14. Transformation of charge+current 4-vector field derived from Electromagnetic 8-vector field F in Cl3,0(R)

(Proof that ρ+**J** = ∇^X **F** transforms in $Cl_{3,0}(R)$ like 4-vectors using hermitian conjugate formula, for the Nabla operator defined as: ∇^X := ∂/∂t - (∂/∂x)**g** - (∂/∂y)**e** - (∂/∂z)**f**)

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Summary

This article presents a derivation of the transformation of the electromagnetic charge and current density vector $\mathcal{J}$ = ρ+**J** , where $\mathcal{J}$ = ∇^X**F** [note [1]] and the electromagnetic Clifford vector field **F** (in $Cl_{3,0}(R)$) is assumed to transform under length preserving coordinate change as $\mathbf{F_2}$ = exp(**p**)***F***(exp(**p**))' , [note[2]] . It is also assumed that the 4-vectors such as coordinates **q**=t+**g**x+**e**y+**f**z and $\mathcal{J}$ must transform as $\mathcal{J}_2$ = exp(**p**)*$\mathcal{J}$*exp(**p**)^T [note[3]].

Introduction

Clifford Algebraic electromagnetic equation has a form of a 4-dimensional (t,x,y,z) differential operator nabla [4] applied to the complex electromagnetic field 4-vector **F** = -S - **E** - *i* **B** [see[5]] , that must equal zero. This can be also expressed as

F = ∇^I**a** (eq.1, see [4])

where **a** is defined as **a** = ϕ+**A** , where ϕ is electric potential and **A** is magnetic vector potential.

The Nabla operator that applies to 4-vector potential field **a** is defined in [4] as[6]:

∇^I = ∂/∂t + (∂/∂x) **g** + (∂/∂y) **e** + (∂/∂z) **f**

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[1] ∇^X := ∂/∂t - (∂/∂x)**g** - (∂/∂y)**e** - (∂/∂z)**f** [this form is assumed, to prove that it leads to correct transform for $\mathcal{J}$ = ∇^X**F**

[2] ()' is quaternion conjugate (example **e'**=-**e** and (**ef**)'=**f'e'**) We restrict **p** to be a linear combination of base vectors {**g,e,f,i,j,k**} only.

[3] $()^T$ is hermitian conjugate (example $\mathbf{e}^T$=**e** and $(\mathbf{ef})^T=\mathbf{f}^T\mathbf{e}^T$)

[4] ∇ = ∂/∂t - **g** ∂/∂x - **e** ∂/∂y - **f** ∂/∂z , ∂/∂t + **g** ∂/∂x + **e** ∂/∂y + **f** ∂/∂z

[5] **F** = generalised EM field tensor = -S - **E** - *i* **B** = -S - E_x**g** - E_y**e** - E_z**f** - *i* (B_x**g** + B_y**e** + B_z**f**) . The general full form is **F** = -S - **E** - *i* **B** + iM **,** however, the imaginary scalar component M of the Clifford EM field vector **F** must be zero or constant in order to cancel the unobservable magnetic monopole charge density and magnetic monopole current density. With M=0, equation ∇**F** = 0 corresponds to the well known Maxwell equation, if we define S as ρ + **J** = ∇S .

[6] Partial derivative operator applies first, then a base vector is right-multiplied to the result. The formula can be better understood if the beses g,e,f are put in the denominators ar right-divisions, that is:
∇^I = ∂/∂t + (∂/∂(x**g**)) + (∂/∂(y**e**)) + (∂/∂(z**f**))

Derivation of transform of $\mathcal{J}$

This assumes that $\mathcal{J}$ is a derivative of some Clifford field **F** that transforms using a formula containing quaternion conjugation). The derivation will be split in two cases: (1) Lorentz transform and (2) 3D Rotation.

1) Lorentz transform of $\mathcal{J}$

Let us take an arbitrary Lorentz transform generator vector **u** that is a linear combination of base vectors **e,f,g** . For the sake of consistency with the previously referenced articles [1] etc, it is assumed that base vector **g** represents axis x, **e** is axis y and **f** is axis z.

