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Assignment #02

System of Linear Equations

By

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(Q#1)

$x + 2y + z = 1$
$2x - 2y + 3z = 1$
$x + 2y - (a^2 - 3)z = a$

Sol

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & -2 & 3 \\ 1 & 2 & 3-a^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ a \end{bmatrix}$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & -2 & 3 & 1 \\ 1 & 2 & 3-a^2 & a \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 2-a^2 & a-1 \end{array}\right] \quad \begin{array}{l} R_2 = R_2 + (-2R_1) \\ R_3 = R_3 + (-R_1) \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 2-a^2 & a-1 \end{array}\right] \quad R_1 = R_1 + (-R_2)$$

(i) No solution if $2 - a^2 = 0$

$a^2 = 2$

$\boxed{a = \pm\sqrt{2}}$

(ii) Exactly one Sol if $2 - a^2 \neq 0$ and $a - 1 \neq 0$

$a \neq \pm\sqrt{2}$ and $a \neq 1$

$\boxed{a = R - (\pm\sqrt{2}, 1)}$

(iii) Infinite Solutions. $a - 1 = 0$

$\boxed{a = 1}$

(Q#2)

a) $x_1 - 4x_2 = 1$
$2x_1 - x_2 = -3$
$-x_1 - 3x_2 = 4$

If the system has solution, that these lines will have intersection point

$$\begin{bmatrix} 1 & -4 \\ 2 & -1 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$$

$$= \left[\begin{array}{cc|c} 1 & -4 & 1 \\ 2 & -1 & -3 \\ -1 & -3 & 4 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 1 & -4 & 1 \\ 0 & 7 & -5 \\ 0 & -7 & 5 \end{array}\right] \quad \begin{array}{l} R_2 = R_2 + (-2R_1) \\ R_3 = R_3 + R_1 \end{array}$$

$$= \left[\begin{array}{cc|c} 1 & -4 & 1 \\ 0 & 7 & -5 \\ 0 & 0 & 0 \end{array}\right] \quad R_3 = R_3 + R_2$$

$$x_2 = \frac{-5}{7}$$

$$x_1 = \frac{7-20}{7} = \frac{-13}{7}$$

So lines intersect at $(-13/7, -5/7)$.

b) Yes, it is

eq① $= 5(3) - 4 + 2(-2)$
$= 7 = $ R·H·S

eq② $= -2(3) + 6(4) + 9(-2)$
$= 0 = $ R·H·S

③ $= -7(3) + 5(4) - 3(-2)$
$= -7 = $ R·H·S

c)

$$a = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -3 & 2 & 0 \\ -4 & 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ -5 \end{bmatrix} = \text{Yes}$$

$$\text{Proof} = \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & -1 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & -3 & 2 & 0 & 0 \\ -4 & 0 & 0 & 7 & -5 \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & -1 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 2 & -3 & 6 \\ 0 & 0 & 0 & 7 & -9 \end{array}\right] \quad \begin{array}{l} R_3 = R_3 + (3R_2) \\ R_4 = R_4 + (4R_1) \end{array}$$

from last row. $7x_4 = -9$

$x_4 = -9/7$

from R_3

$2x_3 - 3x_4 = 6$

$x_3 = \dfrac{6 + 3(-9/7)}{2}$

$x_3 = \dfrac{15}{14}$

from R_2

$x_2 = 2 + x_3$

$x_2 = 43/14$

from R_1 $x_1 = -1 + 2x_3$

So system is consistent

(Q#2, c)

b) $$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & -2 & 3 & 2 \\ 3 & 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 5 \end{bmatrix}$$ Yes,

$$= \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 3 & 0 & 0 & 7 & 5 \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 0 & 0 & -9 & 7 & -1 \end{array}\right] \begin{array}{l} \\ \\ R_3 + 2R_2 \\ R_4 + (-3)R_1 \end{array}$$

$$= \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 0 & 0 & 0 & -5 & 20 \end{array}\right]$$

