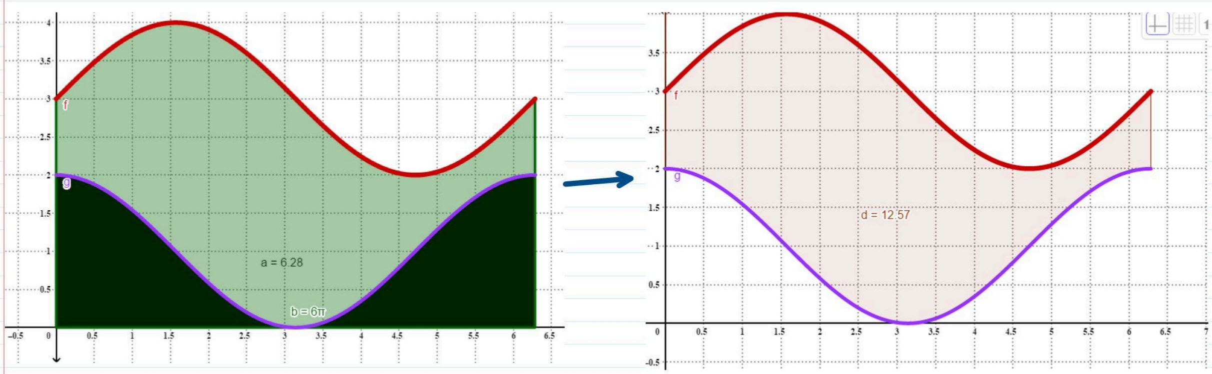


Area Between Curves

If f and g are continuous functions with $f(x) \geq g(x)$ on [a,b], then the area between the curves $y=f(x)$ and $y=g(x)$ from a to b is the integral of $[f-g]$ from **a** to **b**:

$$\int_a^b (f(x)-g(x))\,dx$$

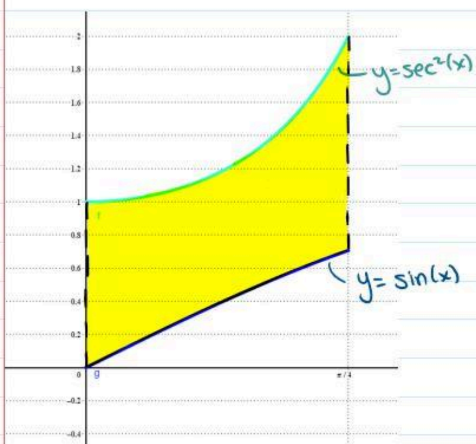
$$\int_a^b (\text{"top - bottom"})\,dx$$

Ex 1

Determine the area of the region between $y = \sec^2(x)$ and $y = \sin(x)$ on $[0, \frac{\pi}{4}]$

Step 1:

Graph the curves to find their relative positions in the plane.



Notice that $y = \sec^2(x)$ is always above the graph $y = \sin(x)$

Step 2

Write an integral and solve

$$A = \int_a^b f(x) - g(x)\,dx$$

$$A = \int_0^{\frac{\pi}{4}} \sec^2(x) - \sin(x)\,dx$$

$$A = \left[\tan(x) + \cos(x)\right]\Big|_0^{\frac{\pi}{4}}$$

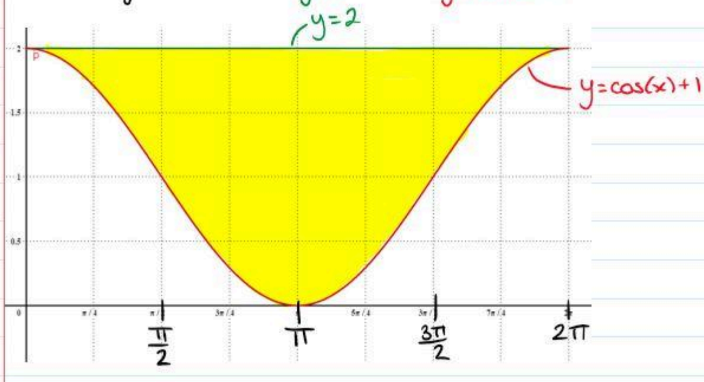
$$A = \left(\tan\left(\tfrac{\pi}{4}\right) + \cos\left(\tfrac{\pi}{4}\right)\right) - \left(\tan(0) + \cos(0)\right)$$