Without losing the generality of the proof, we can assume that a general generator vector **u** (in {**g,e,f**}) is aligned with **g** (that is there is some real U that U**g** = **u**) The reference frame can always be rotated to align **u** with **g** (we leave that without proof)

We will indicate the transformed vectors or coordinates with the subscript 2.
The time-space 4-vector **q**=t+**g**x+**e**y+**f**z transform as

$\mathbf{q_2} = \exp(\mathbf{u})\ \mathbf{q}\ \exp(\mathbf{u})$

$\mathbf{q_2} = \exp(\mathbf{u})\ \mathbf{q}\ \exp(\mathbf{u}) = \exp(2\mathbf{u})t + \exp(2\mathbf{u})x + \mathbf{e}y + \mathbf{f}z =$ [note[7]]

t*cosh2U + x*sinh2U + // t_2

x*cosh2U***g** + t*sinh2U***g** + // $x_2\mathbf{g}$

ey + // $y_2\mathbf{e}$

fz // $z_2\mathbf{f}$

(where U := **u**/**g**)

therefore:

t_2 = t*cosh2U + x*sinh2U

x_2 = x*cosh2U + t*sinh2U

$y_2 = y$

$z_2 = z$

We will transform $\mathcal{J}=\nabla^{x}\mathbf{F} \leftarrow \mathcal{J}_2=\nabla^{x}\mathbf{F_2}$ in reverse, starting with the formula **:**

$$\mathcal{J} = \nabla^{x}\mathbf{F} = \partial\mathbf{F}/\partial t - \partial\mathbf{F}/\partial x\ \mathbf{g} - \partial\mathbf{F}/\partial y\ \mathbf{e} - \partial\mathbf{F}/\partial z\ \mathbf{f}$$

and evaluating the inner terms.

[7] We use the following property of exponential function: exp(**g**U)***g** = **g***exp(**g**U) , For perpendicular vectors (i.e. for anti-commuting vectors), for example **g** and **e** it follows exp(**g**U)***e** = **e***exp(-**g**U)

$$\partial \mathbf{F}/\partial t = (\partial \mathbf{F}/\partial t_2)*(\partial t_2/\partial t) + (\partial \mathbf{F}/\partial x_2)*(\partial x_2/\partial t) + (\partial \mathbf{F}/\partial y_2)*(\partial y_2/\partial t) + (\partial \mathbf{F}/\partial z_2)*(\partial z_2/\partial t) =$$
$$(\partial \mathbf{F}/\partial t_2)*(\cosh 2U) + (\partial \mathbf{F}/\partial x_2)*(\sinh 2U)$$

$$\partial \mathbf{F}/\partial x = (\partial \mathbf{F}/\partial t_2)*(\partial t_2/\partial x) + (\partial \mathbf{F}/\partial x_2)*(\partial x_2/\partial x) + (\partial \mathbf{F}/\partial y_2)*(\partial y_2/\partial x) + (\partial \mathbf{F}/\partial z_2)*(\partial z_2/\partial x) =$$
$$(\partial \mathbf{F}/\partial t_2)*(\sinh 2U) + (\partial \mathbf{F}/\partial x_2)*(\cos 2U)$$

$$\partial \mathbf{F}/\partial y = (\partial \mathbf{F}/\partial t_2)*(\partial t_2/\partial y) + (\partial \mathbf{F}/\partial x_2)*(\partial x_2/\partial y) + (\partial \mathbf{F}/\partial y_2)*(\partial y_2/\partial y) + (\partial \mathbf{F}/\partial z_2)*(\partial z_2/\partial y) = (\partial \mathbf{F}/\partial y_2)$$
$$\partial \mathbf{F}/\partial z = (\partial \mathbf{F}/\partial t_2)*(\partial t_2/\partial z) + (\partial \mathbf{F}/\partial x_2)*(\partial x_2/\partial z) + (\partial \mathbf{F}/\partial y_2)*(\partial y_2/\partial z) + (\partial \mathbf{F}/\partial z_2)*(\partial z_2/\partial z) = (\partial \mathbf{F}/\partial z_2)$$

