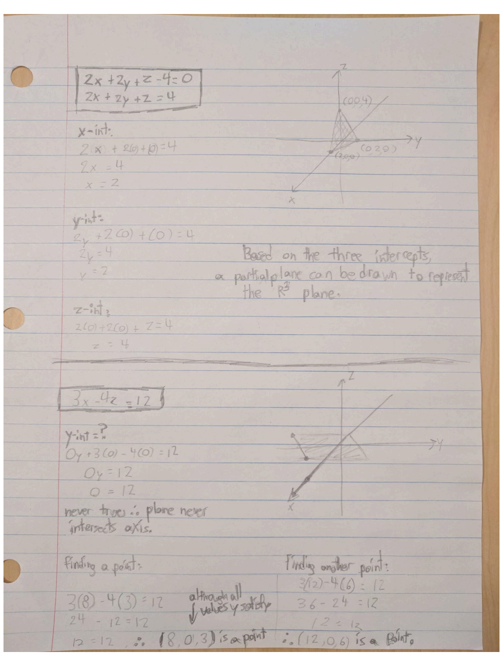
1. xy-plane contains x-axis and y-axis
2. yz-plane contains y-axis and z-axis
3. xz-plane contains x-axis and z-axis

*coordinate axes help determine coordinate planes

Finding points on planes allows for graphing to be done. This can be achieved in a few ways:

- Finding axis intercepts
- Finding specific points that satisfy the equation
- Determining the absence of axis intercepts

Example of plane graphed using determined intercepts (top), and example of plane graphed using specific points as well as an absent intercept:



(save picture to zoom in so it doesn't take up too much space)

Scenarios:

#1a: All 3 variables and D≠0

- Find each *x, y, x* intercept by solving!
- Plot all 3 points and connect!

Ex) 2x + y - 2z = 6

x-intercept: 2x + 0 - 2(0) = 6

x = 3

y-intercept: 2(0) + y - 2(0) = 6

y = 6

z-intercept: 2(0) + (0) - 2z = 6

z = -3

x: (3, 0 ,0)   y: (0,6,0)   z: (0,0,-3)

#1b: All 3 variables and D=0

- Plot one point at origin
- Pick 2 sets of planes, ex) xy and xz plane
- Find 2 sets of points by satisfying the equation

Ex) x + 4y + 3z = 0

(0,0,0) ← origin

xy: (-4) + 4(1) + 3(0) = 0

0 = 0

(-4,1,0) ← point on xy plane

xz: (-3) + 4(0) + 3(1) = 0

0 = 0

(-3, 0, 1) ← point on xz plane

Then graph all 3 points!

#2a: 2 variables and D≠0

- The plane is parallel to the missing plane!
- Find the 2 intercepts of the axes present by solving

Ex) x +2z = 10

x-intercept : x = 10

z-intercept : z = 5

#2b: 2 variables and D = 0

- Point at origin
- Intersects through missing plane
- Find one point on the present plane of the 2 axes

Ex) 5x - 2y = 0

(0,0,0) ← Origin

xy: 5(2) - 2(5) = 0

0=0
(2, 5, 0)
Note: Plot those 2 points, the origin and point picked. They will intersect through the z axis

#3a: 1 variable, D≠ 0
1 variable, D = 0
- Parallel to both the missing planes
- All points on this plane, the coordinate will be equal to D

y = 4
*All points on y coordinate is 4
*parallel to x and z axis

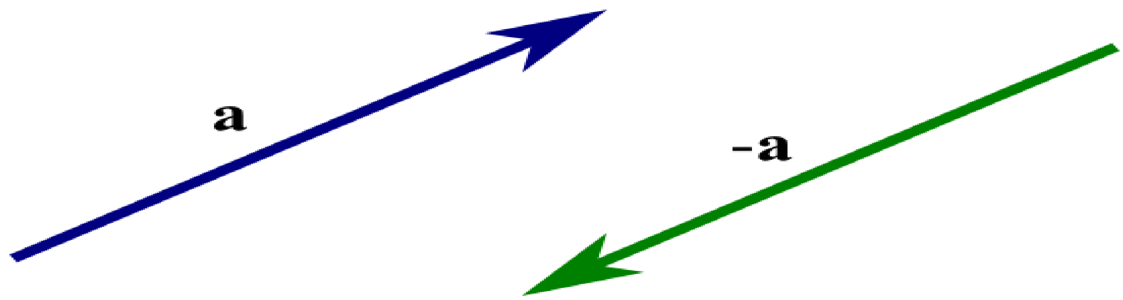
Lesson 1.2 Intro to Vectors

- Numbers with just magnitudes are called scalars.
  - Ex. temperature, time, distance
- Numbers with magnitude and direction are called vectors
  - Ex. displacement, velocity, acceleration, force
  - Magnitude can't be a negative on a vector