So yes system is consistent.
as we can back substitute to get values

c) $$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ -2 & 3 & 2 & 1 & 5 \end{array}\right]$$, Yes

$$= \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 3 & 2 & -3 & -1 \end{array}\right] \quad \begin{array}{l} R_2 \times \frac{1}{2} \\ R_4 + 2R_1 \end{array}$$

$$= \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -3 & -1 \end{array}\right] \quad R_4 + (-3R_2)$$

$$= \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_4 + R_3$$

The sys is consistent and indeterminate.

(Q#2, d)

d) $R_3 + (-4R_2) \longrightarrow R_3 \times \frac{1}{9}$

(Q#3)

a)
$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -3 & -9 \\ 0 & 1 & 0 & 4 & 2 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & -12 \\ 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \quad \begin{array}{l} R_1 + 3R_4 \\ R_2 + (-4R_4) \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -24 \\ 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \quad R_1 + (-2R_2)$$

<u>Sol for system of eq:</u>

$x_1 = -24, \quad x_2 = 6, \quad x_3 = 5, \quad x_4 = -1$

b)

$$\left[\begin{array}{ccc|c} 1 & 7 & 3 & -4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 7 & 3 & -4 \\ 0 & 1 & -1 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad R_3 \leftrightarrow R_4$$

$$\left[\begin{array}{cccc} 1 & 7 & 3 & -4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad R_3 + (-R_2)$$

$x_3 = 1/2$ from R_3

from R_2

$x_2 - x_3 = 3$

$x_2 = 3 + 1/2 = 7/2$

from R_1

$x_1 + 7x_2 + 3x_3 = -4$

$x_1 = -4 - \frac{49}{2} - \frac{9}{2}$

$x_1 = -33$

(Q#3, c)

c) $\begin{bmatrix} 1 & -4 & 9 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & -4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ $R_1 + (-\frac{9}{2}R_3)$, $R_2 + (-\frac{7}{2}R_3)$, $R_3 \times \frac{1}{2}$

$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ $R_1 + (4R_2)$

$S.S = x_1 = x_2 = x_3 = 0$

(Q#4)

(i) Reduced Echelon (Canonical Form) form. OR I

$\{R_2 + (-l_{21}R_1), R_3 + (-l_{31}R_1)\} \to \{R_3 + (-l_{32}R_2)\}$

(ii) Applying the steps in prev question on I

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $R_2 + (l_{21}R_1)$, $R_3 + (-l_{31}R_1)$

$= \begin{bmatrix} 1 & 0 & 0 \\ -l_{21} & 1 & 0 \\ -l_{31} & 0 & 1 \end{bmatrix}$ $R_3 + (-l_{32}R_2)$

$= \begin{bmatrix} 1 & 0 & 0 \\ -l_{21} & 1 & 0 \\ l_{21}l_{32}-l_{31} & -l_{32} & 1 \end{bmatrix}$

This is the Matrix we get when we apply same operations. which is L^{-1}

In part (a) we said doing Gaussian elimination on L gives I. So

$$EL = I \quad — (\mathrm{I})$$

where E is matrix for elimination (Row operations)

Since $EL = I$ so $E = L^{-1}$ — (II)

c) If we do the same steps to $A = LU$ we get

$$EA = ELU$$

$$EA = IU \quad \because \text{from } (\mathrm{I})$$

$$L^{-1}A = U \quad \because \text{from } (\mathrm{II})$$

(Q#5)

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 2 \\ 4 & 4 & 1 & 1 & 11 \\ 1 & -1 & -1 & 2 & 0 \\ 2 & 1 & 2 & -2 & 2 \end{array}\right]$$

Sol:

