Ex 1 Determine the average rate of change of $f(x) = x^3 - x$ over the interval $[1, 3]$.

$$\text{average rate of change} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{f(3) - f(1)}{3 - 1}$$

$$= \frac{24 - 0}{2} = \frac{24}{2} = \boxed{12}$$

Try: $f(x) = x^3 + 1$ over

a) $[2, 3]$ b) $[-1, 1]$

a) $$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a} = \frac{f(3) - f(2)}{3 - 2} = \frac{(27 + 1) - (8 + 1)}{3 - 2}$$

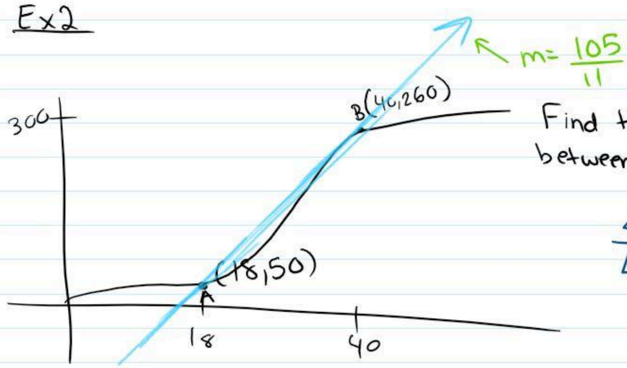
$$= \frac{28 - 9}{1} = \boxed{19}$$

b) $$\frac{f(1) - f(-1)}{1 - (-1)} = \frac{(1^3 + 1) - ((-1)^3 + 1)}{2}$$

$$= \frac{2}{2} = \boxed{1}$$

Recall: A secant line is a line that intersects the line at two places.
The slope of the secant line is the average rate of change between the two points.

Ex2



Find the average rate of change between points A and B.

$$\frac{\Delta y}{\Delta x} = \frac{260-50}{40-18} = \frac{210}{22} = \boxed{\frac{105}{11}}$$

A tangent line is a line that intersects a curve at exactly one point.

The slope of the tangent line is the instantaneous rate of change at a point.

Ex3

Find the slope of the parabola $y = x^2$ at the point $P(2,4)$. Write an equation to the tangent line at this point

Note: a tangent line is a secant line as the two points become infinitely close.
So, in general if one point is x, the other point is $x+h$

Step 1: use def. of instantaneous speed with function

$y = x^2$ $\quad \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ * this is the same as average rate *

$\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$\lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$\lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$\lim_{h \to 0} 2x + h = 2x$$

* This can be used to find slope of the tangent at any x of $y = x^2$ *

Step 2: use x value to find slope

$x = 2$ so $m = 2x = 2(2) = 4$

slope of tangent line at $x=2$ of $y = x^2$

Step 3: create equation of line

$y - y_1 = m(x - x_1)$ $\quad m = 4 \quad P(2, 4)$

$$\left.\begin{array}{l} y - 4 = 4(x-2) \checkmark \\ y - 4 = 4x - 8 \\ y = 4x - 4 \checkmark \end{array}\right\} \text{either works}$$

Try: $f(x) = \frac{1}{x-1}$ at $x=2$

Determine the equation of the tangent line at $x=2$.

1) use limit!

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\lim_{h\to 0} \frac{\frac{1}{x+h-1} - \frac{1}{x-1}}{h}$$

$$\lim_{h\to 0} \frac{\frac{1}{x+h-1}\frac{(x-1)}{(x-1)} - \frac{1}{x-1}\frac{(x+h-1)}{(x+h-1)}}{h}$$

$$\lim_{h\to 0} \frac{\frac{x-1-(x+h-1)}{(x+h-1)(x-1)}}{h} \cdot \frac{\frac{1}{h}}{\frac{1}{h}}$$

$$\lim_{h\to 0} \frac{\cancel{x}-\cancel{1}-\cancel{x}-h+\cancel{1}}{(x+h-1)(x-1)h}$$

$$\lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}(x+h-1)(x-1)}$$

$$\lim_{h\to 0} \frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)(x-1)} = \frac{-1}{(x-1)^2}$$