Vector Notation:
- Proper vector notation involves putting a half arrow on top of the name of the line to indicate it is a vector quantity
- The arrow of the vector diagram indicates direction of motion
- Vectors are named by indicating first the starting point and then the direction it moves in (ex if you start at point A and are moving towards point B, you name this vector AB)

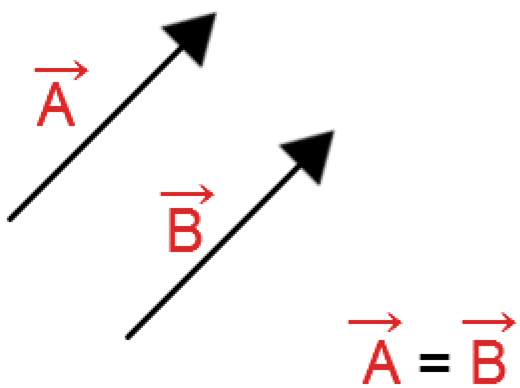
$$\vec{a} \quad \overrightarrow{AB}$$

Opposite Vectors:
- Vectors with the same MAGNITUDE but opposite directions are considered opposite vectors
- These lines must be parallel to each other, directed in opposite directions



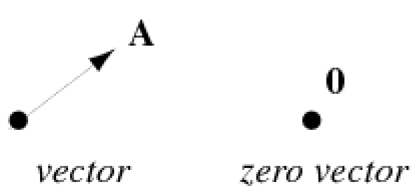
Equal Vectors:

- Two vectors are equal if (and only if) they are parallel to each other, pointing in the same direction and the magnitude of both vectors are equal
- Parallelism is needed to have both vectors in the same direction. If they were not parallel, they would end up crossing and not be travelling in the same direction.



Zero Vector:

- A vector with no magnitude or direction
- Written as $\vec{0}$
- Direction is undefined
- Magnitude is zero



Lesson 1.3 Vector Addition

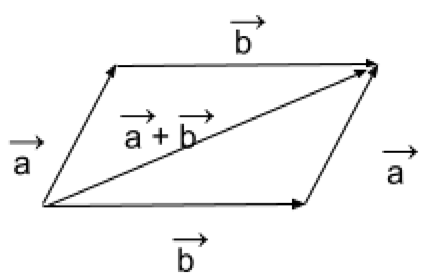
A useful and concise video lesson detailing the concepts to follow :)
[Adding and Subtracting Vectors]

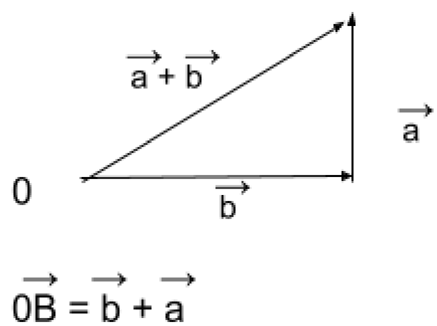
When vectors are added, new ones are created.

To add vectors, we draw them “tail to head”. That is, connecting to one another.
Drawing this produces a triangle, with the longest side being the resultant vector while the shorter ones are the added vectors! (this is a generalization, and assumes vectors added are more or less traveling in the same direction. Other configurations can break this rule.)

The parallelogram below represents two possible triangle representations of vector addition:

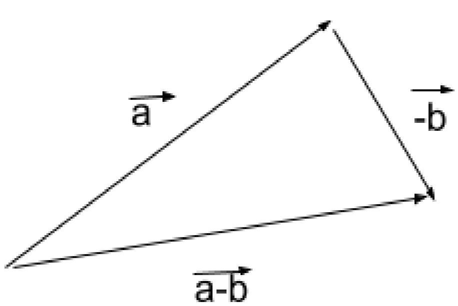
No matter the order of addition, the resultant vector will still be the same magnitude and direction in reference to origin



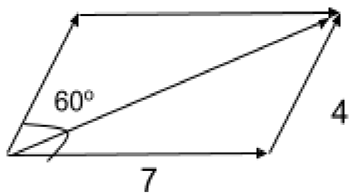


Subtracting vectors is done in the same way as addition. Instead of adding the second vector, its opposite is added to satisfy subtraction rules:

$$\vec{a} - \vec{b} = \vec{a} + \left(-\vec{b}\right)$$



Determining resultant vector magnitude and exact direction can be done with triangle properties, including cosine law and sine law as well as SOH CAH TOA.