Now, we will substitute the **F** expressed in terms of the reverse transformed $\mathbf{F_2}$ into the above formulae, that is: $\mathbf{F} = \exp(-\mathbf{u})\ \mathbf{F_2}\ \exp(+\mathbf{u})$. This will yield:

$$\mathcal{J} = \nabla^X \mathbf{F} = \exp(-\mathbf{u})*((\partial \mathbf{F_2}/\partial t_2)*\cosh 2U + (\partial \mathbf{F_2}/\partial x_2)*\sinh 2U)*\exp(+\mathbf{u}) +$$
$$-\exp(-\mathbf{u})*((\partial \mathbf{F_2}/\partial t_2)*\sinh 2U + (\partial \mathbf{F_2}/\partial x_2)*\cosh 2U)*\exp(+\mathbf{u})*\mathbf{g} +$$
$$-\exp(-\mathbf{u})*(\partial \mathbf{F_2}/\partial y_2)*\exp(+\mathbf{u})*\mathbf{e} +$$
$$-\exp(-\mathbf{u})*(\partial \mathbf{F_2}/\partial z_2)*\exp(+\mathbf{u})*\mathbf{f}$$
$$=$$
$$\exp(-\mathbf{u})*((\partial F_2/\partial t_2)*(\cosh 2U - \sinh 2U*\mathbf{g})) +$$
$$- \exp(-\mathbf{u})*((\partial F_2/\partial x_2)*(\cosh 2U - \sinh 2U*\mathbf{g})*\mathbf{g})*\exp(+\mathbf{u}) +$$
$$- \exp(-\mathbf{u})*(\partial F_2/\partial y_2)*\mathbf{e}*\exp(-\mathbf{u}) +$$
$$- \exp(-\mathbf{u})*(\partial F_2/\partial z_2)*\mathbf{f}*\exp(-\mathbf{u})$$
$$=$$
$$\exp(-\mathbf{u})*((\partial F_2/\partial t_2) * \exp(-2\mathbf{u}))*\exp(+\mathbf{u}) +$$
$$-\exp(-\mathbf{u})*((\partial F_2/\partial x_2)*\mathbf{g}*\exp(-2\mathbf{u}))*\exp(+\mathbf{u}) +$$
$$-\exp(-\mathbf{u})*(\partial F_2/\partial y_2)*\mathbf{e}*\exp(-\mathbf{u}) +$$
$$-\exp(-\mathbf{u})*(\partial F_2/\partial z_2)*\mathbf{f}*\exp(-\mathbf{u})$$
$$= \exp(-\mathbf{u})*((\partial F_2/\partial t_2) +(\partial F_2/\partial x_2)\mathbf{g} +(\partial F_2/\partial y_2)\mathbf{e} +(\partial F_2/\partial z_2)\mathbf{f})*\exp(-\mathbf{u}) =$$
$$= \exp(-\mathbf{u})*(\nabla^X{}_2\mathbf{F_2})*\exp(-\mathbf{u})$$
$$=$$
$$\exp(-\mathbf{u})*(\mathcal{J}_2)*\exp(-\mathbf{u})$$

therefore, finally:

$$\boxed{\mathcal{J}_2 = \exp(\mathbf{u})*(\mathcal{J})*\exp(\mathbf{u})}$$

Therefore, Lorentz transform of $\mathcal{J}$ follows 4-vector transform and is consistent with the hermitian conjugate formula.

2) Rotation transform of $\mathcal{J}$. [note[8]]

We take as previously, a generator vector **v** aligned with the base vector **g** (x axis), except this time **v** has imaginary coefficient. **v** = *i* V **g** where V is a real number and *i* is imaginary unit (also *i*=**efg**)

$$\mathbf{q_2} = \exp(\mathbf{v})\ \mathbf{q}\ \exp(-\mathbf{v})$$

$\mathbf{q_2} = \exp(\mathbf{v})\ \mathbf{q}\ \exp(-\mathbf{v}) = t + \mathbf{g}x + \exp(2\mathbf{v})\mathbf{e}y + \exp(2\mathbf{v})\mathbf{f}z =$ [note[9]]

$t +$
$\mathbf{g}x +$
$\mathbf{e}y*\cos(2V) - \mathbf{f}y*\sin(2V) +$
$\mathbf{f}z*\cos(2V) + \mathbf{e}z*\sin(2V)$
$=$
$t +$ // t_2
$\mathbf{g}x +$ // $x_2\mathbf{g}$
$\mathbf{e}y*\cos(2V) + \mathbf{e}z*\sin(2V) +$ // $y_2\mathbf{e}$
$\mathbf{f}z*\cos(2V) - \mathbf{f}y*\sin(2V)$ // $z_2\mathbf{f}$

therefore:

$t_2 = t$
$x_2 = x$
$y_2 = y*\cos(2V) + z*\sin(2V)$
$z_2 = z*\cos(2V) - y*\sin(2V)$

We will transform now (in reverse) the $\mathcal{J} = \nabla^{I}\mathbf{F}$:

$$\mathcal{J} = \nabla^{X}\mathbf{F} = \partial\mathbf{F}/\partial t - (\partial\mathbf{F}/\partial x)\mathbf{g} - (\partial\mathbf{F}/\partial y)\mathbf{e} - (\partial\mathbf{F}/\partial z)\mathbf{f}$$

$\partial\mathbf{F}/\partial t = (\partial\mathbf{F}/\partial t_2)*(\partial t_2/\partial t) + (\partial\mathbf{F}/\partial x_2)*(\partial x_2/\partial t) + (\partial\mathbf{F}/\partial y_2)*(\partial y_2/\partial t) + (\partial\mathbf{F}/\partial z_2)*(\partial z_2/\partial t) = (\partial\mathbf{F}/\partial t_2)$

$\partial\mathbf{F}/\partial x = (\partial\mathbf{F}/\partial t_2)*(\partial t_2/\partial x) + (\partial\mathbf{F}/\partial x_2)*(\partial x_2/\partial x) + (\partial\mathbf{F}/\partial y_2)*(\partial y_2/\partial x) + (\partial\mathbf{F}/\partial z_2)*(\partial z_2/\partial x) = (\partial\mathbf{F}/\partial x_2)$

$\partial\mathbf{F}/\partial y = (\partial\mathbf{F}/\partial t_2)*(\partial t_2/\partial y) + (\partial\mathbf{F}/\partial x_2)*(\partial x_2/\partial y) + (\partial\mathbf{F}/\partial y_2)*(\partial y_2/\partial y) + (\partial\mathbf{F}/\partial z_2)*(\partial z_2/\partial y) =$
$(\partial\mathbf{F}/\partial y_2)*(\cos 2V) - (\partial\mathbf{F}/\partial z_2)*(\sin 2V)$

$\partial\mathbf{F}/\partial z = (\partial\mathbf{F}/\partial t_2)*(\partial t_2/\partial z) + (\partial\mathbf{F}/\partial x_2)*(\partial x_2/\partial z) + (\partial\mathbf{F}/\partial y_2)*(\partial y_2/\partial z) + (\partial\mathbf{F}/\partial z_2)*(\partial z_2/\partial z) =$
$(\partial\mathbf{F}/\partial z_2)*(\cos 2V) + (\partial\mathbf{F}/\partial y_2)*(\sin 2V)$

Now, we will substitute the **F** expressed in terms of the inverse-transformed $\mathbf{F_2}$ into the above formulae: $\mathbf{F} = \exp(-\mathbf{v})\ \mathbf{F_2}\ \exp(\mathbf{v})$. This will yield:

$$\mathcal{J} = \nabla^{X}\mathbf{F} = \exp(-\mathbf{v})*(\ (\partial\mathbf{F_2}/\partial t_2))*\exp(\mathbf{v}) +$$

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[8] This chapter is almost identical as in ref [13], because rotation transform is the same for EM field Clifford 8-vector **F** as for 4-vectors such as **a** and $\mathcal{J}$.

[9] We use the following property of exponential function: exp(**g**U)***g** = **g***exp(**g**U) , For perpendicular vectors (i.e. for anti-commuting vectors), for example **g** and **e** it follows exp(**g**U)***e** = **e***exp(-**g**U)

$-\exp(-\mathbf{v})*((\partial\mathbf{F}_2/\partial x_2))*\exp(\mathbf{v})*\mathbf{g} +$

$-\exp(-\mathbf{v})*((\partial\mathbf{F}_2/\partial y_2)*(\cos 2V) - (\partial\mathbf{F}_2/\partial z_2)*(\sin 2V))*\exp(\mathbf{v})*\mathbf{e} +$