$$A = \left(1 + \tfrac{\sqrt{2}}{2}\right) - (0 + 1)$$

$$A = 1 + \tfrac{\sqrt{2}}{2} - 1$$

$$\boxed{A = \tfrac{\sqrt{2}}{2} \text{ units squared}}$$

Try: Determine the area of the region bounded by the curves $y = 2$ and $y = \cos(x) + 1$ on $[0, 2\pi]$



$$\int_0^{2\pi} (2) - (\cos(x) + 1)\,dx$$

$$= \int_0^{2\pi} 1 - \cos(x)\,dx$$

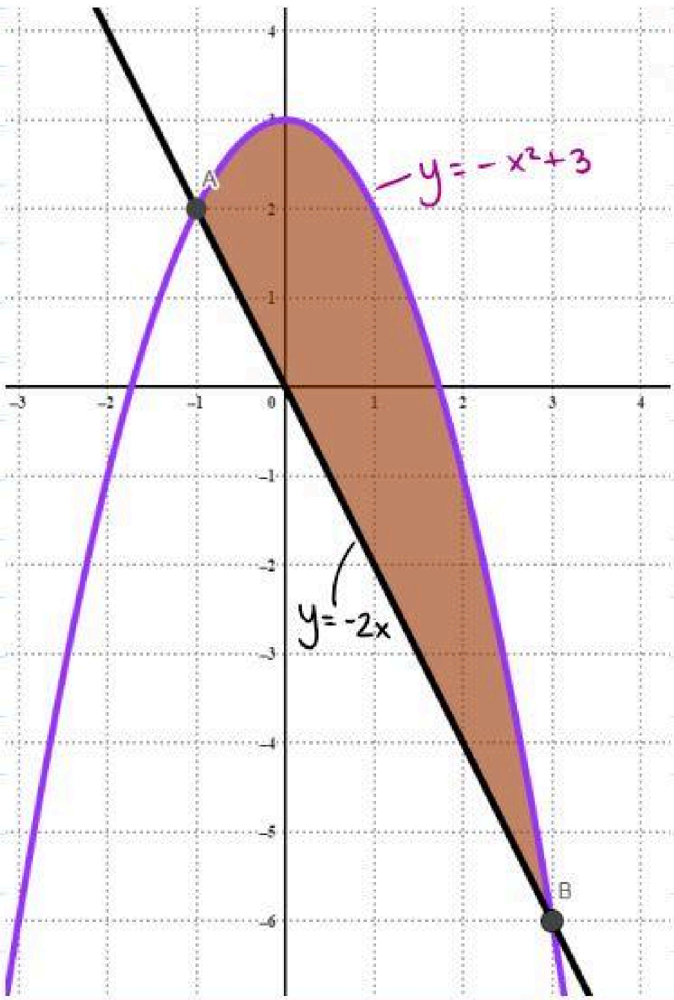
$$= \left[x - \sin(x)\right]\Big|_0^{2\pi}$$

$$= \left(2\pi - \sin(2\pi)\right) - \left(0 - \sin(0)\right)$$

$$= 2\pi$$

Areas Enclosed by Intersecting Curves

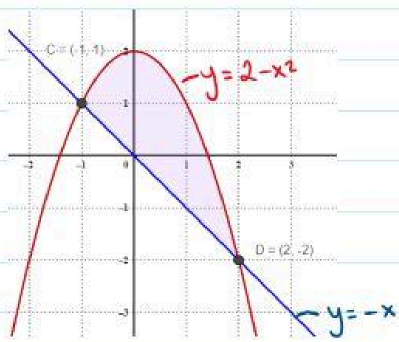
When a region is enclosed by intersecting curves, the points of intersection give the limits of integration



Ex 2

Determine the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$

Step 1: Graph the curves



Step 2: Determine the intersection

either algebraically or using a calculator

set functions equal to each other & solve

$y = 2 - x^2 \qquad y = -x$

$$2 - x^2 = -x$$
$$0 = x^2 - x - 2$$
$$0 = (x-2)(x+1)$$
$$\underbrace{x = 2 \quad x = -1}_{\text{so these are the}}$$

