**Analysis I**
**MAT320/MAT640**
**with Professor Sormani**
**Spring 2022**

**Lesson 3**
**Supremum, Infimum, and Continuoum Property**
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**Your work for today's lesson will go in a googledoc you create entitled MAT320S22-Lesson3-Lastname-Firstname with your last name and your first name. The googledoc will be shared with the professor sormanic@gmail.com as an editor. Put any questions you have inside your doc and email me to let me know it is there.**

**Today's lesson has four parts each with its own set of videos and notes. Do the classwork and all the homework as you proceed through the lesson. There are seven homework problems. This lesson expects you to already know how to do proofs of upper and lower bounds, proofs about maxima and minima, and proofs by contradiction. You may wish to scan through your notes on these topics before starting.**

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## Lesson 3 Part 1 Supremum

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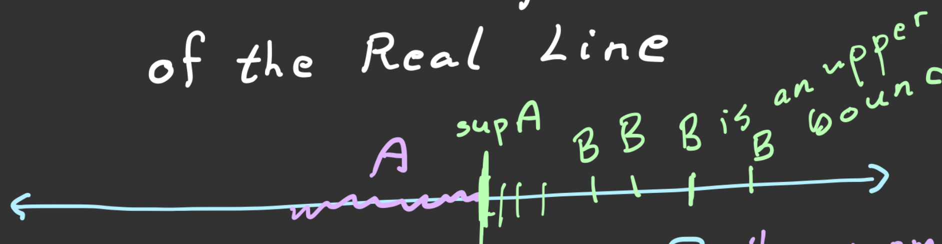
**Watch the Playlist MAT320S21-3-1to9 being sure to do the classwork yourself before watching the solutions.**

# Sup and Inf and the Continuum Property of the Real Line




Definition: Given a set $A \subset \mathbb{R}$ the supremum

$$\sup(A) = \min\{B \mid B \text{ is an upper bound for } A\}$$

The Continuum Property of the Real Line:

If a set, $A \subset \mathbb{R}$, has an upper bound then $A$ has a supremum in $\mathbb{R}$

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MAT320S21

Scratchwork MAT320S21

Note that the supremum is an upper bound and it is the smallest upper bound.

Given a set $A$, how do we prove $t = \sup(A)$?

Part I: Prove $t$ is an upper bound for $A$

Given any $x \in A$, show $x \leq t$.

Part II: Prove $t$ is the smallest upper bound

Prove this by contradiction

Assume on the contrary that there is a smaller upper bound $B < t$

So $\forall x \in A,\ x \leq B$. Contradict this somehow.

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MAT320S21

Scratchwork | MAT320S21

Defn $B$ is an upper Bound for $A$
$\forall x \in A \quad x \leq B$

Defn of supremum:
$\sup A = \min\{B \mid B \text{ is an upper bound for } A\}$
the smallest upper bound

Useful for Justifications

Rules of Inequalities

I $a \geq b$ AND $b \geq c \Rightarrow a \geq c$

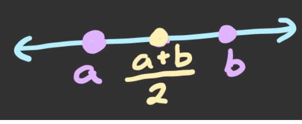
II $a \geq b$ AND $c \in \mathbb{R} \Rightarrow a+c \geq b+c$

III $a \geq b$ AND $c > 0 \Rightarrow ac \geq bc$

IV $a \geq b$ AND $c < 0 \Rightarrow ac \leq bc$

Average Theorem

$a < b \Rightarrow a < \frac{a+b}{2} < b$




Prove $\sup(A) = t$
where $A =$

Part I: Show $t$ is an upper bound
① Given $x \in A$

final $x \leq t$

Part II: Show $t$ is the smallest
① Assume on the contrary that
$\exists$ a smaller upper bound $B < t$.

$\otimes$ Something.

Defn $B$ is an upper Bound for $A$
$\forall x \in A \quad x \leq B$

Defn of supremum:
$\sup A = \min\{B \mid B \text{ is an upper bound for } A\}$
the smallest upper bound

Useful for Justifications

Rules of Inequalities

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Average Theorem

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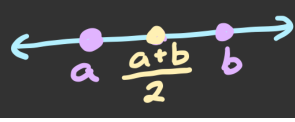



Prove $\sup(A) = t$
where $A =$

Part I: Show $t$ is an upper bound

① Given $x \in A$

⋮

final $x \leq t$

Upper bound proof like last week

Part II: Show $t$ is the smallest

① Assume on the contrary that $\exists$ a smaller upper bound $B < t$.

name the smaller upper bound

②

③

⋮

⊗ Something.

