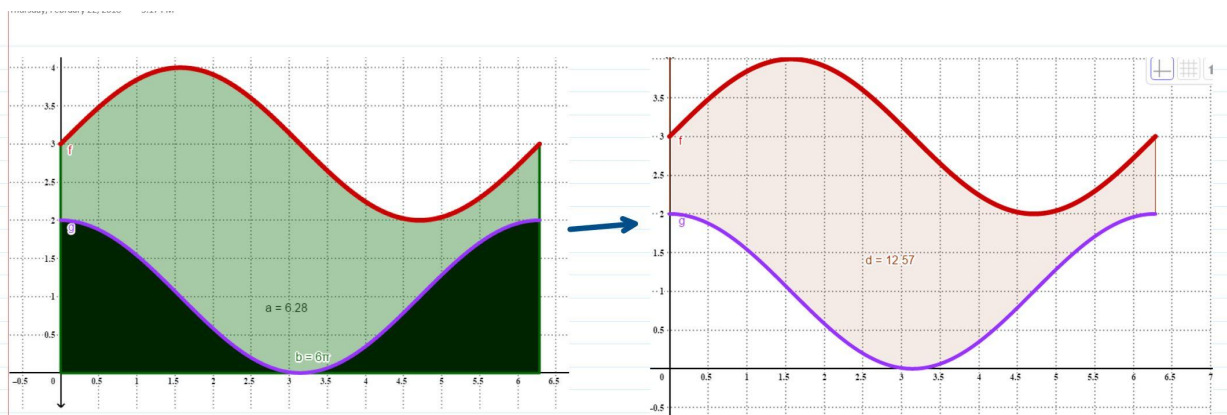




Area Between Curves

If f and g are continuous functions with $f(x) \geq g(x)$ on $[a, b]$, then the area between the curves $y=f(x)$ and $y=g(x)$ from a to b is the integral of $[f-g]$ from a to b :

$$\int_a^b (f(x) - g(x)) dx$$

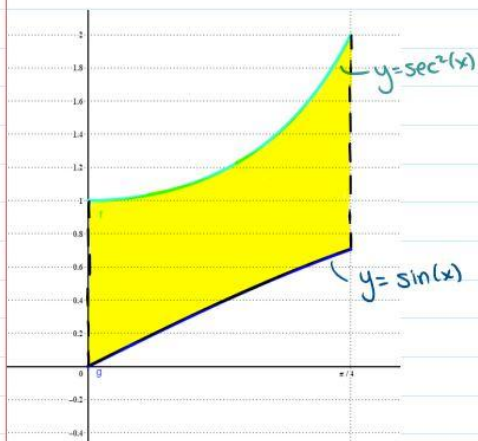
$$\int_a^b (\text{"top" - "bottom"}) dx$$

Ex 1

Determine the area of the region between $y = \sec^2(x)$ and $y = \sin(x)$ on $[0, \frac{\pi}{4}]$

Step 1:

Graph the curves to find their relative positions in the plane.



Notice that $y = \sec^2(x)$ is always above the graph $y = \sin(x)$

Step 2

Write an integral and solve

$$A = \int_a^b f(x) - g(x) dx$$

$$A = \int_0^{\frac{\pi}{4}} \sec^2(x) - \sin(x) dx$$

$$A = [\tan(x) + \cos(x)] \Big|_0^{\frac{\pi}{4}}$$

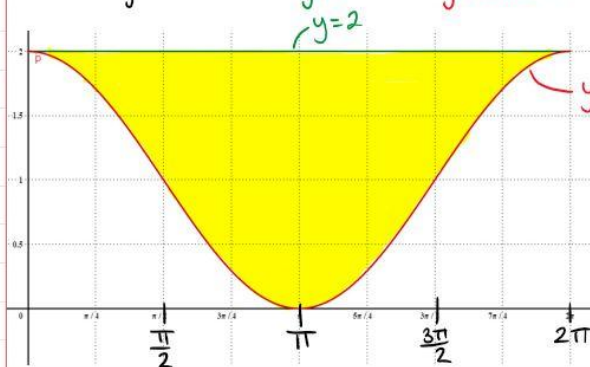
$$A = (\tan(\frac{\pi}{4}) + \cos(\frac{\pi}{4})) - (\tan(0) + \cos(0))$$