$-\exp(-\mathbf{v})*((\partial\mathbf{F}_2/\partial z_2)*(\cos 2V) + (\partial\mathbf{F}_2/\partial y_2)*(\sin 2V))*\exp(\mathbf{v})*\mathbf{f}$

=

$\exp(-\mathbf{v})*((\partial\mathbf{F}_2/\partial t_2))*\exp(\mathbf{v}) +$

$-\exp(-\mathbf{v})*((\partial\mathbf{F}_2/\partial x_2))*\exp(\mathbf{v})*\mathbf{g} +$

$-\exp(-\mathbf{v})*((\partial\mathbf{F}_2/\partial y_2)*(\cos 2V)*\mathbf{e} - (\partial\mathbf{F}_2/\partial z_2)*(\sin 2V)*\mathbf{e} +$

$(\partial F_2/\partial z_2)*(\cos 2V)*\mathbf{f} + (\partial\mathbf{F}_2/\partial y_2)*(\sin 2V)*\mathbf{f})*\exp(-\mathbf{v})$

=

$\exp(-\mathbf{v})*((\partial\mathbf{F}_2/\partial t_2))*\exp(\mathbf{v}) +$

$-\exp(-\mathbf{v})*((\partial\mathbf{F}_2/\partial x_2))*\mathbf{g}*\exp(\mathbf{v}) +$

$-\exp(-\mathbf{v})*((\partial\mathbf{F}_2/\partial y_2)*(\cos 2V)\mathbf{e} + (\partial\mathbf{F}_2/\partial y_2)*(\sin 2V)*\mathbf{ggf} +$

$(\partial\mathbf{F}_2/\partial z_2)*(\cos 2V)\mathbf{f} - (\partial\mathbf{F}_2/\partial z_2)*(\sin 2V)*\mathbf{gge})*\exp(-\mathbf{v})$

=

$\exp(-\mathbf{v})*((\partial\mathbf{F}_2/\partial t_2))*\exp(\mathbf{v}) +$

$-\exp(-\mathbf{v})*((\partial\mathbf{F}_2/\partial x_2))*\mathbf{g}*\exp(\mathbf{v}) +$

$-\exp(-\mathbf{v})*((\cos 2V)*(\partial\mathbf{F}_2/\partial y_2)\mathbf{e} + (\sin 2V)*(\partial\mathbf{F}_2/\partial y_2)*\mathbf{g(gf)} +$ //note[10]

$(\cos 2V)*(\partial\mathbf{F}_2/\partial z_2)\mathbf{f} - (\sin 2V)*(\partial\mathbf{F}_2/\partial z_2)*\mathbf{g(ge)})*\exp(-\mathbf{v})$

=

$\exp(-\mathbf{v})*((\partial\mathbf{F}_2/\partial t_2))*\exp(\mathbf{v}) +$

$-\exp(-\mathbf{v})*((\partial\mathbf{F}_2/\partial x_2))*\mathbf{g}*\exp(\mathbf{v}) +$

$-\exp(-\mathbf{v})*((\cos 2V)*(\partial\mathbf{F}_2/\partial y_2)*\mathbf{e} - (\sin 2V)*(\partial\mathbf{F}_2/\partial y_2)*\mathbf{g}(\mathbf{e}i) +$

$(\cos 2V)*(\partial\mathbf{F}_2/\partial z_2)*\mathbf{f} - (\sin 2V)*(\partial\mathbf{F}_2/\partial z_2)*\mathbf{g}(i\mathbf{f}))*\exp(-\mathbf{v})$

=

$\exp(-\mathbf{v})*((\partial\mathbf{F}_2/\partial t_2))*\exp(\mathbf{v}) +$

$-\exp(-\mathbf{v})*((\partial\mathbf{F}_2/\partial x_2))*\mathbf{g}*\exp(\mathbf{v}) +$

$-\exp(-\mathbf{v})*((\partial\mathbf{F}_2/\partial y_2)*(\cos 2V - \mathbf{g}i*\sin 2V)*\mathbf{e} +$

$(\partial\mathbf{F}_2/\partial z_2)*(\cos 2V - \mathbf{g}i*\sin 2V)*\mathbf{f})*\exp(-\mathbf{v})$