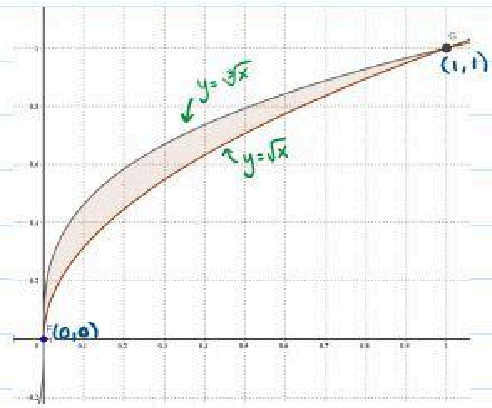
Step 3: set up the integral & solve

$$\int_{-1}^{2} \left(2 - x^2 - (-x)\right)dx = \int_{-1}^{2} (2 + x - x^2)\,dx$$

$$= \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]\Bigg|_{-1}^{2}$$
$$= \left[2(2) + \frac{(2)^2}{2} - \frac{(2)^3}{3}\right] - \left[2(-1) + \frac{(-1)^2}{2} - \frac{(-1)^3}{3}\right]$$
$$= \left[4 + \frac{4}{2} - \frac{8}{3}\right] - \left[-2 + \frac{1}{2} + \frac{1}{3}\right]$$
$$= \left[\frac{24}{6} + \frac{12}{6} - \frac{16}{6}\right] - \left[\frac{-12}{6} + \frac{3}{6} + \frac{2}{6}\right]$$
$$= \left[\frac{20}{6}\right] - \left[\frac{-7}{6}\right]$$
$$= \frac{20}{6} + \frac{7}{6}$$
$$= \frac{27}{6}$$
$$= \frac{9}{2} \text{ units squared}$$

Try: Determine the area of the region between the curves $x = y^3$ and $x = y^2$

Hint: solve for y



$\sqrt[3]{x} = \sqrt[3]{y^3}$ $\qquad$ $\sqrt[2]{x} = \sqrt[2]{y^2}$

$y = \sqrt[3]{x}$ $\qquad$ $y = \pm\sqrt{x}$

$$\sqrt[3]{x} = \sqrt{x}$$

$$(x^{1/3})^6 = (x^{1/2})^6$$

$$x^2 = x^3$$

$$0 = x^3 - x^2$$

$$0 = x^2(x-1)$$

$$\underbrace{x = 0 \quad x = 1}_{\text{limits of integration}}$$

$$\int_0^1 \left[(x^{1/3}) - (x^{\frac{1}{2}})\right] dx$$

$$\left[\frac{3}{4}x^{4/3} - \frac{2}{3}x^{3/2}\right]\Big|_0^1$$

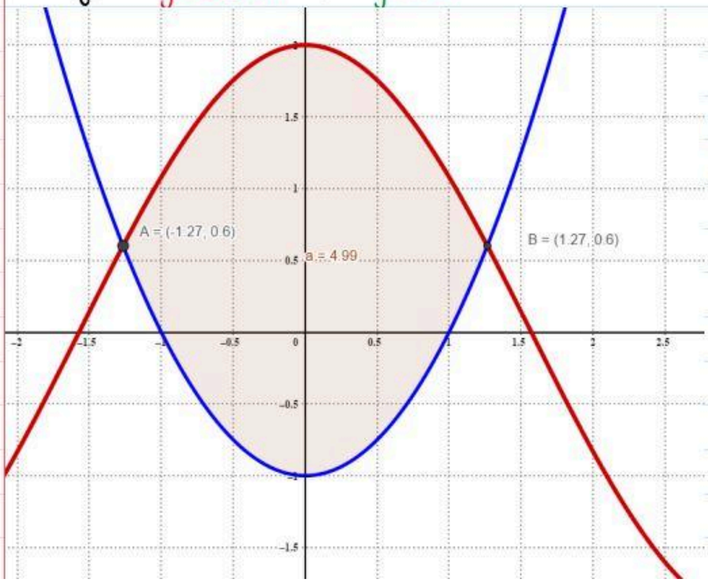
$$\left[\frac{3}{4}(1)^{4/3} - \frac{2}{3}(1)^{3/2}\right] - \left[\frac{3}{4}(0)^{4/3} - \frac{2}{3}(0)^{3/2}\right]$$

$$= \frac{3}{4} - \frac{2}{3}$$

$$= \frac{9}{12} - \frac{8}{12} = \boxed{\frac{1}{12}\ \text{units}^2}$$

Ex 3

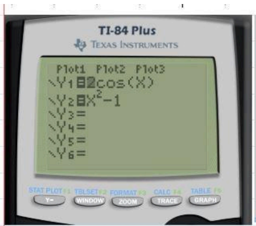
Using a calculator to determine the area of the region $y = 2\cos(x)$ and $y = x^2 - 1$