$$A = (1 + \frac{\sqrt{2}}{2}) - (0 + 1)$$

$$A = 1 + \frac{\sqrt{2}}{2} - 1$$

$$A = \frac{\sqrt{2}}{2} \text{ units squared}$$

Try: Determine the area of the region bounded by the curves $y = 2$ and $y = \cos(x) + 1$ on $[0, 2\pi]$



$$\int_0^{2\pi} (2) - (\cos(x) + 1) dx$$

$$= \int_0^{2\pi} 1 - \cos(x) dx$$

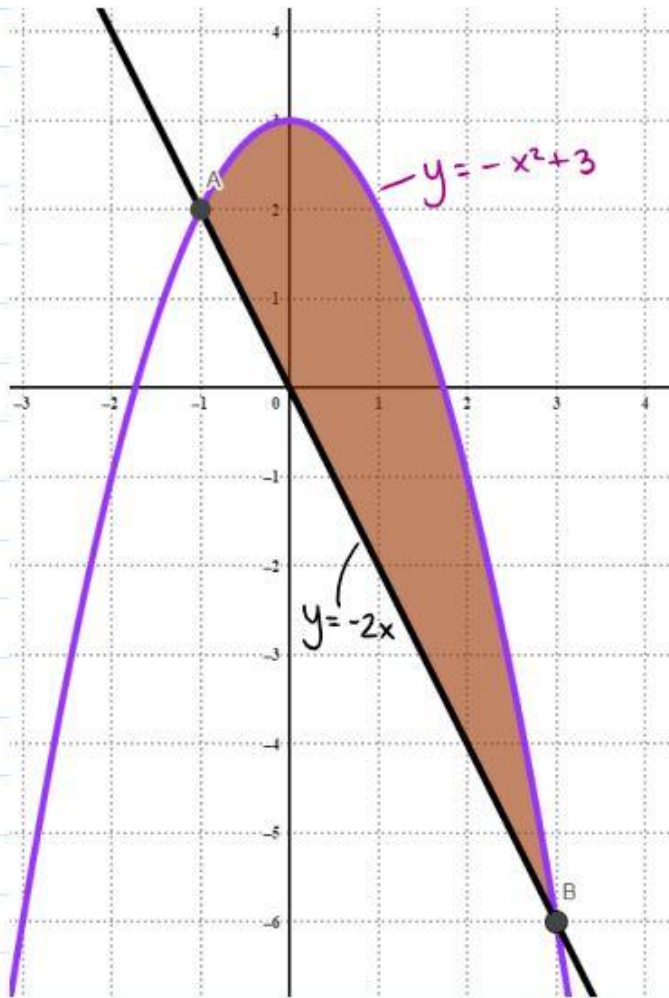
$$= [x - \sin(x)] \Big|_0^{2\pi}$$

$$= (2\pi - \sin(2\pi)) - (0 - \sin(0))$$

$$= 2\pi$$

Areas Enclosed by Intersecting Curves

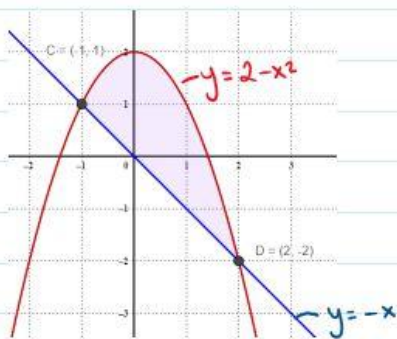
When a region is enclosed by intersecting curves, the points of intersection give the limits of integration



Ex 2

Determine the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$

Step 1: Graph the curves



Step 2: Determine the intersection either algebraically or using a calculator

set functions equal to each other & solve

$$y = 2 - x^2 \quad y = -x$$

$$2 - x^2 = -x$$

$$0 = x^2 - x - 2$$

$$0 = (x - 2)(x + 1)$$

$$x = 2 \quad x = -1$$

so these are the

Step 3: set up the integral & solve

$$\int_{-1}^2 (2 - x^2 - (-x)) dx = \int_{-1}^2 (2 + x - x^2) dx$$

$$= \left[2x + \frac{x^2}{2} - \frac{x^3}{3} \right]_{-1}^2$$

$$= \left[2(2) + \frac{(2)^2}{2} - \frac{(2)^3}{3} \right] - \left[2(-1) + \frac{(-1)^2}{2} - \frac{(-1)^3}{3} \right]$$