Defn $B$ is an upper Bound for $A$
$\forall x \in A \quad x \le B$

Defn of supremum:
$\sup A = \min\{B \mid B \text{ is an upper bound for } A\}$
the smallest upper bound

Useful for Justifications

Rules of Inequalities

I $a \ge b$ AND $b \ge c \Rightarrow a \ge c$

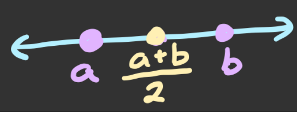
II $a \ge b$ AND $c \in \mathbb{R} \Rightarrow a+c \ge b+c$

III $a \ge b$ AND $c > 0 \Rightarrow ac \ge bc$

IV $a \ge b$ AND $c < 0 \Rightarrow ac \le bc$

Average Theorem

$a < b \Rightarrow a < \frac{a+b}{2} < b$




Prove $\sup(A) = t = 5$
where $A = (3,5)$

Part I: Show $t$ is an upper bound

① Given $x \in A = (3,5)$ — ① Given

② $3 < x < 5$ — ② By defn $(a,b)$

final $x \le 5$ — final defn of $\le$

Done with Part I

Part II: Show $t$ is the smallest

① Assume on the contrary that $\exists$ a smaller upper bound $B < 5$

② $\forall x \in A = (3,5) \quad x \le B$ — ① defn of upper bound

Try for a contradiction

⊗ something.

Try this! Classwork!

I $a \geq b$ AND $b \geq c \Rightarrow a \geq c$

II $a \geq b$ AND $c \in \mathbb{R} \Rightarrow a+c \geq b+c$

III $a \geq b$ AND $c > 0 \Rightarrow ac \geq bc$

IV $a \geq b$ AND $c < 0 \Rightarrow ac \leq bc$

Average Theorem

$$a < b \Rightarrow a < \frac{a+b}{2} < b$$

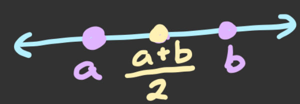
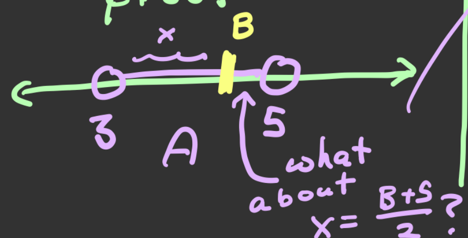



Diagram for the proof:




Part II: Show t is the smallest

(1) Assume on the contrary that $\exists$ a smaller upper bound $B < 5$

(2) $\forall x \in A = (3,5)$ $x \leq B$ (1) defn of upper bound

(3) Use $x = \frac{B+5}{2}$ to reach a contradiction

Try for a contradiction

$\otimes$ Something.

Try this! classwork!

I $a \geq b$ AND $b \geq c \Rightarrow a \geq c$

II $a \geq b$ AND $c \in \mathbb{R} \Rightarrow a+c \geq b+c$

III $a \geq b$ AND $c > 0 \Rightarrow ac \geq bc$

IV $a \geq b$ AND $c < 0 \Rightarrow ac \leq bc$

Average Theorem

$a < b \Rightarrow a < \frac{a+b}{2} < b$

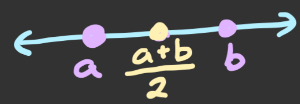
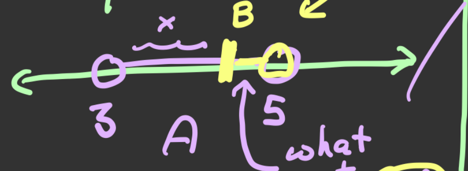



Diagram for the proof:




what about $y = \frac{B+5}{2}$?

## Part II: Show 5 is the smallest

(1) Assume on the contrary that $\exists$ a smaller upper bound $B < 5$ — (1) Indirect hyp.

(2) $\forall x \in A = (3,5)$ $x \leq B$ — (2) defn of upper bound

(3) $4 \leq B$ — (3) $4 \in (3,5)$

(4) Let $y = \frac{B+5}{2}$ then $\boxed{B < y} < 5$ — (4) $a < b \Rightarrow a < \frac{a+b}{2} < b$

(5) $3 < 4 \leq B < y < 5$ — (5) by step 3 and step 4

(6) $y \in (3,5)$ — (6) defn of $(a,b)$

(7) $y \leq B$ — (7) by step 2 taking $x=$

$\otimes$ $B < y$ in step 4

Thus our indirect hypothesis is false. So 5 is the smallest upper bound. Part II is done.