(i) Naive

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 2 \\ 0 & 0 & -3 & 5 & 3 \\ 0 & -2 & -2 & 3 & -2 \\ 0 & -1 & 0 & 0 & -2 \end{array}\right] \begin{array}{l} R_2 + (-4R_1) \\ R_3 + (-R_1) \\ R_4 + (-R_1) \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & -3 & 5 & 3 \\ 0 & 0 & -2 & 3 & 2 \\ 0 & -1 & 0 & 0 & -2 \end{array}\right] \begin{array}{l} R_1 + (-R_4) \\ R_3 + (-2R_4) \\ \\ \\ \end{array}$$

$$\therefore \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & -2 & 3 & 2 \\ 0 & -1 & 0 & 0 & -2 \end{array}\right] \quad R_2 + \left(-\frac{3}{2}R_3\right)$$

from R_2

$$x_4 = 0$$

from R_3

$$-2x_3 + 3x_4 = 2$$
$$x_3 = -1$$

from R_4

$$-x_2 = -2$$
$$x_2 = 2$$

from R_1

$$x_1 + x_3 - x_4 = 0$$
$$x_1 - 1 = 0$$
$$x_1 = 1$$

(ii) Partial Pivoting.

$$= \begin{bmatrix} 1 & 1 & 1 & -1 & 2 \\ 4 & 4 & 1 & 1 & 11 \\ 1 & -1 & -1 & 2 & 0 \\ 2 & 1 & 2 & -2 & 2 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$= \begin{bmatrix} 4 & 4 & 1 & 1 & 11 \\ 1 & 1 & 1 & -1 & 2 \\ 1 & -1 & -1 & 2 & 0 \\ 2 & 1 & 2 & -2 & 2 \end{bmatrix} \quad \begin{array}{l} R_4 + \left(-\frac{1}{2}R_1\right) \\ R_2 + \left(-\frac{1}{4}R_1\right) \\ R_3 + \left(-\frac{1}{4}R_1\right) \end{array}$$

$$= \begin{bmatrix} 4 & 4 & 1 & 1 & 11 \\ 0 & 0 & 0.75 & -1.25 & -0.75 \\ 0 & -2 & -1.25 & 1.75 & -2.75 \\ 0 & -1 & 1.5 & -2.5 & 1.5 \end{bmatrix} \quad R_2 \leftrightarrow R_4$$

$$\begin{bmatrix} 4 & 4 & 1 & 1 & 11 \\ 0 & -1 & 1.5 & -2.5 & 1.5 \\ 0 & -2 & -1.25 & 1.75 & -2.75 \\ 0 & 0 & 0.75 & -1.25 & -0.75 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 4 & 4 & 1 & 1 & 11 \\ 0 & -2 & -1.25 & 1.75 & -2.75 \\ 0 & -1 & 1.5 & -2.5 & 1.5 \\ 0 & 0 & 0.75 & -1.25 & -0.75 \end{bmatrix} \quad R_3 + (-1/2\, R_2)$$

$$\begin{bmatrix} 4 & 4 & 1 & 1 & 11 \\ 0 & -2 & -1.25 & 1.75 & -2.75 \\ 0 & 0 & 2.125 & -3.375 & -3.5 \\ 0 & 0 & 0.75 & -1.25 & -0.75 \end{bmatrix} \quad R_4 + (-0.3529\, R_3)$$

$$\begin{bmatrix} 4 & 4 & 1 & 1 & 11 \\ 0 & -2 & 1.25 & 1.75 & -2.75 \\ 0 & 0 & 2.125 & -3.375 & -3.5 \\ 0 & 0 & 0 & -0.0588 & 0 \end{bmatrix}$$