=

$\exp(-\mathbf{v})*((\partial\mathbf{F}_2/\partial t_2))*\exp(\mathbf{v}) +$

$-\exp(-\mathbf{v})*((\partial\mathbf{F}_2/\partial x_2))*\mathbf{g}*\exp(\mathbf{v}) +$

$-\exp(-\mathbf{v})*((\partial\mathbf{F}_2/\partial y_2)\ \exp(-2\mathbf{v})\ *\mathbf{e}* + (\partial\mathbf{F}_2/\partial z_2)*\exp(-2\mathbf{v})*\mathbf{f})*\exp(-\mathbf{v})$

=

$\exp(-\mathbf{v})*((\partial\mathbf{F}_2/\partial t_2))*\exp(\mathbf{v}) +$

$\exp(-\mathbf{v})*(- (\partial\mathbf{F}_2/\partial x_2))*\mathbf{g}*\exp(\mathbf{v}) +$

$\exp(-\mathbf{v})*(- (\partial\mathbf{F}_2/\partial y_2)*\mathbf{e}*\exp(+2\mathbf{v}) - (\partial\mathbf{F}_2/\partial z_2)*\mathbf{f}*\exp(+2\mathbf{v}))*\exp(-\mathbf{v}) =$

$\exp(-\mathbf{v})*((\partial\mathbf{F}_2/\partial t_2)) - (\partial\mathbf{F}_2/\partial x_2))*\mathbf{g} - (\partial\mathbf{F}_2/\partial y_2)*\mathbf{e} - (\partial\mathbf{F}_2/\partial z_2)*\mathbf{f})*\exp(+\mathbf{v})$

$= \exp(-\mathbf{v})*(\nabla^{X}{}_2\mathbf{F}_2)*\exp(\mathbf{v})$

$= \exp(-\mathbf{v})*(\mathcal{J}_2)*\exp(+\mathbf{v})$

Finally

[10] $\mathbf{gf} = -\mathbf{fg} = -\mathbf{eefg} = -\mathbf{e}i$ and $\mathbf{ge} = \mathbf{geff} = \mathbf{efgf} = i\mathbf{f}$

$\mathcal{J}_2$ = exp(+**v**)*($\mathcal{J}$)*exp(-**v**)

Conclusion

In a general case, we can combine the **u** and **v** cases in one formula that employs the Hermitian conjugate $( )^T$ [note[11]]

$\mathcal{J}_2$ = exp(**p**)*($\mathcal{J}$)*(exp(**p**))^T

References:

1)
"Representing 3D vectors, 4-vectors, rotation and Lorentz transform using Clifford Algebra $Cl_{3,0}$(R).", Stan Bleszynski, 4-March-2018 https://goo.gl/wuB43b

2) reserved

3)
"Zero Length Bases in Clifford Algebra $Cl_{3,0}$({1,h})", Stan Bleszynski 16-April-2018 (Updated 15/06/2018) link: https://goo.gl/kTh5jb

4)
"Electromagnetics in Clifford Algebra"
Stan Bleszynski, David Coffey, 13-Jan-2019,
https://docs.google.com/document/d/1ki-N2au3AVE5Mo7KR6CVY6TZDwwkxcnJYg0mJ--K58M/edit

5) "Hamilton Equation in Clifford $CL_{3,0}$(R) ", Stan Bleszynski
17/10/2019]
https://docs.google.com/document/d/1_jWOF3cku0K8FwpMcgCUtkz69uJy0_emsQdblJdF9lg/edit?usp=sharing

6) "Standard model physics from an algebra?" by C. Furey
PhD thesis, Waterloo, Ontario, Canada, 2015
https://arxiv.org/pdf/1611.09182.pdf

[11] exp(**p**) := exp(**u**)*exp(**v**) , and exp(**u**)*exp(**v**))' = exp(**v'**)*exp(**u'**)

13) Proof that EM Clifford 8-vector field F transforms in $Cl_{3,0}$(R) differently, using quaternion conjugate formula, unlike Clifford 4-vectors transforming using hermitian conjugate formula.
https://docs.google.com/document/d/1e37GgeSA9JOr9nNNRrosEGmPmlhhJX1dBQzuvS8B_04/edit?usp=sharing