The value and orientation of the diagonal resultant vector can be determined with a simple sequence of these tools.

- Since vectors are equivalent no matter where they are translated, they can be moved around to construct diagrams more conveniently for solving.
- The angle between the vectors of magnitude 7 and 4 can be found to be 120 degrees, by the use of supplementary angle properties.
- Using the angle of 120, cosine law can be used to determine the magnitude of the resultant.
    - $c^2 = a^2 + b^2 - 2ab\cos\theta$
- Knowing the resultant magnitude, sine law can be used to determine the resultant angle relative to the horizontal axis.
    - $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$
- Unit vectors are magnitude 1

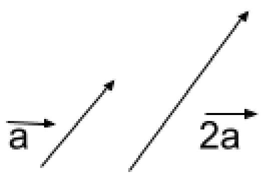
Lesson 1.4 Properties of Vectors

- (Basic Math Rules xD)

Scalar Multiplication can only happen with the magnitudes of the vectors and not the directions.

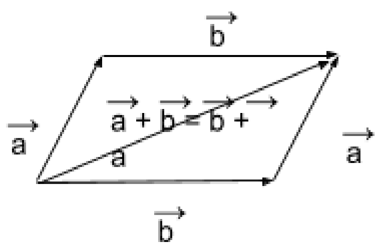
Any vector can be multiplied by any real scalar to “stretch” or “compress” it.

- ➔ If the scalar is positive, direction is unaffected.
- ➔ The vector becomes opposite if the scalar is negative
- ➔ The scalar stretches or compresses the vector based on its value(i.e If a vector is multiplied by a scalar of two the vector is stretched to two times its original length)



Important: all multiples of a vector are collinear (parallel).

Commutative Law: $a + b = b + a$

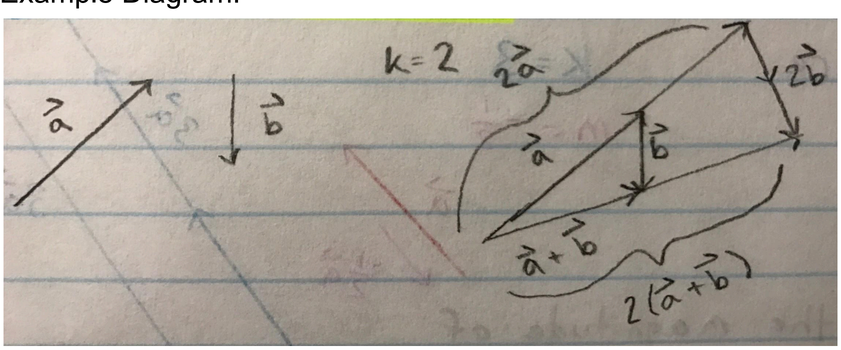


Distributive Law:

$$k(\vec{a}+\vec{b}) = k\vec{a} + k\vec{b}$$

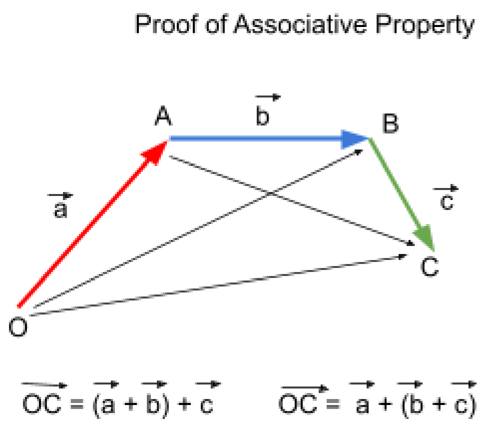
- Solved using FOIL

Example Diagram:



Associative Law:
(a+b) + c = a + (b + c)

- Diagrams can look different → resultant vector will be the same



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Associative Law of Scalars:
m(na) = (mn)a
https://www.youtube.com/watch?v=sQWM1TgI1yI
proof

Adding Zero:
a + 0 = a

Identity Law
1a = a

Distributive Law of Scalars
(m+n)a = ma + na
https://www.youtube.com/watch?v=sQWM1TgI1yI
proof

Unit Vectors:
Base vectors with magnitudes of 1. These allow for simpler and exact calculations and
Unit vector for "a" is $\frac{1}{|a|}$a.
Full solutions to worksheet

*****POSSIBLE TEST QUESTION → given property/equation, show that it is true*****

Lesson 1.5 Operations with Vectors in R^2 and R^3
R^2 → 2 dimensions, real numbers
R^3 → 3 dimensions, just add z coordinate