from R_4

$x_4 = 0$

from R_3

$x_3 = -1$

from R_2

$x_2 = 2$

from R_1

$x_1 = 1$

Verifying

eq: 1 $\Rightarrow (1) + (2) + (-1) - (0) = 2$

eq: 2 $\Rightarrow 4(1) + 4(2) + (-1) - (0) = 11$

eq: 3 $\Rightarrow (1) - (2) - (-1) + 2(0) = 0$

eq: 4 $\Rightarrow 2(1) + (2) + 2(-1) - 2(0) = 2$

(iii) Complete Pivoting

$$\begin{bmatrix} 1 & 1 & 1 & -1 & 2 \\ 4 & 4 & 1 & 1 & 11 \\ 1 & -1 & -1 & 2 & 0 \\ 2 & 1 & 2 & -2 & 2 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 4 & 4 & 1 & 1 & 11 \\ 1 & 1 & 1 & -1 & 2 \\ 1 & -1 & -1 & 2 & 0 \\ 2 & 1 & 2 & -2 & 2 \end{bmatrix} \quad \begin{matrix} R_2 + (-0.25)R1 \\ R_3 + (-0.25)R1 \\ R_4 + (-0.5)R1 \end{matrix}$$

$$\begin{bmatrix} 4 & 4 & 1 & 1 & 11 \\ 0 & 0 & 0.75 & -1 & -0.75 \\ 0 & -2 & -1.25 & 1 & -2.75 \\ 0 & -1 & 1.5 & -2 & -3.5 \end{bmatrix} \quad \begin{matrix} R_2 \leftrightarrow R_4 \\ C_2 \leftrightarrow C_4 \end{matrix}$$

$[x1, x4, x3, x2]$

$$\begin{bmatrix} 4 & 1 & 1 & 4 & 11 \\ 0 & -2.5 & 1.5 & -1 & -3.5 \\ 0 & 1.75 & -1.25 & -2 & -2.75 \\ 0 & -1.25 & 0.75 & 0 & -0.75 \end{bmatrix} \quad \begin{matrix} R_3 + 0.7R_2 \\ R_4 + -0.5R_2 \end{matrix}$$

$$\begin{bmatrix} 4 & 1 & 4 & 1 & 11 \\ 0 & -2.5 & -1 & 1.5 & -3.5 \\ 0 & 0 & -2.7 & -0.2 & -5.2 \\ 0 & 0 & 0.5 & 0 & 1 \end{bmatrix} \quad C_3 \leftrightarrow C_4$$

$[x1, x4, x2, x3]$

$$\begin{bmatrix} 4 & 1 & 4 & 1 & 11 \\ 0 & -2.5 & -1 & 1.5 & -3.5 \\ 0 & 0 & -2.7 & -0.2 & -5.2 \\ 0 & 0 & 0.5 & 0 & 1 \end{bmatrix}$$

$R_4 + (0.1851851851)R3$

$$\left[\begin{array}{ccccc} 4 & 1 & 4 & 1 & 11 \\ 0 & -2.5 & -1 & 1.5 & -3.5 \\ 0 & 0 & -2.7 & -0.2 & -5.2 \\ 0 & 0 & 0 & -0.037 & 0.037 \end{array}\right]$$

Since order $[x_1, x_4, x_2 x_3]$

So,

from R_4

$x_3 = -1$

from R_3

$x_2 = 2$

from R_2

$x_4 = 0$

from R_1

$x_1 = 1$

Hence Proved, same set of ans $\{1, 2, -1, 0\}$ from all methods.

(iv)

Complete Pivoting method is prefered for its stability.
But I prefer partial pivoting on computer due to less row operations.
By hand I prefer Naive due to it's simplicity and easy calculations.