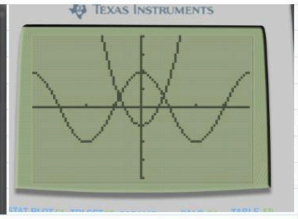
We want to determine the area between the two curves!

$$\int_A^B \left[2\cos(x) - (x^2-1)\right] dx$$

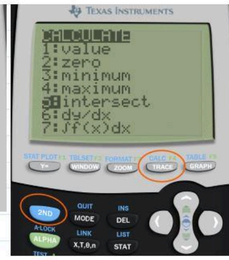
$$= \int_A^B 2\cos(x) - x^2 + 1 \, dx$$



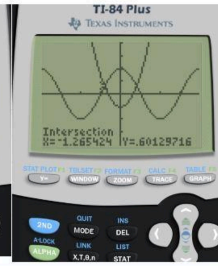
• Type both functions in your calculator



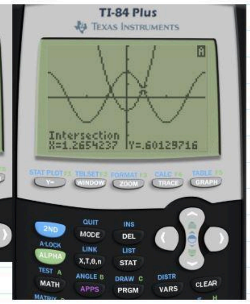
• Press graph



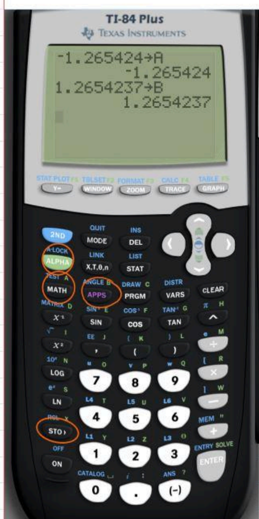
• 2nd
• Calc
• (5) Intersect



• Pick 1st curve
• Pick 2^{nd} curve
• guess close to intersecting point



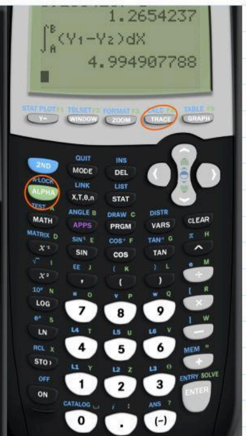
• Repeat for second intersection



• Type in X-value (all decimal values provided)

• Store left endpoint as "A"

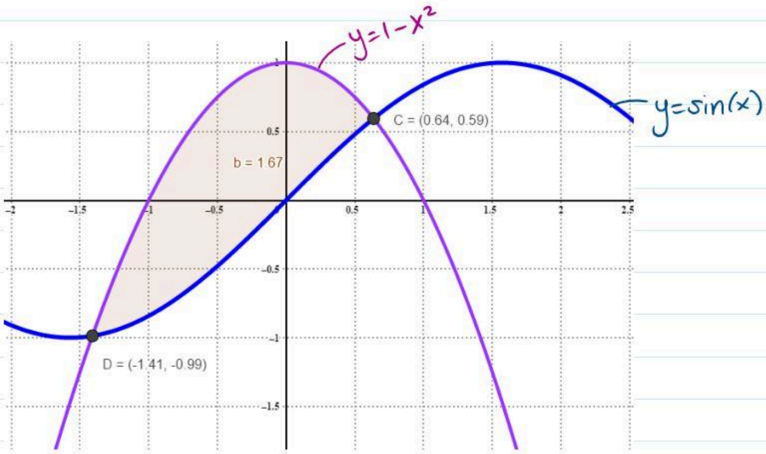
• Store right endpoint as "B"



• Math 9
• A is lower limit
• B is upper limit
• Alpha – Trace – Y_1
• Alpha – Trace – Y_2
• Enter!