$$= \left[4 + \frac{4}{2} - \frac{8}{3} \right] - \left[-2 + \frac{1}{2} + \frac{1}{3} \right]$$

$$= \left[\frac{24}{6} + \frac{12}{6} - \frac{16}{6} \right] - \left[\frac{-12}{6} + \frac{3}{6} + \frac{2}{6} \right]$$

$$= \left[\frac{20}{6} \right] - \left[\frac{-7}{6} \right]$$

$$= \frac{20}{6} + \frac{7}{6}$$

$$= \frac{27}{6}$$

$$= \frac{9}{2} \text{ units squared}$$

Try: Determine the area of the region between
the curves $x=y^3$ and $x=y^2$
Hint: solve for y

$$\sqrt[3]{x} = \sqrt[3]{y^3}$$

$$y = \sqrt[3]{x}$$

$$\sqrt[3]{x} = \sqrt[3]{y^2}$$

$$y = \pm \sqrt[3]{x}$$

$$\sqrt[3]{x} = \sqrt{x}$$

$$(x^{1/3})^6 = (x^{1/2})^6$$

$$x^2 = x^3$$

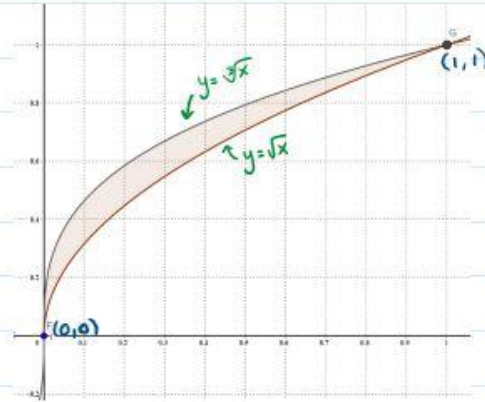
$$0 = x^3 - x^2$$

$$0 = x^2(x-1)$$

$$x = 0 \quad x = 1$$

limits of

integration



$$\int_0^1 [(x^{1/3}) - (x^{1/2})] dx$$

$$\left[\frac{3}{4} x^{4/3} - \frac{2}{3} x^{3/2} \right]_0^1$$

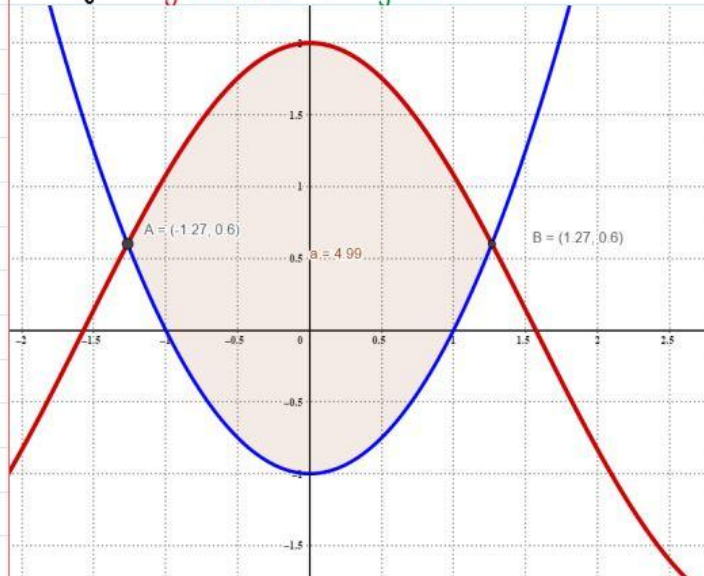
$$\left[\frac{3}{4} (1)^{4/3} - \frac{2}{3} (1)^{3/2} \right] - \left[\frac{3}{4} (0)^{4/3} - \frac{2}{3} (0)^{3/2} \right]$$