1 Question # 5 Part B

1.1 Source Code Partial Pivoting

```
import numpy as np # Only using to copying on arrays

# Method to display matrix
def display(matrix):
    for i in range(N):
        print("[", end="")
        for j in range(N+1):
            print("{:10.4f} ".format(matrix[i][j]), end="")
        print("]")

def echelon(augmented_matrix, output=True):

    # Converting orignal matrix to np array
    # To avoid changes in orignal matrix
    # And improve calculation speed
    matrix = np.array(augmented_matrix).tolist()

    for i in range(N-1):

        # Finding the pivot
        m = 0 # Storing max as pivot
        mr = 0 # Storing pivot row
        for j in range(i, N):
            if abs(m) < abs(matrix[j][i]):
                m = matrix[j][i]
                mr = j
        # IF Max value is 0.
        # Then we skip the procedure
        # As we will not be able to find
        # all the unknows because equations
        # are dependent
        if m == 0:
             return None

        #Showing New Pivot
        if output:
            print("Pivot: ", m)

        # Checking if swap is required
        if mr != i:
            # Swapping
            temp = matrix[i]
            matrix[i] = matrix[mr]
            matrix[mr] = temp
            if output:
                print("Swapping R{} with R{}".format(i+1, mr+1))
                display(matrix)
```

```
        # making all entries below pivot 0
        print("Row Operation:")
        for j in range(i+1, N):

            # Fetching Multiplying factor
            multiplying_factor = matrix[j][i] / m

            # Skip Row Operation if multiplying factor is 0
            if abs(multiplying_factor) == 0:
                continue

            if output:
                # Showing the row operation to be performed
                print("R"+str(j+1)+" +
("+str(multiplying_factor)+")R"+str(i+1))

            # Multiplying and Adding element
            for k in range(i, N+1):
                matrix[j][k] += multiplying_factor * matrix[i][k] * 1.0

        if output:
            # Showing matrix
            display(matrix)

    # Returning reduced matrix
    return matrix

def back_substitution_echelon(echelon_matrix, output=True):

    # Converting orignal matrix to np array
    # To avoid changes in orignal matrix
    # And improve calculation speed
    matrix = np.array(echelon_matrix)

    # Performing Back Substitution
    results = []

    # Getting last unknown's value
    results.append(matrix[N-1][N] / matrix[N-1][N-1])

    # Looping through all other rows for unknows
    for i in range(N-2, -1, -1):

        # Constant on RHS of Row
        x = matrix[i][N]

        # Iterate through row elements
        k = 0
        for j in range(N-1, i, -1):

            # Multiply row element with respective variable
            # Then subtract it on RHS
```

```
            x -= matrix[i][j] * results[k]

            # Change the unknown
            k += 1

        # Divide the last co-efficient of unknown on RHS
        # To get the value of unknown
        results.append(x / matrix[i][i])

    # Return Results in Array
    return results


# Getting Number of equations
N = 4

# Augmented Matrix
augmented_matrix = [
    [1,  1,  1, -1,  2],
    [4,  4,  1,  1, 11],
    [1, -1, -1,  2,  0],
    [2,  1,  2, -2,  2]
]


display(augmented_matrix)

back_substitution_echelon(echelon(augmented_matrix))
```