$$\boxed{\int_A^B (2\cos(x) - x^2 + 1)\,dx = 4.994}$$

Try: Use a calculator to determine the area of the region between $y = \sin(x)$ and $y = 1 - x^2$

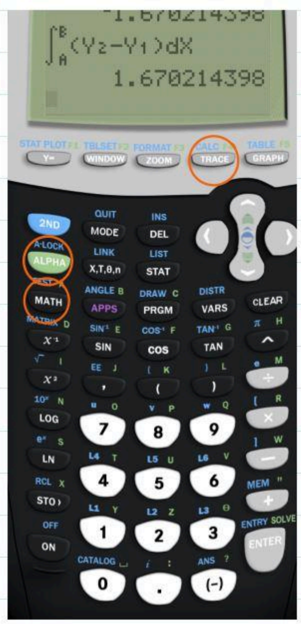


$$\int_A^B (\text{"top"} - \text{"bottom"})\,dx$$

$$\int_A^B (1 - x^2 - \sin(x))\,dx$$

$X = -1.409624 \rightarrow A$

$X = 0.6367365 \rightarrow B$



$$\int_A^B (Y_2 - Y_1)\,dx$$

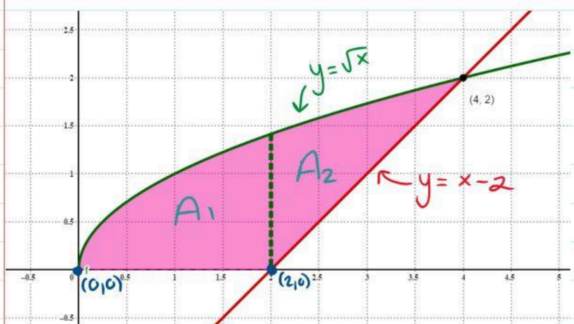
$$\int_A^B (1 - x^2 - \sin(x))\,dx$$

$$\boxed{\approx 1.67}$$

Ex 4 Boundaries with changing functions

Determine the area of the region R in the first quadrant bounded above by $y=\sqrt{x}$ and below by the x-axis and $y = x-2$

Step 1: Graph



$A_R = A_1 + A_2$

$y = 0 \quad y = x-2$

$0 = x - 2$

$x = 2$

$y=\sqrt{x} \quad y = x-2$

$(\sqrt{x})^2 = (x-2)^2$

$x = x^2 - 4x + 4$

$0 = x^2 - 5x + 4$

$0 = (x-4)(x-1)$

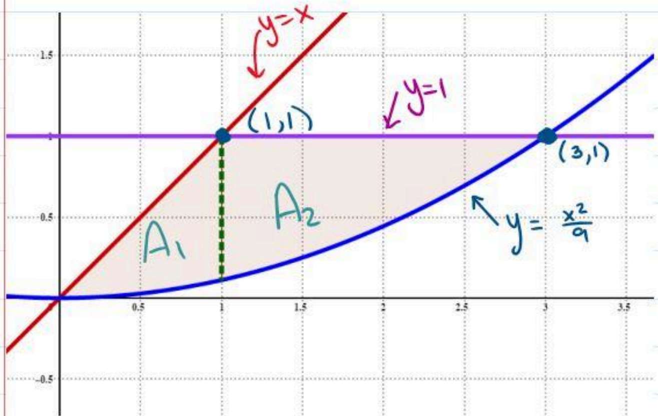
$x = 4 \quad \cancel{x=1}$ doesn't work

$$A_1 = \int_0^2 \sqrt{x}\,dx$$
$$= \int_0^2 x^{1/2}\,dx$$
$$= \frac{2}{3}x^{3/2}\Big|_0^2$$
$$= \frac{2}{3}(2)^{3/2}$$
$$= \frac{2}{3}(\sqrt{8})$$
$$= \frac{2}{3}\cdot 2\sqrt{2}$$
$$= \frac{4\sqrt{2}}{3}$$