**Defn** $B$ is an upper Bound for $A$
$\forall x \in A \quad x \leq B$

**Defn** of supremum:

$\sup A = \min\{B \mid B \text{ is an upper bound for } A\}$

the smallest upper bound

**Useful for Justifications**

Rules of Inequalities

I $a \geq b$ AND $b \geq c \Rightarrow a \geq c$

II $a \geq b$ AND $c \in \mathbb{R} \Rightarrow a+c \geq b+c$

III $a \geq b$ AND $c > 0 \Rightarrow ac \geq bc$

IV $a \geq b$ AND $c < 0 \Rightarrow ac \leq bc$

Average Theorem

$a < b \Rightarrow a < \frac{a+b}{2} < b$

$a \quad \frac{a+b}{2} \quad b$

Prove $\sup(A) = 4$ where $A = [2, 4]$

Part I: Show 4 is an upper bound

Classwork

Part II: Show 4 is the smallest

Classwork

Defn $B$ is an upper Bound for $A$
$\forall x \in A \quad x \leq B$

Defn of supremum:
$\sup A = \min\{B \mid B \text{ is an upper bound for } A\}$
the smallest upper bound

## Useful for Justifications

Rules of Inequalities

I $a \geq b$ AND $b \geq c \Rightarrow a \geq c$
II $a \geq b$ AND $c \in \mathbb{R} \Rightarrow a+c \geq b+c$
III $a \geq b$ AND $c > 0 \Rightarrow ac \geq bc$
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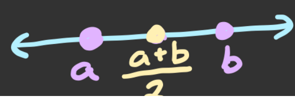
Average Theorem
$a < b \Rightarrow a < \frac{a+b}{2} < b$




Prove $\sup(A) = 4$
where $A = [2,4]$

Part I: Show 4 is an upper bound

① Given $x \in [2,4]$ — ① Given
② $2 \leq x \leq 4$ — ② by defn $[a,b]$
final $x \leq 4$ — final by step 2

Part II: Show 4 is the smallest

① Assume on the contrary that there is a smaller upper bound $B < 4$ — ① Ind H/P
② $- - - - - -$ — ③ Defn upper bound

Continue as Class work.

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SCREEN RECORDING now

Screen Recording video saved to Photos

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I $a \geq b$ AND $b \geq c \Rightarrow a \geq c$

II $a \geq b$ AND $c \in \mathbb{R} \Rightarrow a+c \geq b+c$

III $a \geq b$ AND $c > 0 \Rightarrow ac \geq bc$

IV $a \geq b$ AND $c < 0 \Rightarrow ac \leq bc$

Average Theorem

$$a < b \Rightarrow a < \frac{a+b}{2} < b$$

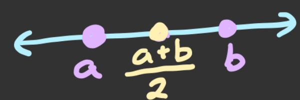
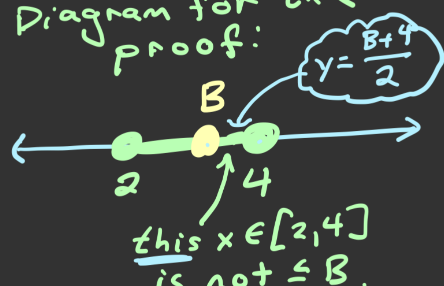



Diagram for the proof:

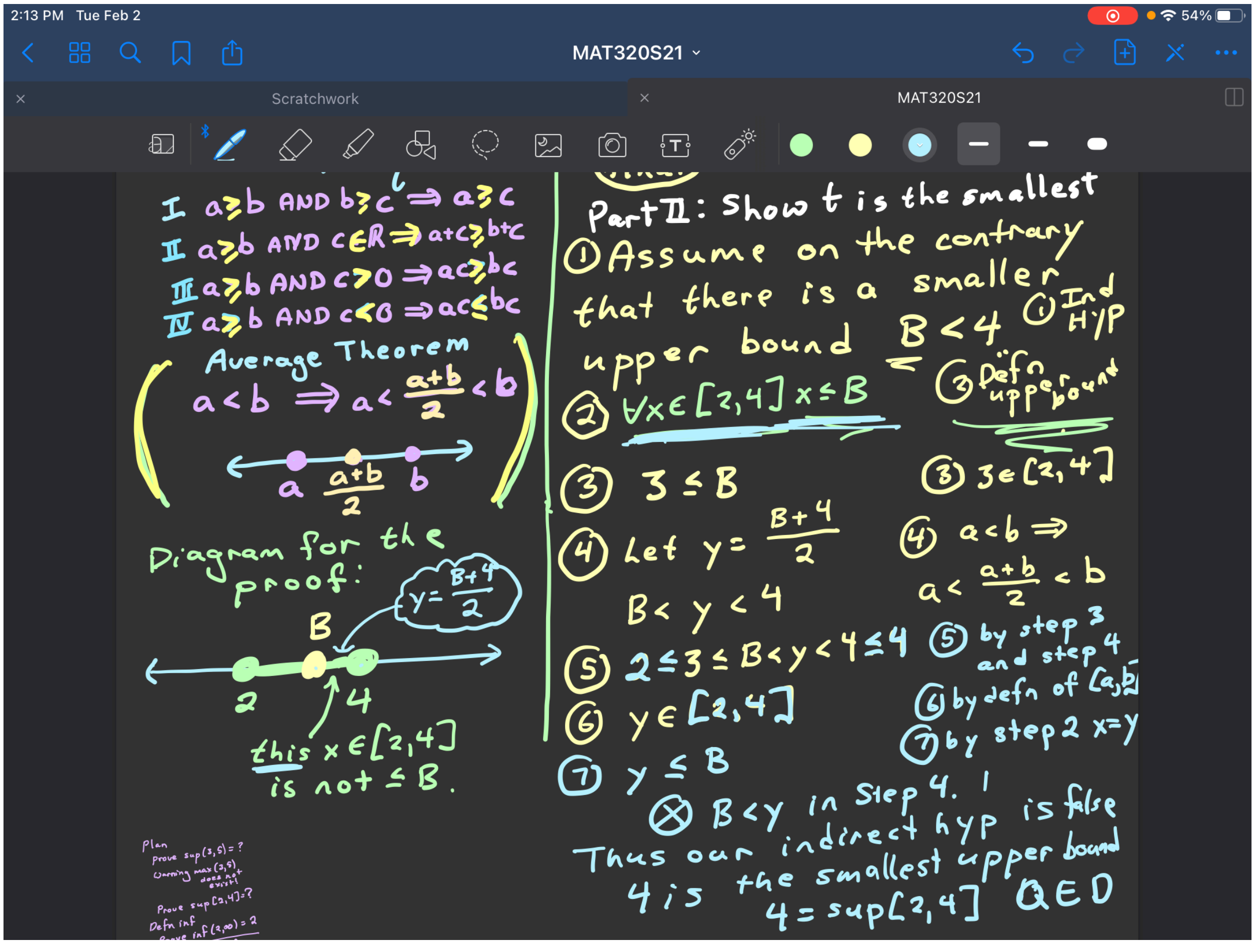



Part II: Show 4 is the smallest

(1) Assume on the contrary that there is a smaller upper bound $B < 4$ (1) Ind H/P

(2) $\forall x \in [2,4]\ x \leq B$ (3) Defn upper bound

Try for classwork




**Be sure to complete the classwork above and replay videos as needed until it is well understood before continuing.**

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## Lesson 3 Part 2 Infimum

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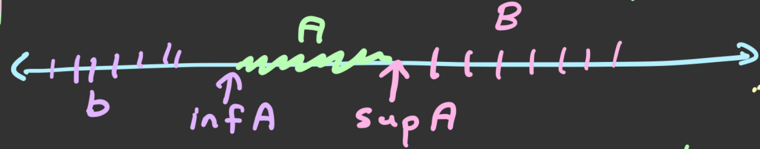
**Watch the [Playlist MAT320S21-3-10-15] being sure to do the classwork yourself before watching the videos.**