1.2 Output

```
[    1.0000     1.0000     1.0000    -1.0000     2.0000 ]
[    4.0000     4.0000     1.0000     1.0000    11.0000 ]
[    1.0000    -1.0000    -1.0000     2.0000     0.0000 ]
[    2.0000     1.0000     2.0000    -2.0000     2.0000 ]
Pivot:  4
Swapping R1 with R2
[    4.0000     4.0000     1.0000     1.0000    11.0000 ]
[    1.0000     1.0000     1.0000    -1.0000     2.0000 ]
[    1.0000    -1.0000    -1.0000     2.0000     0.0000 ]
[    2.0000     1.0000     2.0000    -2.0000     2.0000 ]
Row Operation:
R2 + (-0.25)R1
R3 + (-0.25)R1
R4 + (-0.5)R1
[    4.0000     4.0000     1.0000     1.0000    11.0000 ]
[    0.0000     0.0000     0.7500    -1.2500    -0.7500 ]
[    0.0000    -2.0000    -1.2500     1.7500    -2.7500 ]
[    0.0000    -1.0000     1.5000    -2.5000    -3.5000 ]
Pivot:  -2.0
Swapping R2 with R3
[    4.0000     4.0000     1.0000     1.0000    11.0000 ]
[    0.0000    -2.0000    -1.2500     1.7500    -2.7500 ]
[    0.0000     0.0000     0.7500    -1.2500    -0.7500 ]
[    0.0000    -1.0000     1.5000    -2.5000    -3.5000 ]
Row Operation:
R4 + (-0.5)R2
[    4.0000     4.0000     1.0000     1.0000    11.0000 ]
[    0.0000    -2.0000    -1.2500     1.7500    -2.7500 ]
[    0.0000     0.0000     0.7500    -1.2500    -0.7500 ]
[    0.0000     0.0000     2.1250    -3.3750    -2.1250 ]
Pivot:  2.125
Swapping R3 with R4
[    4.0000     4.0000     1.0000     1.0000    11.0000 ]
[    0.0000    -2.0000    -1.2500     1.7500    -2.7500 ]
[    0.0000     0.0000     2.1250    -3.3750    -2.1250 ]
[    0.0000     0.0000     0.7500    -1.2500    -0.7500 ]
Row Operation:
R4 + (-0.35294117647058826)R3
[    4.0000     4.0000     1.0000     1.0000    11.0000 ]
[    0.0000    -2.0000    -1.2500     1.7500    -2.7500 ]
[    0.0000     0.0000     2.1250    -3.3750    -2.1250 ]
[    0.0000     0.0000     0.0000    -0.0588     0.0000 ]
Out[2]:
[-0.0, -1.0, 2.0, 1.0]
```

(Q#6)

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 2 \\ 4 & 4 & 1 & 1 & 11 \\ 1 & -1 & -1 & 2 & 0 \\ 2 & 1 & 2 & -2 & 2 \end{array}\right]$$

(ii) Hybrid Method.

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 2 \\ 0 & 0 & -3 & 5 & 3 \\ 0 & -2 & -2 & 3 & -2 \\ 0 & -1 & 0 & 0 & -2 \end{array}\right] \begin{array}{l} R_2 + (-4R_1) \\ R_3 + (-R_1) \\ R_4 + (-2R_1) \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & -3 & 5 & 3 \\ 0 & 0 & -2 & 3 & 2 \\ 0 & -1 & 0 & 0 & -2 \end{array}\right] \begin{array}{l} R_1 + (R_4) \\ R_3 + (-2R_4) \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & -2 & 3 & 2 \\ 0 & -1 & 0 & 0 & -2 \end{array}\right] \quad R_2 + (-3/2)R_3$$

$R_2 \longleftrightarrow R_4$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 & -2 \\ 0 & 0 & -2 & 3 & 2 \\ 0 & 0 & 0 & 1/2 & 0 \end{array}\right] \text{——— ① Triangular form} \quad \begin{array}{l} R_3 \times (-1/2) \\ R_4 \times (2) \\ R_2 \times (-1) \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -3/2 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} R_1 + R_4 \\ R_3 + (\frac{3}{2}) R_4 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad R_1 - R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

from R_1 $x_1 = 1$

R_2 $x_2 = 2$

R_3 $x_3 = -1$

R_4 $x_4 = 0$

(i) Gauss – Jordan Method

Triangular form

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 & -2 \\ 0 & 0 & -2 & 3 & 2 \\ 0 & 0 & 0 & 1/2 & 0 \end{array}\right]$$