$$A_2 = \int_2^4 \sqrt{x} - (x-2)\,dx$$
$$= \int_2^4 x^{1/2} - x + 2\,dx$$
$$= \frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + 2x \Big|_2^4$$
$$= \left[\frac{2}{3}(4)^{3/2} - \frac{1}{2}(4)^2 + 2(4)\right] - \left[\frac{2}{3}(2)^{3/2} - \frac{1}{2}(2)^2 + 2(2)\right]$$
$$= \left[\frac{2}{3}(8) - \frac{1}{2}(16) + 8\right] - \left[\frac{4\sqrt{2}}{3} - 2 + 4\right]$$
$$= \left[\frac{16}{3} - \cancel{8} + \cancel{8}\right] - \left[\frac{4\sqrt{2}}{3} + 2\right]$$
$$= \frac{16}{3} - \frac{4\sqrt{2}}{3} - 2$$
$$= \frac{16}{3} - \frac{4\sqrt{2}}{3} - \frac{6}{3}$$
$$= \frac{10}{3} - \frac{4\sqrt{2}}{3}$$

$$A_R = A_1 + A_2$$
$$= \frac{4\sqrt{2}}{3} + \frac{10}{3} - \frac{4\sqrt{2}}{3}$$
$$= \boxed{\frac{10}{3}\text{ units}^2}$$

Try: Determine the area enclosed by $y=x$, $y=1$, $y=\frac{x^2}{9}$



$A_R = A_1 + A_2$

$y=x \quad y=\frac{x^2}{9}$

$x = \frac{x^2}{9}$

$9x = x^2$

$x^2 - 9x = 0$

$x(x-9) = 0$

$x=0 \quad \cancel{x=9}$

$y=1$ & $y=x$

$1 = x$

$y=1 \quad y=\frac{x^2}{9}$

$1 = \frac{x^2}{9}$

$9 = x^2$

$\boxed{x=3}$

$A_1 = \int_0^1 (x) - \left(\frac{x^2}{9}\right) dx$

$= \int_0^1 x - \frac{x^2}{9}\, dx$

$= \frac{1}{2}x^2 - \frac{1}{27}x^3 \Big|_0^1$

$= \frac{1}{2}(1)^2 - \frac{1}{27}(1)^3$

$= \frac{1}{2} - \frac{1}{27}$

$= \frac{27}{54} - \frac{2}{54}$

$= \frac{25}{54}$

$A_2 = \int_1^3 (1) - \left(\frac{x^2}{9}\right) dx$

$= \int_1^3 1 - \frac{x^2}{9}\, dx$

$= \left(x - \frac{1}{27}x^3\right)\Big|_1^3$

$= \left(3 - \frac{1}{27}(3)^3\right) - \left(1 - \frac{1}{27}(1)^3\right)$

$= (3-1) - \left(1 - \frac{1}{27}\right)$

$= 2 - 1 + \frac{1}{27}$

$= 1 + \frac{1}{27}$

$= \frac{28}{27}$

$A_R = A_1 + A_2$

$= \frac{25}{54} + \frac{28}{27}$

$= \frac{25}{54} + \frac{56}{54}$

$= \frac{81}{54} = \boxed{\frac{3}{2} \text{ units}^2}$

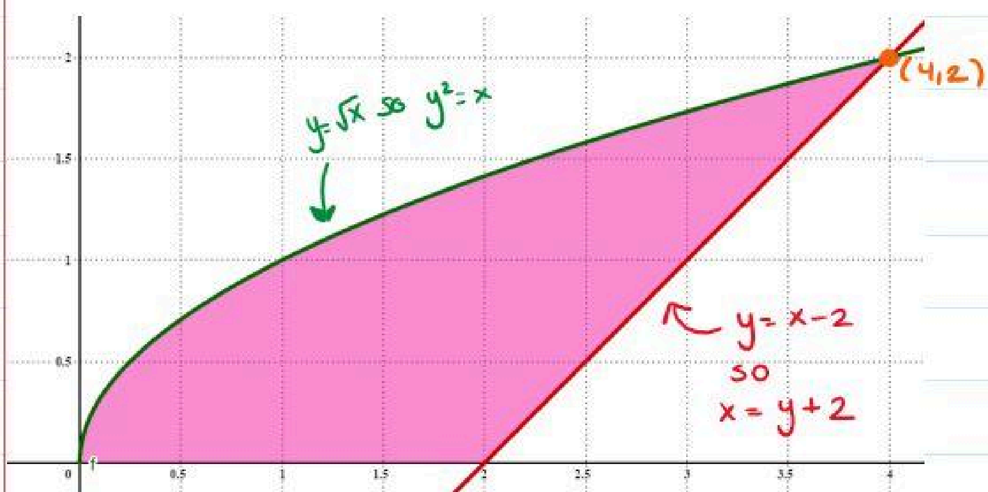
Integrating with Respect to "y"