# Supremum and Infimum

**Defn:** $\sup(A) = \min\{B \mid B \text{ is an upper bound for } A\}$




**Defn:** $\inf(A) = \max\{b \mid b \text{ is a lower bound for } A\}$

Classwork Show $\inf(2,4) = 2$

Classwork Show $\inf[5,\infty) = 5$

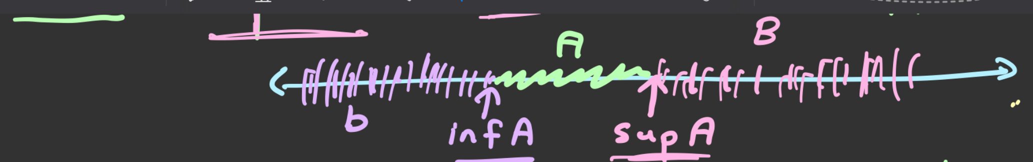
# Supremum and Infimum

**Defn:** $\sup(A) = \min\{B \mid B \text{ is an upper bound for } A\}$


**Defn:** $\inf(A) = \max\{b \mid b \text{ is a lower bound for } A\}$

Classwork Show $\inf(2,4) = 2$

Classwork Show $\inf[5,\infty) = 5$

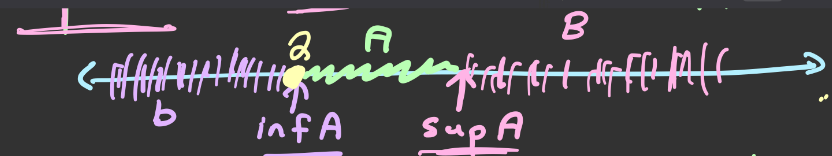



**Defn:** $\inf(A) = \max\{\, b \mid b \text{ is a lower bound for } A \,\}$

- Classwork Show $\inf(2,4) = 2$

  Part I $2$ is a lower bound for $(2,4)$

  Part II $2$ is the biggest lower bound.




**Defn:** $\inf(A) = \max\{b \mid b \text{ is a lower bound for } A\}$

- Classwork Show $\inf(2,4) = 2$

Part I $2$ is a lower bound for $(2,4)$

Part II $2$ is the biggest lower bound.

[ Part I: Given: $x \in (2,4)$ Show $2 \le x$ ] ← Defn of lower bound

Proof: (easy so you do yourself)

Part II: $2$ is the biggest lower bound for $(2,4)$

① Assume on the contrary that fill in.

Part II: 2 is the biggest lower bound for $(2,4)$

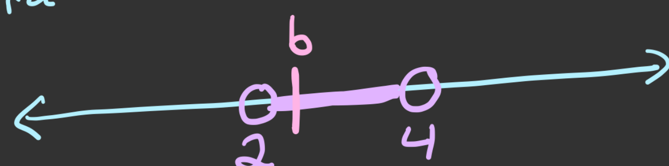
① Assume on the contrary that there is a bigger lower bound $b>2$

① Ind. Hyp.

② $\forall x \in (2,4)\ b \leq x$

② defn of lower bound

Draw to Plan Proof




can you find an $x$ in $(2,4)$ smaller than $b$?

$b \leq x$
$x \geq b$

Contradiction

Call this bad point $y$ and then get a ⊗

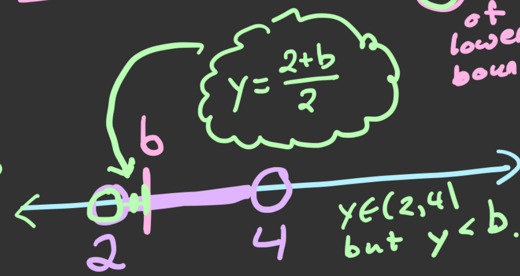
Part II: 2 is the biggest lower bound for $(2,4)$

① Assume on the contrary that there is a bigger lower bound $b>2$ — ① Ind. Hyp.

② $\forall x\in(2,4)\ \ b\le x$ — ② defn of lower bound

③ $b\le 3$ — ③ $3\in(2,4)$

④ Let $y=\frac{2+b}{2}$ so $2<y<b$ — ④ $a<b \Rightarrow a<\frac{a+b}{2}<b$

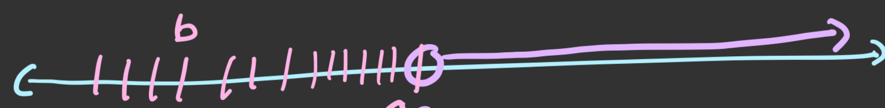



⑤ $2<y<b\le 3<4$ — ⑤ by steps 3 and 4

⑥ $y\in(2,4)$ — ⑥ by defn of interval

Try to write the Contradiction yourself.

Classwork $\inf(2,\infty) = 2$




max of the lower bounds is $2$ = infimum.