2) $x=2$

$$\frac{-1}{(2-1)^2} = \frac{-1}{1^2} = -1$$

slope at $x=2$

3) $y-y_1 = m(x-x_1)$

need to find the point when $x=2$

$f(2) = \frac{1}{2-1} = \frac{1}{1} = 1$ $(2,1)$

$y-1 = -1(x-2)$

$y-1 = -x+2$

$y = -x+3$ ✓

Slope of a curve at a point

The slope of the curve $y = f(x)$ at the point $P(a, f(a))$ is the number $m = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

The tangent line to the curve at P is the line through P with this slope.

Ex 4 Let $f(x) = \dfrac{1}{x}$

a) find the slope of the curve at $x = a$

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} = \lim_{h \to 0} \frac{\frac{a}{(a+h)a} - \frac{a+h}{a(a+h)}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{a-(a+h)}{a(a+h)}}{\frac{h}{1}} \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h} \cdot a(a+h)}$$

$$= \lim_{h \to 0} \frac{-1}{a(a+h)} = \frac{-1}{a^2}$$

$a(a+h)$ $\qquad$ a

b) Where does the slope equal $-\frac{1}{4}$?

The slope equals $-\frac{1}{4}$ when $-\dfrac{1}{a^2} = -\dfrac{1}{4}$

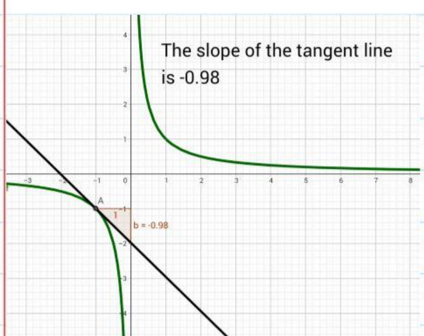
$$-a^2 = -4$$
$$a^2 = 4$$
$$a = \pm 2$$

Explination: The curve has a slope of $-\frac{1}{4}$ at points $(2, \frac{1}{2})$ and $(-2, -\frac{1}{2})$.

$x = 2$
$f(2) = \dfrac{1}{(2)} = \dfrac{1}{2}$

$x = -2$
$f(-2) = \dfrac{1}{(-2)} = -\dfrac{1}{2}$

c) What happens to the tangent to the curve at the points $(a, \frac{1}{a})$ for different values of a?



$m = -\dfrac{1}{a^2}$ is always negative. Also, as a approaches $\pm\infty$, m approaches 0, so the graph gets increasingly horizontal.

As a approaches 0 from either direction the tangent gets increasingly steeper because m is approaching $-\infty$.

Try: For $y = x^2+2$ at $x = a$

a) $m = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{[a+h]^2+2-(a^2+2)}{h} = \lim_{h\to 0} \frac{\cancel{a^2}+2ah+h^2+\cancel{2}-\cancel{a^2}-\cancel{2}}{h}$

$= \lim_{h\to 0} \frac{h(2a+h)}{h}$

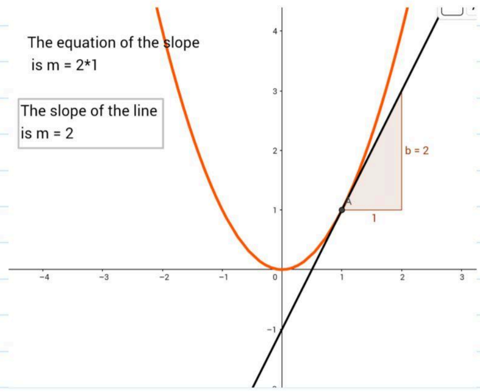
$= \lim_{h\to 0} 2a+h$

$= \boxed{2a}$

b) As a approaches 0 from either side, m approaches "0" so becomes increasingly horizontal.