$$= \frac{3}{4} - \frac{2}{3}$$

$$= \frac{9}{12} - \frac{8}{12} = \frac{1}{12} \text{ units}^2$$

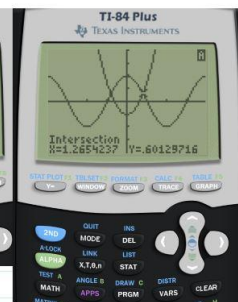
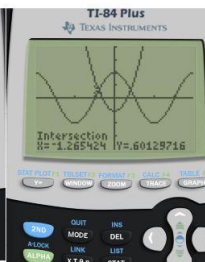
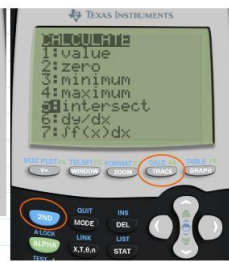
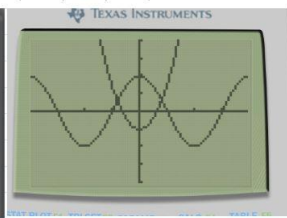
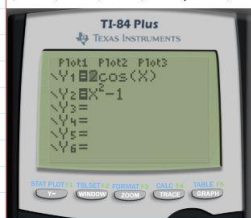
Ex 3

using a calculator to determine the area of the region $y = 2\cos(x)$ and $y = x^2 - 1$



We want to determine the area between the two curves!

$$\int_A^B [2\cos(x) - (x^2 - 1)] dx$$
$$= \int_A^B 2\cos(x) - x^2 + 1 dx$$



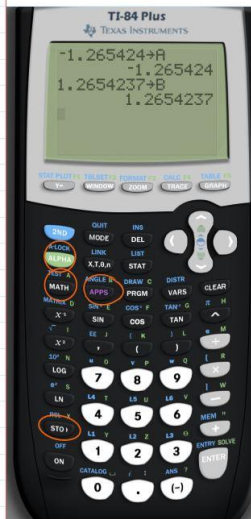
• Type both functions in your calculator

• Press graph

- 2nd
- Calc
- (5) Intersect

- Pick 1st curve
- Pick 2nd curve
- guess close to intersecting point

• Repeat for second intersection



- Math 9
- A is lower limit
- B is upper limit
- Alpha - Trace - Y₁
- Alpha - Trace - Y₂
- Enter!

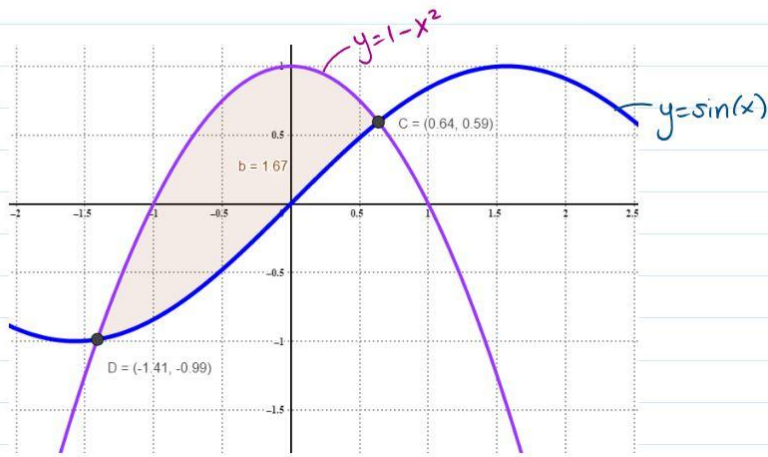
• Type in X-value (all decimal values provided)

• Store left endpoint as "A"

• Store right endpoint as "B"

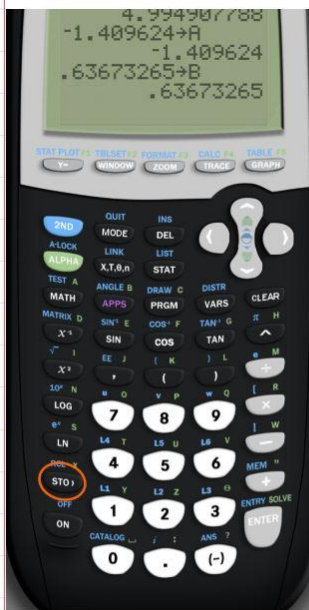
$$\int_A^B (2\cos(x) - x^2 + 1) dx = 4.994$$

Try: use a calculator to determine the area of the region between $y = \sin(x)$ and $y = 1 - x^2$