Part I ?

Part II ?

pause + try

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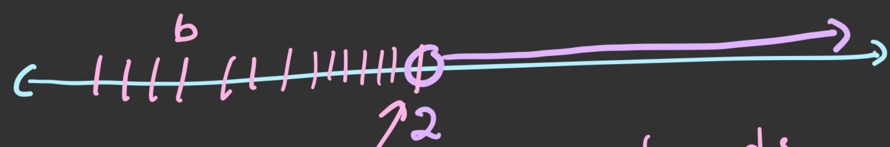
Classwork $\inf(2,\infty) = 2$




max of the lower bounds is $2$ = infimum.

Part I Show $2$ is a lower bound for $(2,\infty$

Given

Show

so easy 2 lines

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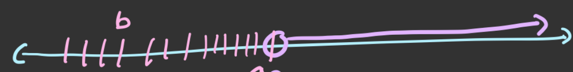
Classwork $\inf(2,\infty) = 2$




max of the lower bounds is $2$ = infimum.

Part I Show $2$ is a lower bound for $(2,\infty)$

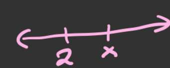
$(2,\infty) = \{x \mid x > 2\}$

Given $x \in (2,\infty)$

Show $2 \le x$

| | |
|---|---|
| ① $x \in (2,\infty)$ | ① Given |
| ② $2 < x$ | ② Defn of $(a,\infty)$ |
| ③ $2 \le x$ | ③ Defn of $\le$ |

Done w/ Step I





Part II Show $2$ is the biggest lower bound for $(2,\infty)$

① Assume on the contrary ____ . Try!

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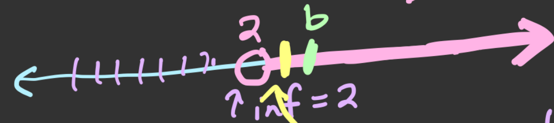


Part II Show 2 is the biggest lower bound for $(2,\infty)$

① Assume on the contrary that there is a bigger lower bound $b>2$. — ① Indirect Hypothesis

② $\forall x \in (2,\infty)\ b \le x$ — ② defn of lower bound

③ $5 \le x$ — ③ $5 \in (2,\infty)$

What point in $(2,\infty)$ is smaller than $b$ In our picture?

$$y = \frac{2+b}{2}$$

Finish the proof!


inf = 2

Part II Show 2 is the biggest lower bound for $(2,\infty)$

① Assume on the contrary that there is a bigger lower bound $b>2$. — ① Indirect Hypothesis

② $\forall x \in (2,\infty)\ b \leq x$ — ② defn of lower bound

③ $5 \leq x$ — ③ $5 \in (2,\infty)$

$$y = \frac{2+b}{2}$$

What point in $(2,\infty)$ is smaller than $b$ In our picture?

Finish the proof!

$\boxed{\text{HW1}}$ Prove $\sup(-\infty, 5) = 5$

$\boxed{\text{HW2}}$ Prove $\inf[4, 8] = 4$

Questions like this will be on Exam I.

**The above homeworks 1-2 may be done after watching the rest of the videos..**

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## Lesson 3 Part 3 Max vs Sup

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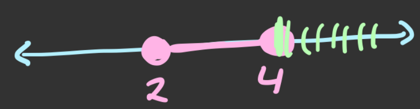
**Watch the [Playlist MAT320S21-3-16-18]**

# Max vs Sup




$A=[2,4]$

· $\text{Max}(A)$ = is an upper bound for A that is in A

Here $\text{Max}(A)=4$ because $\forall x\in[2,4]\ x\leq 4$ and $4\in[2,4]$

$\text{Sup}(A)$ = smallest upper bound

Here $\text{Sup}(A)=4$ because 4 is an upper bound and 4 is the smallest.

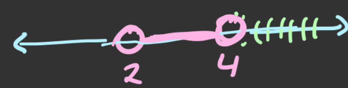
Continuoum Property of the Reals

Any set A with an upper bound has sup A.

but A may not have a max.

# Max vs Sup

New $A = (2,4)$




Max(A) = is an upper bound for A that is in A

No Max

Here ~~Max(A) = 4 because $\forall x \in$~~ $[2,4]$ $x \le 4$ ~~and $4 \in$~~ $[2,4]$

Sup(A) = smallest upper bound

Here Sup(A) = 4 because 4 is an upper bound and 4 is the smallest.