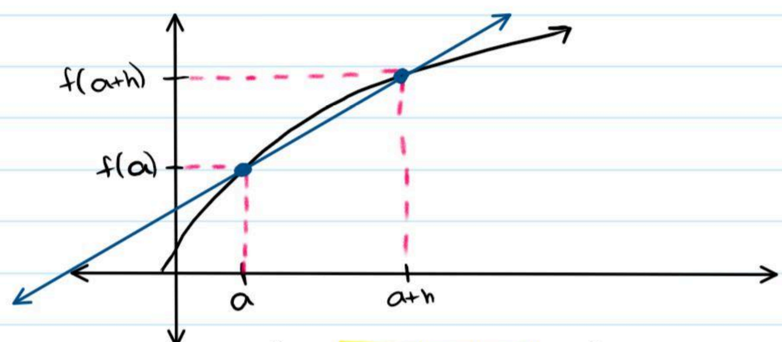
As a approaches ∞, m becomes increasingly vertical with a positive slope.

As a approaches $-\infty$, m becomes increasingly vertical with a negative slope.



Difference Quotient

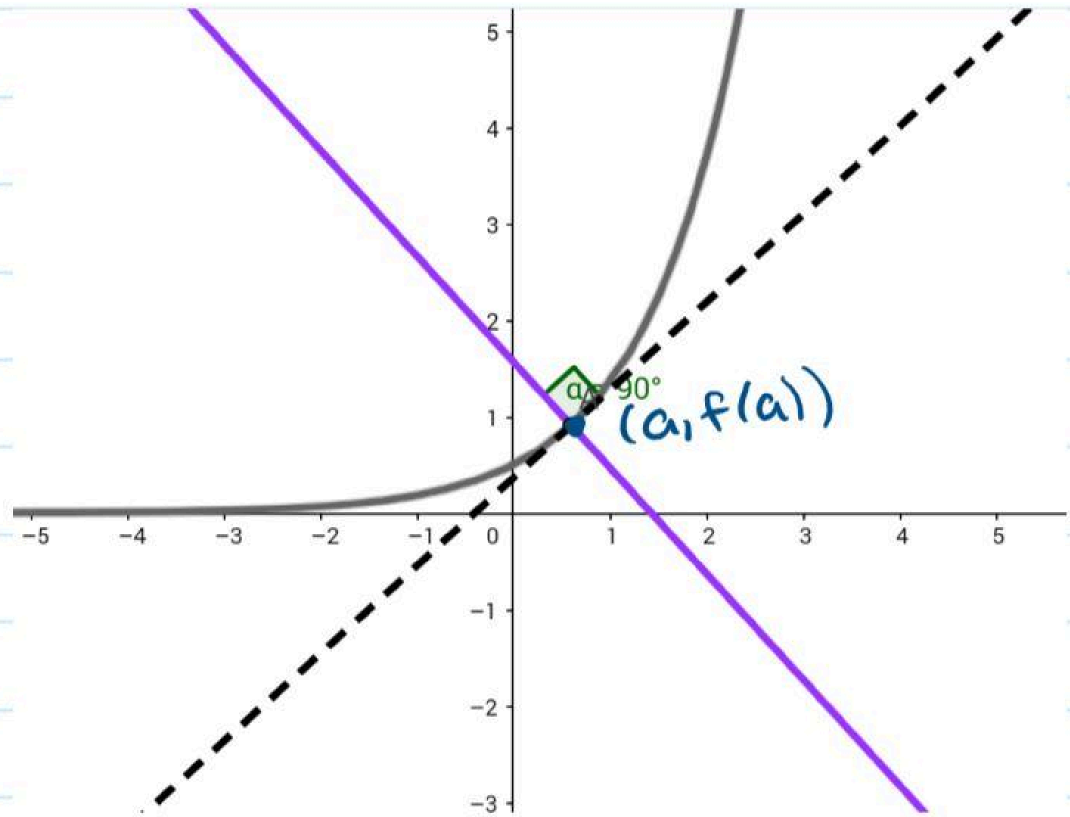
$$\frac{f(a+h)-f(a)}{h}$$



The difference quotient represents the secant line of a curve from the point $(a, f(a))$ to $(a+h, f(a+h))$. Which is also the average rate of change from $x=a$ to $x=a+h$.