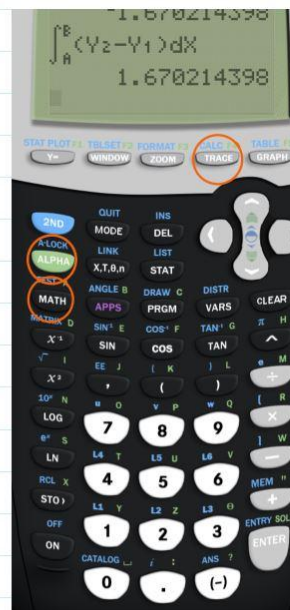
$$\int_A^B (\text{"top"} - \text{"bottom"}) dx$$

$$\int_A^B (1 - x^2 - \sin(x)) dx$$



$$X = -1.409624 \rightarrow A$$

$$X = 0.63673265 \rightarrow B$$



$$\int_A^B (Y_2 - Y_1) dx$$

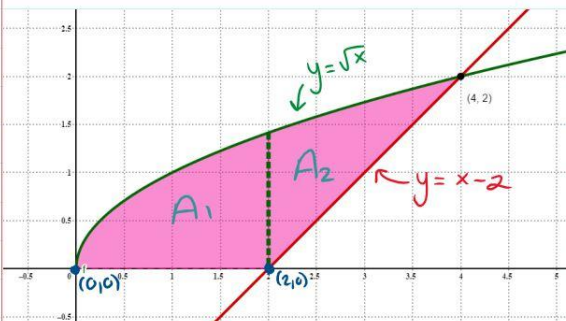
$$\int_A^B (1 - x^2 - \sin(x)) dx$$

$$\approx 1.67$$

Ex 4 Boundaries with changing functions

Determine the area of the region R in the first quadrant bounded above by $y = \sqrt{x}$ and below by the x -axis and $y = x - 2$

Step 1: Graph



$$A_R = A_1 + A_2$$

$$y = 0 \quad y = x - 2$$
$$0 = x - 2$$
$$\boxed{x = 2}$$

$$y = \sqrt{x} \quad y = x - 2$$
$$(\sqrt{x})^2 = (x - 2)^2$$
$$x = x^2 - 4x + 4$$
$$0 = x^2 - 5x + 4$$
$$0 = (x - 4)(x - 1)$$
$$\boxed{x = 4} \quad \boxed{x = 1}$$

doesn't work

$$A_1 = \int_0^2 \sqrt{x} \, dx$$
$$= \int_0^2 x^{1/2} \, dx$$
$$= \left. \frac{2}{3} x^{3/2} \right|_0^2$$
$$= \frac{2}{3} (2)^{3/2}$$
$$= \frac{2}{3} (\sqrt{8})$$
$$= \frac{2}{3} \cdot 2\sqrt{2}$$
$$= \frac{4\sqrt{2}}{3}$$

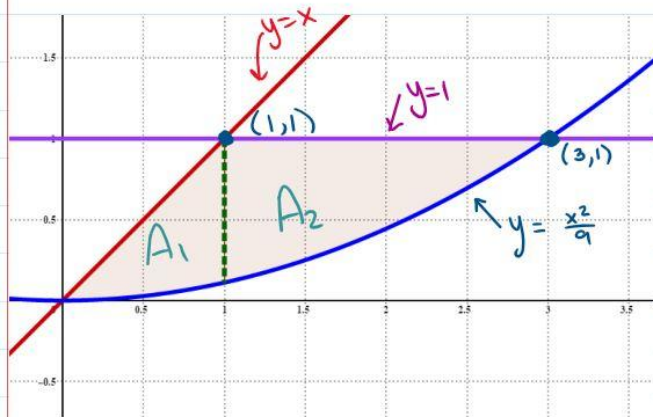
$$A_2 = \int_2^4 \sqrt{x} - (x - 2) \, dx$$
$$= \int_2^4 x^{1/2} - x + 2 \, dx$$
$$= \left. \frac{2}{3} x^{3/2} - \frac{1}{2} x^2 + 2x \right|_2^4$$
$$= \left[\frac{2}{3} (4)^{3/2} - \frac{1}{2} (4)^2 + 2(4) \right] - \left[\frac{2}{3} (2)^{3/2} - \frac{1}{2} (2)^2 + 2(2) \right]$$
$$= \left[\frac{2}{3} (8) - \frac{1}{2} (16) + 8 \right] - \left[\frac{4\sqrt{2}}{3} - 2 + 4 \right]$$
$$= \left[\frac{16}{3} - 8 + 8 \right] - \left[\frac{4\sqrt{2}}{3} + 2 \right]$$
$$= \frac{16}{3} - \frac{4\sqrt{2}}{3} - 2$$
$$= \frac{16}{3} - \frac{4\sqrt{2}}{3} - \frac{6}{3}$$
$$= \frac{10}{3} - \frac{4\sqrt{2}}{3}$$