Continuoum Property of the Reals

Any set A with an upper bound has sup A.

but A may not have a max.

Write a proof that $(2,4)$ has no max.

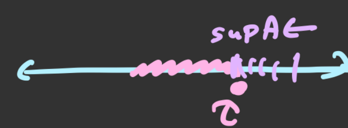
We know $\sup(2,4)$ exists by Continuoum Prop.

**Hint:** For this classwork above you need to use proof by contradiction and then the definition of max.   See [Lesson 2 video 20] after trying.

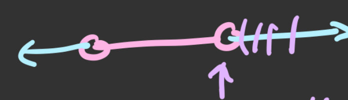


Theorem if max $A$ exists then $\max A = \sup A$.


Theorem if $\sup A \in A$ then max $A$ exists.




Careful $A = (2, 4)$ has no max



$\sup A = 4$
notice $\sup A \notin A$

**Theorem** if $\max A$ exists
then $\max A = \sup A$.


**Theorem** if $\sup A \in A$
then $\max A$ exists.


Careful $A = (2,4)$
has no max

$\sup A = 4$
notice
$\sup A \notin A$

**Theorem** if $\max A$ exists
then $\underline{\max A} = \sup A$.

we are proving
$\sup A = t$
$\max A = t$

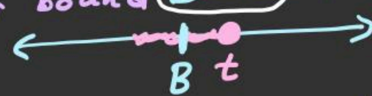
Part I: Show $t = \max A$ is an upper bound.

① $\max(A)$ is an upper bound — ① defn of max

Done with Part I

Part II: Show $t = \max A$ is the smallest upper bound.

① Assume on the contrary there is an upper bound $B < t = \max A$ — ① Indirect Hyp.




② $\forall x \in A \;\; x \le B$ — ② Defn upper bound

③ $t = \max A$ is in $A$ — ③ by defn of max

④ $\boxed{t \le B}$ $\otimes$ $B < t$ in Step 1 — ④ by Step 3

Done with Part II

The Theorem is proven QED

$\boxed{\text{HW3}}$ Show if $\min A$ exists
then $\min A = \inf A$

$\boxed{\text{HW4}}$ Find a set $A$ with an inf but no min

Theorem if $\sup A \in A$ then $\max A = M = \sup A$

Proof Part I $M$ is an upper bound $\Big\}$ Defn of Max

Part II $M \in A$

Part I follows immediately for $M = \sup A$

① $M$ is an upper bound ① defn of sup

done with Part I

Part II follows from given $M = \sup A \in A$.

done with Part II

① $M \in A$ ① Given $M = \sup A$ is in $A$

QED

Hint

$\boxed{\text{HW5}}$ Prove if $\inf A \in A$ then $\min A = \inf A$

Find a set $A \subset \mathbb{R}$ where $\inf A \notin A$ and $\min A$ does not exist.

**Above we have HW3, HW4, and HW5. Don't forget HW1 and HW2 in Part 2 as well. If you are tired you may wish to do the last part on another day.**

**Hint for HW5:**
**Given: m=inf A and m in A**
**Show: m=min A**

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## Lesson 3 Part 4 Sup and Inf of functions

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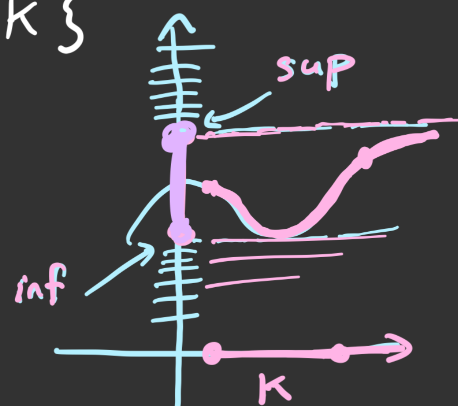
**Watch the [Playlist MAT320S21-3-19-22]**