Sometimes the boundaries of the regions are more easily described by functions in terms of y rather than in terms of x

Ex 5

Determine the area of the region in Ex 4 by integrating with respect to y

$y = \sqrt{x}$, $y = x - 2$



Step 1

Rename curves in terms of y by solving for x

$y = \sqrt{x}$ $\qquad\qquad$ $y = x - 2$

$x = y^2$ $\qquad\qquad$ $y + 2 = x$

which function is "larger"?

$$\underset{\substack{f(y) \\ \text{"right"}}}{y+2} \geq \underset{\substack{g(y) \\ \text{"left"}}}{y^2} \quad \text{on the interval } [0,2]$$

Step 3

set up integral & solve

$$\int_0^2 (y+2) - (y^2)\,dy$$

$$\int_0^2 y+2-y^2\,dy$$

$$= \left[\tfrac{1}{2}y^2 + 2y - \tfrac{1}{3}y^3\right]\Big|_0^2$$

$$= \left[\tfrac{1}{2}(2)^2 + 2(2) - \tfrac{1}{3}(2)^3\right]$$

$$= 2 + 4 - \tfrac{8}{3}$$

$$= 6 - \tfrac{8}{3}$$

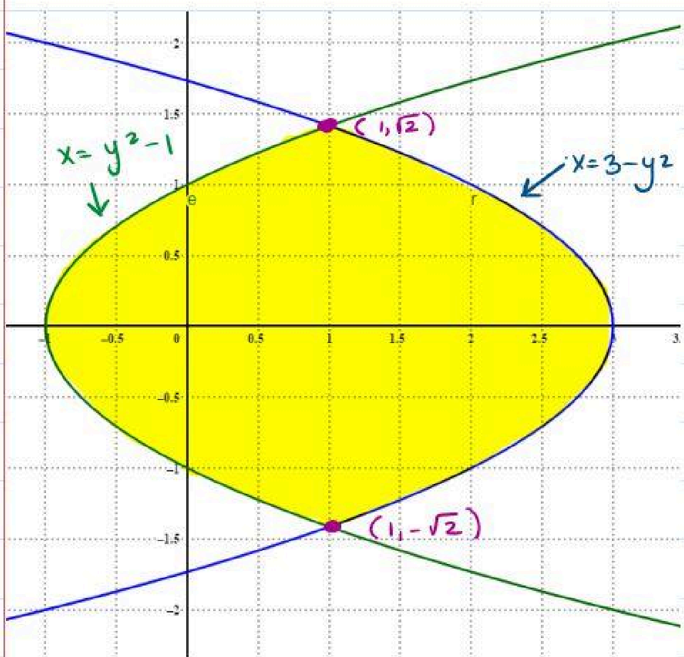
$$= \tfrac{18}{3} - \tfrac{8}{3}$$

$$\boxed{= \tfrac{10}{3} \text{ units}^2}$$

Try: Determine the area enclosed in the

Try: Determine the area enclosed in the region between $y^2 = x+1$ and $y^2 = 3-x$

Hint: Integrate with respect to y



$x = y^2 - 1 \qquad -x = y^2 - 3$

$\qquad\qquad\qquad\quad x = 3 - y^2$

$3 - y^2 > y^2 - 1$

right $\qquad$ left

intersection points

$y^2 - 1 = 3 - y^2$

$2y^2 = 4$

$y^2 = 2$

$y = \pm\sqrt{2}$

$y = -\sqrt{2}$ lower limit $\qquad y = \sqrt{2}$ upper limit

$$\int_{-\sqrt{2}}^{\sqrt{2}} (3-y^2) - (y^2 - 1)\, dy$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} 3 - y^2 - y^2 + 1\, dy$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} -2y^2 + 4\, dy$$

$$= \left[-\frac{2}{3} y^3 + 4y \right]_{-\sqrt{2}}^{\sqrt{2}}$$

$$= \left[-\frac{2}{3}(\sqrt{2})^3 + 4\sqrt{2} \right] - \left[-\frac{2}{3}(-\sqrt{2})^3 + 4(-\sqrt{2}) \right]$$

$$= -\frac{2}{3} \cdot 2\sqrt{2} + 4\sqrt{2} - \left(\frac{2}{3} 2\sqrt{2} - 4\sqrt{2} \right)$$

$$= -\frac{4\sqrt{2}}{3} + 4\sqrt{2} - \frac{4\sqrt{2}}{3} + 4\sqrt{2}$$

Ex 6 How to choose!?!?