$$A_R = A_1 + A_2$$

$$= \frac{4\sqrt{2}}{3} + \frac{10}{3} - \frac{4\sqrt{2}}{3}$$

$$= \boxed{\frac{10}{3} \text{ units}^2}$$

Try: Determine the area enclosed by $y=x$, $y=1$, $y=\frac{x^2}{9}$



$$A_1 = \int_0^1 (x) - \left(\frac{x^2}{9}\right) dx$$

$$= \int_0^1 x - \frac{x^2}{9} dx$$

$$= \left. \frac{1}{2}x^2 - \frac{1}{27}x^3 \right|_0^1$$

$$= \frac{1}{2}(1)^2 - \frac{1}{27}(1)^3$$

$$= \frac{1}{2} - \frac{1}{27}$$

$$= \frac{27}{54} - \frac{2}{54}$$

$$= \frac{25}{54}$$

$$A_2 = \int_1^3 (1) - \left(\frac{x^2}{9}\right) dx$$

$$= \int_1^3 1 - \frac{x^2}{9} dx$$

$$= \left(x - \frac{1}{27}x^3 \right) \Big|_1^3$$

$$= \left(3 - \frac{1}{27}(3)^3 \right) - \left(1 - \frac{1}{27}(1)^3 \right)$$

$$= (3 - 1) - \left(1 - \frac{1}{27} \right)$$

$$= 2 - 1 + \frac{1}{27}$$

$$= 1 + \frac{1}{27}$$

$$= \frac{28}{27}$$

$$A_R = A_1 + A_2$$

$$= \frac{25}{54} + \frac{28}{27}$$

$$= \frac{25}{54} + \frac{56}{54}$$

$$= \frac{81}{54} = \boxed{\frac{3}{2} \text{ units}^2}$$

$$A_R = A_1 + A_2$$

$$y=x \quad y=\frac{x^2}{9}$$

$$y=1 \neq y=x$$

$$(1=x)$$

$$x = \frac{x^2}{9}$$

$$9x = x^2$$

$$x^2 - 9x = 0$$

$$x(x-9) = 0$$

$$(x=0) \quad x=9$$

$$y=1 \quad y=\frac{x^2}{9}$$

$$1 = \frac{x^2}{9}$$

$$9 = x^2$$

$$(x=3)$$

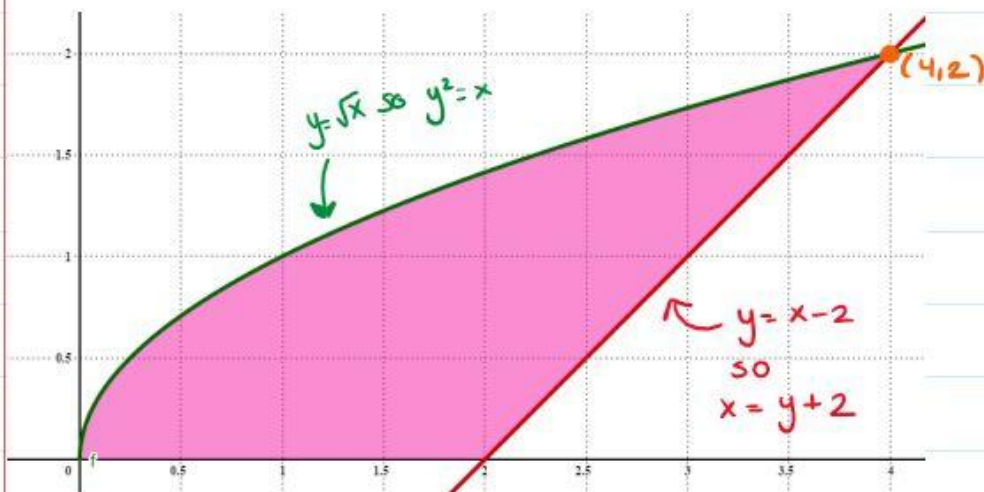
Integrating with Respect to "y"