Normal Line

The line normal to a curve is the line perpendicular to the tangent at that point



(a, f(a))
α = 90°

Ex 5 : Finding a Normal Line

Write an equation for the normal to the curve $f(x) = 4-x^2$ at $x=1$.

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{4-(x+h)^2-(4-x^2)}{h} = \lim_{h\to 0} \frac{4-(x^2+2xh+h^2)-4+x^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{4}-\cancel{x^2}-2xh-h^2-\cancel{4}+\cancel{x^2}}{h}$$

$$= \lim_{h\to 0} \frac{h(-2x-h)}{h}$$

$$= \lim_{h\to 0} -2x-h = \boxed{-2x}$$

Use $x=1$ to find slope of tangent

$m = -2x = -2(1) = -2$

So, $m=-2$ is the slope of the tangent line, and $\perp m = \frac{1}{2}$

is the slope of the normal line

at the point $(1,3)$
we have:

$f(x) = 4-x^2$
$f(1) = 4-(1)^2$
$= 3$
$(1,3)$

$y-3 = \frac{1}{2}(x-1)$
$y-3 = \frac{1}{2}x - \frac{1}{2}$
$y = \frac{1}{2}x - \frac{1}{2}+3$
$\boxed{y = \frac{1}{2}x + \frac{5}{2}}$

Try: Determine the normal of the function $y = \frac{1}{x-1}$ at $x = 2$
(look at example 3)

$$m = \frac{-1}{(x-1)^2} = \frac{-1}{(2-1)^2} = \frac{-1}{1^2} = -1$$

$\perp m = -1$

so slope of normal is 1

at $x = 2$ the point is $(2, 1)$

$$y - 1 = 1(x-2)$$
$$y - 1 = x - 2$$
$$\boxed{y = x - 1}$$

Recall: The Difference Quotient gives average rate of change
i.e. → average speed

The instantaneous rate of change or instantaneous speed is given by:

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

Ex 6: Recall, a rock falling from rest is given by the equation

$$y = 16t^2$$

Find the speed of a falling rock at $t = 1$ seconds

$$\lim_{h\to 0} \frac{f(t+h)-f(t)}{h} = \lim_{h\to 0} \frac{16(t+h)^2 - 16t^2}{h}$$

$$= \lim_{h\to 0} \frac{16(t^2+2th+h^2) - 16t^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{16t^2} + 32th + 16h^2 - \cancel{16t^2}}{h}$$

$$= \lim_{h\to 0} \frac{h(32t + 16h)}{h}$$

$$= \lim_{h\to 0} 32t + 16h$$

$$= 32t$$

now let $t = 2$

$32(2) = 64$

The rock speed at the istant $t=1$ was 32 ft/sec.

Try: Freefall on Mars is $y = 1.86t^2$

A rock dropped from 200 meters. Find the rocks speed at $t = 1$.

$$\lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

$$= \lim_{h \to 0} \frac{1.86(t+h)^2 - 1.86(t)^2}{h}$$

$$= \lim_{h \to 0} \frac{1.86(t^2 + 2th + h^2) - 1.86t^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{1.86t^2} + 3.72th + 1.86h^2 - \cancel{1.86t^2}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(3.72t + 1.86h)}{\cancel{h}}$$

$$= \lim_{h \to 0} 3.72t + 1.86h = 3.72t$$

use $t = 1$

$$3.72(1) = \boxed{3.72 \text{ m/sec}}$$