Vector Reference System Using Coordinates:
Algebraic Vectors = Vectors denoted by coordinate sets, on x-y plane (e.g., (4,1) (7,1))
This system allows for standardized vectors in reference to the landmarks of the cartesian plane

Position Vectors = Vectors starting from the origin, represented by only one set of coordinates due to second one being known as (0,0) (e.g., (7,6), or (5,6,7))
Starting all vectors from the origin allows for standardization. To achieve this:
- Subtract "Tail" coordinates of a geometric vector from its "Head" coordinates
- Difference of coordinates indicates the location of an algebraic (starting from the origin) vector's head
- $A\,(x1, y1) \quad B\,(x2, y2)$

Geometric Vectors = Vectors without coordinates
The vectors we used for properties lesson are examples of these :)

<u>Determining Magnitude of Position Vectors</u>

$$\text{Magnitude} = \sqrt{x^2 + y^2} \quad \text{(for 2D vectors)}$$

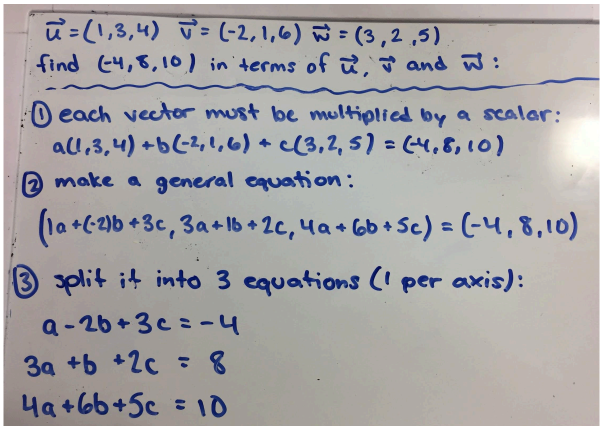
$$\text{Magnitude} = \sqrt{x^2 + y^2 + z^2} \quad \text{(for 3D vectors)}$$

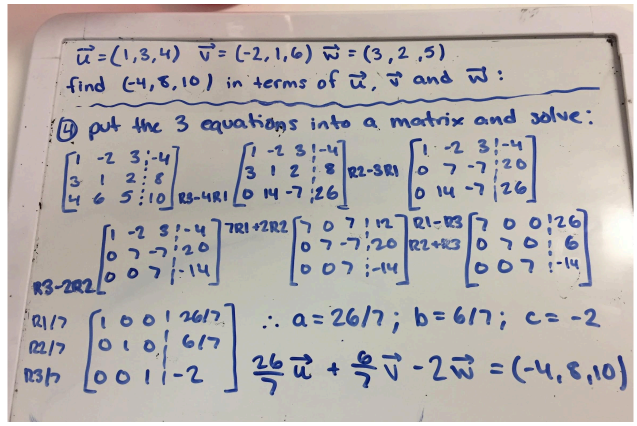
- Due to the single set of coordinates that represents a position vector, determining magnitude is a simple application of the distance formula we know for algebraic vectors.
- Angle (for the direction of the vector) can be determined through SOH CAH TOA

<u>Expressing a Position Vector Through Other, Non-parallel Vectors</u>

Interestingly, any one position vector can be expressed through other, non-parallel vectors. To do this, the use of matrices can be highly useful!

Example of vector expression problem:





Standard Basis Vectors:

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\overrightarrow{i}$ = x-axis

$\overrightarrow{j}$ = y-axis

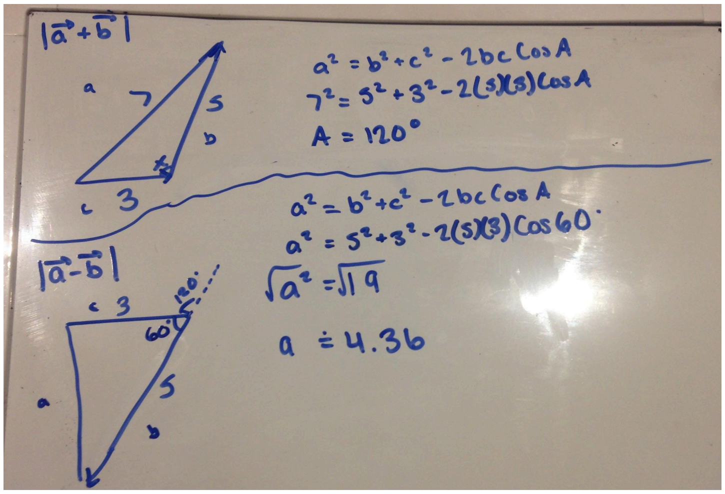
$\overrightarrow{k}$ = z-axis

Example:
Express the vector (4, -6, 9) as standard basis vectors.