Sometimes the boundaries of the regions are more easily described by functions in terms of y rather than in terms of x

Ex5

Determine the area of the region in Ex 4 by integrating with respect to y

$$y = \sqrt{x}, \quad y = x - 2$$



Step 1

Rename curves in terms of y by solving for x

$$y = \sqrt{x} \\ x = y^2$$

$$y = x - 2 \\ y + 2 = x$$

which function is "larger"?

$$\begin{array}{ccc} y+2 & \geq & y^2 \quad \text{on the interval } [0, 2] \\ \uparrow & & \uparrow \\ f(y) & & g(y) \\ \text{"right"} & & \text{"left"} \end{array}$$

Step 3

set up integral & solve

$$\int_0^2 (y+2) - (y^2) dy$$

$$\int_0^2 y+2-y^2 dy$$

$$= \left[\frac{1}{2}y^2 + 2y - \frac{1}{3}y^3 \right]_0^2$$

$$= \left[\frac{1}{2}(2)^2 + 2(2) - \frac{1}{3}(2)^3 \right]$$

$$= 2 + 4 - \frac{8}{3}$$

$$= 6 - \frac{8}{3}$$

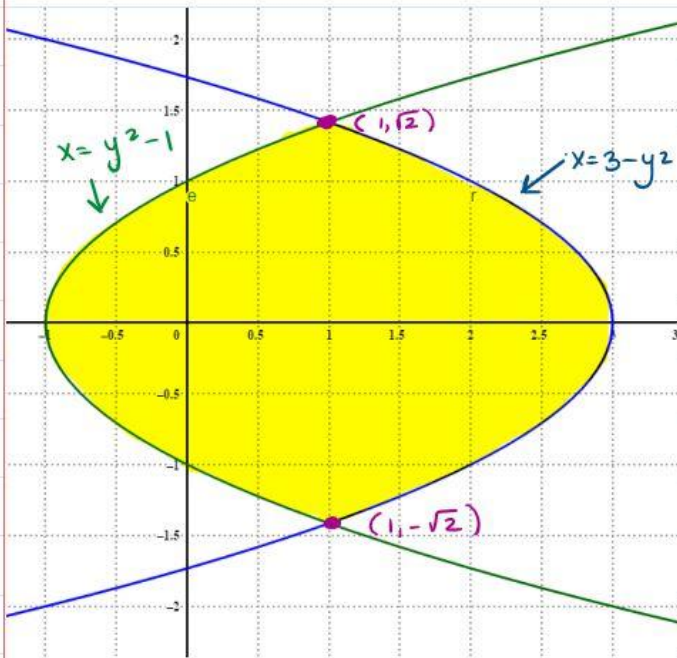
$$= \frac{18}{3} - \frac{8}{3}$$

$$= \frac{10}{3} \text{ units}^2$$

Try: Determine the area enclosed in the

Try: Determine the area enclosed in the region between $y^2 = x+1$ and $y^2 = 3-x$

Hint: Integrate with respect to y



$$x = y^2 - 1 \quad -x = y^2 - 3$$

$$x = 3 - y^2$$

$$3 - y^2 > y^2 - 1$$

$\uparrow$ right $\uparrow$ left

intersection points

$$y^2 - 1 = 3 - y^2$$

$$2y^2 = 4$$

$$y^2 = 2$$

$$y = \pm \sqrt{2}$$

$$y = -\sqrt{2}$$

lower limit

$$y = \sqrt{2}$$

upper limit

$$\int_{-\sqrt{2}}^{\sqrt{2}} (3 - y^2) - (y^2 - 1) dy$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} 3 - y^2 - y^2 + 1 dy$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} -2y^2 + 4 dy$$

$$= \left[-\frac{2}{3} y^3 + 4y \right]_{-\sqrt{2}}^{\sqrt{2}}$$

$$= \left[-\frac{2}{3} (\sqrt{2})^3 + 4\sqrt{2} \right] - \left[-\frac{2}{3} (-\sqrt{2})^3 + 4(-\sqrt{2}) \right]$$

$$= -\frac{2}{3} \cdot 2\sqrt{2} + 4\sqrt{2} - \left(\frac{2}{3} 2\sqrt{2} - 4\sqrt{2} \right)$$

$$= -\frac{4\sqrt{2}}{3} + 4\sqrt{2} - \frac{4\sqrt{2}}{3} + 4\sqrt{2}$$

Ex 6 How to choose!?!?