Determine the area of the region enclosed by

$y = x^3$, $x = y^2 - 2$

If we integrate in terms of x, we need:

$y = x^3 \leftarrow$ easy

$y = \pm\sqrt{x+2} \rightarrow y = \sqrt{x+2}$

$\rightarrow y = -\sqrt{x+2}$

If we integrate in terms of y, we need:

$$\left.\begin{array}{l} x = \sqrt[3]{y} \\ \\ x = y^2 - 2 \end{array}\right\} \text{easier for this problem}$$

Next step: determine limits!

$y^{1/3} = y^2 - 2$

$x^{1/3} = x^2 - 2 \rightarrow$ the calculator only works in "x"

graph & solve for intersections

$y = -1$ & $y = 1.793003715 \rightarrow A$

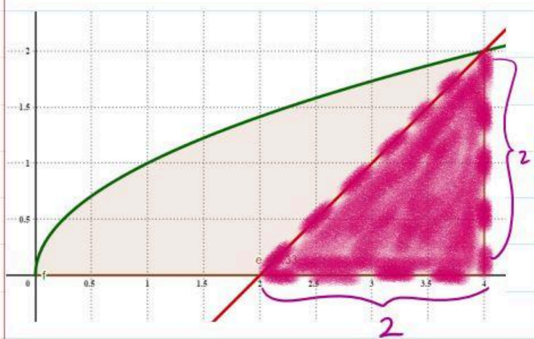
Store this as A

$$\int_{-1}^{A} y^{1/3} - (y^2 - 2)\,dy$$

$$= \int^{A} y^{1/3} - y^2 + 2\,dy$$

Ex 7 How to use geometry to save time

Recall Ex 4 ...



we can find $\int_0^4 \sqrt{x}\,dx$ and subtract the triangle we don't want

$$A = \int_0^4 x^{\frac{1}{2}}dx - \frac{1}{2}(2)(2)$$

$$= \frac{2}{3}x^{\frac{3}{2}}\Big|_0^4 - 2$$

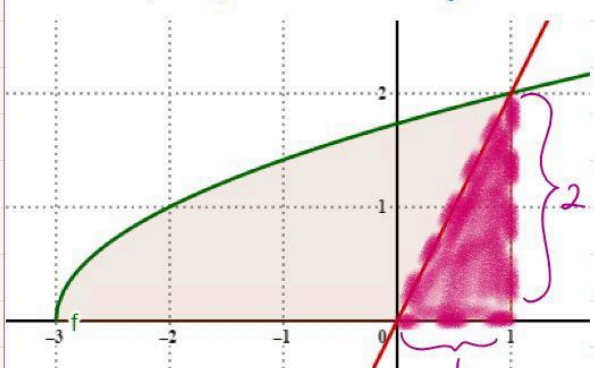
$$= \frac{2}{3}(4)^{3/2} - 2$$

$$= \frac{2}{3}(8) - 2$$

$$= \frac{16}{3} - \frac{6}{3}$$

$$\boxed{= \frac{10}{3} \text{ units}^2}$$

Try: Use geometry to determine the area enclosed by $y^2 = x+3$ and $y = 2x$ and the x-axis



$$\left(\int_{-3}^{1} \sqrt{x+3}\,dx\right) - \frac{1}{2}(1)(2)$$

$$\int_{-3}^{1} (x+3)^{1/2}dx - 1$$

$$\frac{2}{3}(x+3)^{3/2}\Big|_{-3}^{1} - 1$$

$$\frac{2}{3}\left((1+3)^{3/2} - (-3+3)^{3/2}\right) - 1$$

$$\frac{2}{3}(4)^{3/2} - 1$$

$$\frac{2}{3}(8) - 1$$

$$\frac{16}{3} - \frac{3}{3}$$

$$\frac{13}{3} = \boxed{4.333 \text{ units}^2}$$