$$\sup_K f(x) = \sup \{ f(x) \mid x \in K \}$$

$$\inf_K f(x) = \inf \{ f(x) \mid x \in K \}$$

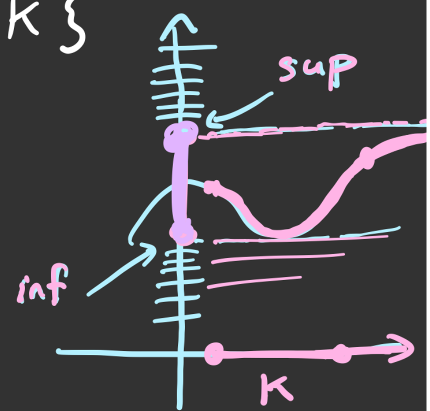


$$\sup_K f(x) = \sup\{f(x) \mid x \in K\}$$

$$\inf_K f(x) = \inf\{f(x) \mid x \in K\}$$

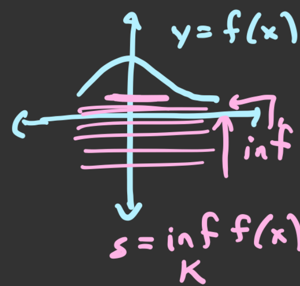
Thm: $\sup_K -f(x) = -\inf_K f(x)$

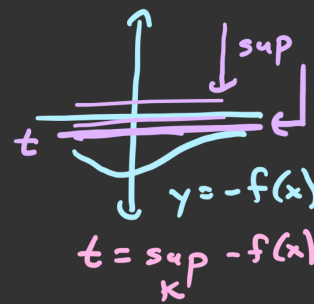
Thm: $\sup_K (f(x)+g(x)) \leq \sup_K f(x) + \sup_K g(x)$



Thm: $\underbrace{\sup_K -f(x)}_{t} = -\underbrace{\inf_K f(x)}_{s}$

Show $t = -s$



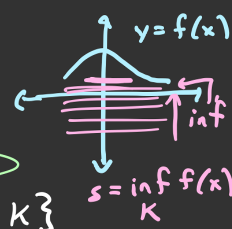



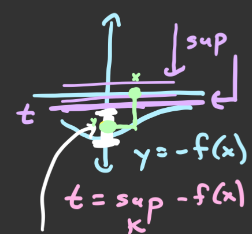

Prove this!

Think about how to do this.

Thm: $\underbrace{\sup_K -f(x)}_{t} = -\underbrace{\inf_K f(x)}_{s}$

Show $t = -s$




Prove $t = \sup\{-f(x) \mid x \in K\}$

Part I: $t$ is an upper bound for $A = \{-f(x) \mid x \in K\}$

Part II: $t$ is the smallest upper bound.

Proof of Part I: Given $y \in A$ Show $y \le t$.

| | |
|---|---|
| ① Given any $y \in A$ | ① Given |
| ② $y = -f(x)$ for some $x \in K$ <br> $\exists x \in K$ s.t. $y = -f(x)$. | ② Defn of $A$ |
| ③ $s \le f(u)\ \forall u \in K$ | ③ $\inf_K f(x) = s$ and defn of inf; $s$ is a lower bound |
| ④ $s \le f(x)$ | ④ by $x \in K$ in step 2 |
| ⑤ $-s \ge -f(x)$ | ⑤ $a \le b$ + $c = -1 < 0 \Rightarrow ac \ge bc$ |
| final $y \le t$ | final sub $y = -f(x)$ from Step 2, sub $t = -s$. |

Proof of Part II: Show $t=-s$ is the smallest upper bound for $A=\{-f(x) \mid x \in K\}$

① Assume on the contrary that $\exists$ a smaller upper bound $B \leq t = -s$

① Indirect Hypothesis

②

Use that $s = \inf_{K} f(x)$.

Complete this proof for $\boxed{HW6}$ CHALLENGE!

⊗ Contradiction

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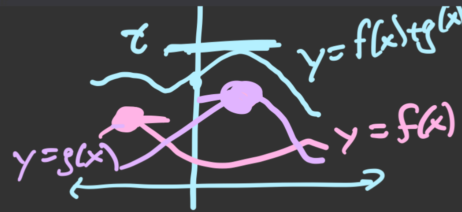
MAT320S21

Scratchwork | MAT320S21

Thm: $\underbrace{\sup_K (f(x)+g(x))}_{t} \le \sup_K f(x) + \sup_K g(x)$

This is also a challenging homework $\boxed{HW7}$




$$t = \sup_K (f(x)+g(x)) = \sup\{f(x)+g(x) \mid x \in K\}$$

Show $t \le \sup_K f(x) + \sup_K g(x)$

To do this use the defn of sup says $t$ is the smallest upper bound.

It will be enough to show $\sup_K f(x) + \sup_K g(x)$ is an upper bound.

**Above we have the extra challenging homework problems HW6 and HW7. This homework is required, not extra credit.**