Determine the area of the region enclosed by

$$y = x^3, \quad x = y^2 - 2$$

If we integrate in terms of x , we need:

$$\begin{aligned} y &= x^3 \leftarrow \text{easy} \\ y &= \pm \sqrt{x+2} \end{aligned} \quad \begin{aligned} &\rightarrow y = \sqrt{x+2} \\ &\rightarrow y = -\sqrt{x+2} \end{aligned}$$

If we integrate in terms of y , we need:

$$\left. \begin{aligned} x &= \sqrt[3]{y} \\ x &= y^2 - 2 \end{aligned} \right\} \begin{aligned} &\text{easier for this} \\ &\text{problem} \end{aligned}$$

Next step: determine limits!

$$y^{1/3} = y^2 - 2$$

$x^{1/3} = x^2 - 2 \rightarrow$ the calculator only works in "x"
graph & solve for intersections

$$y = -1 \quad \& \quad y = 1.793003715 \rightarrow A$$

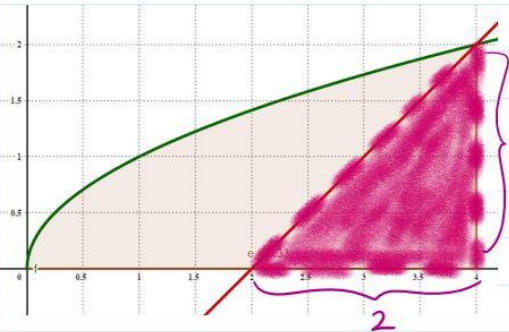
↑
Store this
as A

$$\int_{-1}^A y^{1/3} - (y^2 - 2) dy$$

$$= \int_{-1}^A y^{1/3} - y^2 + 2 dy$$

Ex7 How to use geometry to save time

Recall Ex 4 ...



We can find $\int_0^4 \sqrt{x} dx$
and subtract the triangle
we don't want

$$A = \int_0^4 x^{\frac{1}{2}} dx - \frac{1}{2}(2)(2)$$

$$= \frac{2}{3} x^{\frac{3}{2}} \Big|_0^4 - 2$$

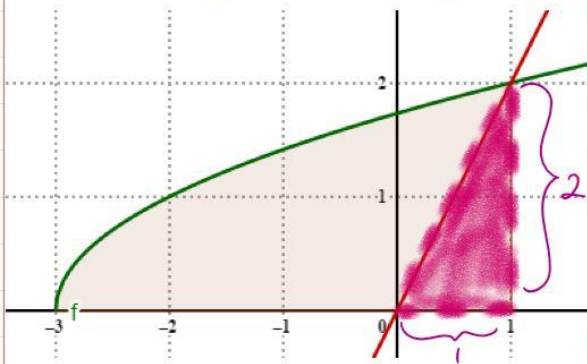
$$= \frac{2}{3}(4)^{\frac{3}{2}} - 2$$

$$= \frac{2}{3}(8) - 2$$

$$= \frac{16}{3} - \frac{6}{3}$$

$$= \frac{10}{3} \text{ units}^2$$

Try: Use geometry to determine the area enclosed
by $y^2 = x+3$ and $y=2x$ and the x-axis



$$\left(\int_{-3}^1 \sqrt{x+3} dx \right) - \frac{1}{2}(1)(2)$$

$$\int_{-3}^1 (x+3)^{\frac{1}{2}} dx - 1$$

$$\frac{2}{3}(x+3)^{\frac{3}{2}} \Big|_{-3}^1 - 1$$

$$\frac{2}{3}((1+3)^{\frac{3}{2}} - (-3+3)^{\frac{3}{2}}) - 1$$

$$\frac{2}{3}(4)^{\frac{3}{2}} - 1$$

$$\frac{2}{3}(8) - 1$$

$$\frac{16}{3} - \frac{3}{3}$$

$$\frac{13}{3} = 4.333 \text{ units}^2$$