4i - 6j + 9k = (4, -6, 9)

*****Try the last question from the homework!! It's pretty challenging and seems like something that might be a thinking question for the test!*****

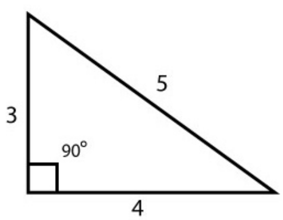
15. Given $|\vec{a}| = 3$, $|\vec{b}| = 5$, and $|\vec{a} + \vec{b}| = 7$, determine $|\vec{a} - \vec{b}|$.



Solving Vectors with Trigonometry

To solve some questions involving vectors, trigonometry is necessary to fill in the gaps. Here is some review to solve potential questions in the future:

Pythagorean Theorem: Only works on right triangles, the “c” value represents the hypotenuse, where “a” and “b” are either of the side lengths.



$c = \sqrt{a^2 + b^2}$

$c = \sqrt{3^2 + 4^2}$

$c = \sqrt{25}$

$c = 5$

Cosine Law:
When given either 3 side lengths, or 2 side lengths and an angle, Cosine Law can be used to solve for the remaining value (only on right triangles)

Given the same triangle, solve for the hypotenuse using this method:

$c^2 = a^2 + b^2 - 2ab\ cosC$

$c = \sqrt{3^2 + 4^2 - 2(3)(4)\ cos90}$

$c = \sqrt{9 + 16 - 0}$

$c = \sqrt{25}$
$c = 5$

SOH-CAH-TOA: This method can be used on any right triangle, it states that:

$sin\theta = \frac{Opp}{Hyp}$
$cos\theta = \frac{Adj}{Hyp}$
$tan\theta = \frac{Opp}{Adj}$

Application:
Having these three methods in your toolbelt will help after making a diagram to visualize a vector, rather than measuring it by hand. Keep in mind that all of these equations only work in degrees though!

I have created a unit summary for Unit 1 - Intro To Vectors

- **I hope it helps!**

Vectors Unit 1 Review- Intro to Vectors

Matricies

- coefficients of lines
- Coefficient matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 0 & 5 \end{bmatrix}$$

- Augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right]$$

- Row Echelon Form
  - └ rows w/ ∅ bottom
  - └ 1st not ∅ in row is to right of 1st not ∅ in previous row

3-D

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

- Sketching planes
  - └ Find intercepts
  - └ cartesian form

$$Ax+By+Cz+D=0$$

Vectors

- $\vec{v}$ or $\overrightarrow{AB}$
- Opposite vectors, opposite direction
- Equal vectors, same mag and ↑
- Zero vector, $\vec{\emptyset}$ mg=0 direction, undefined
- Adding/Subtracting
  - └ tip to tail
  - └ cos law $c^2=a^2+b^2-2ab\cos C$
  - └ $\vec{a}-\vec{b}=\vec{a}+(-\vec{b})$

Properties

1) Scalar Multiplication
   - └ $k\vec{a}$ is scalar multiple
     - $k=0$, $\vec{\emptyset}$
     - $k-1$, $-\vec{a}$
   - └ $\vec{a}$ and $k\vec{a}$ are collinear
2) Commutative Law
   - └ $\vec{a}+\vec{b}=\vec{b}+\vec{a}$
3) Associative Law
   - └ $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$
4) Distributive Law
   - └ $k(\vec{a}+\vec{b})=k\vec{a}+k\vec{b}$
5) Adding zero
   - └ $\vec{a}+\vec{0}=\vec{a}$
6) Identity law
   - └ $1\vec{a}=\vec{a}$
7) Associative law of Scalars
   - └ $m(n\vec{a})=(mn)\vec{a}$
8) Distributive Law of Scalars
   - └ $(m+n)\vec{a}=m\vec{a}+n\vec{a}$
9) Unit Vectors
   - └ vector w/ mag. of 1

$$\hat{v}=\frac{1}{|\vec{v}|}\cdot\vec{v}$$

Opperations w/ Vectors

- $\vec{v}$ between 2 points
  - └ $\overrightarrow{AB}=(x_2-x_1, y_2-y_1)$
  - └ x_2 y_2 are tip
- Position $\vec{v}$, start @ origin
  - └ w/ position $\vec{v}$
  - └ $l=\sqrt{x^2+y^2+z^2}$...
  - w/ Unit vector, multiply coordinates by $|\vec{v}|^{-1}$
- Forming a basis as combination of two non parallel vectors
- Standard basis
  - $\vec{i}, \vec{j}, \vec{k}$
  - x y z