from R_4 $\boxed{x_4 = 0}$

R_3 $-2x_3 + 3x_4 = 2$

$\boxed{x_3 = -1}$

R_2 $-x_2 = -2$

$\boxed{x_2 = 2}$

R_1 $x_1 + x_3 - x_4 = 0$

$x_1 + (-1) = 0$

$\boxed{x_1 = -1}$

2 Question # 6 Part d

2.1 Source Code (We use echelon method from previous question for Triangular Matrix)

```
# Hybrid Method
def reduced_echelon(echelon_matrix, output=True):

    print("Now going reduced echelon")

    #If the matrix is not valid
    if echelon_matrix == None:
        return None

    # Copying orignal matrix to new list
    # To avoid changes in orignal matrix
    # And improve calculation speed
    matrix = np.array(echelon_matrix).tolist()

    # Solutions to store answers
    results = []

    for i in range(N-1, 0, -1):
        for j in range(i-1, -1, -1):

            # Calculating Multiplying factor
            multiplying_factor = (matrix[j][i] / matrix[i][i])

            # Skip Row Operation if multiplying factor is 0
            if abs(multiplying_factor) == 0:
                continue

            if output:
                # Showing the row operation to be performed
                print("R"+str(j)+" + ("+str(multiplying_factor)+")R"+str(i))

            # Multiplying and Adding element
            for k in range(N, -1, -1):
                matrix[j][k] -= multiplying_factor * matrix[i][k] * 1.0

            if output:
                # Showing matrix
                display(matrix)

    # Dividing coeff to get unknowns
    for i in range(N):
        results.append(matrix[i][N] / matrix[i][i])

        # Updating Matrix
        matrix[i][i] = 1
        matrix[i][N] = results[i]
```

```
    # Final Reduced Echelon Matrix
    print("Final Reduced Echelon Matrix")
    display(matrix)

    # Returning results
    return results

# Getting Number of equations
N = 4

# Augmented Matrix
augmented_matrix = [
    [1,  1,  1, -1,  2],
    [4,  4,  1,  1, 11],
    [1, -1, -1,  2,  0],
    [2,  1,  2, -2,  2]
]


display(augmented_matrix)

reduced_echelon(echelon(augmented_matrix))
```

2.2 Output

```
|    1.0000     1.0000     1.0000    -1.0000     2.0000 |
|    4.0000     4.0000     1.0000     1.0000    11.0000 |
|    1.0000    -1.0000    -1.0000     2.0000     0.0000 |
|    2.0000     1.0000     2.0000    -2.0000     2.0000 |
Pivot:  4
Swapping R1 with R2
|    4.0000     4.0000     1.0000     1.0000    11.0000 |
|    1.0000     1.0000     1.0000    -1.0000     2.0000 |
|    1.0000    -1.0000    -1.0000     2.0000     0.0000 |
|    2.0000     1.0000     2.0000    -2.0000     2.0000 |
Row Operation:
R2 + (0.25)R1
R3 + (0.25)R1
R4 + (0.5)R1
|    4.0000     4.0000     1.0000     1.0000    11.0000 |
|    0.0000     0.0000     0.7500    -1.2500    -0.7500 |
|    0.0000    -2.0000    -1.2500     1.7500    -2.7500 |
|    0.0000    -1.0000     1.5000    -2.5000    -3.5000 |
Pivot:  -2.0
Swapping R2 with R3
|    4.0000     4.0000     1.0000     1.0000    11.0000 |
|    0.0000    -2.0000    -1.2500     1.7500    -2.7500 |
|    0.0000     0.0000     0.7500    -1.2500    -0.7500 |
|    0.0000    -1.0000     1.5000    -2.5000    -3.5000 |
Row Operation:
R4 + (0.5)R2
|    4.0000     4.0000     1.0000     1.0000    11.0000 |
|    0.0000    -2.0000    -1.2500     1.7500    -2.7500 |
|    0.0000     0.0000     0.7500    -1.2500    -0.7500 |
|    0.0000     0.0000     2.1250    -3.3750    -2.1250 |
Pivot:  2.125
Swapping R3 with R4
|    4.0000     4.0000     1.0000     1.0000    11.0000 |
|    0.0000    -2.0000    -1.2500     1.7500    -2.7500 |
|    0.0000     0.0000     2.1250    -3.3750    -2.1250 |
|    0.0000     0.0000     0.7500    -1.2500    -0.7500 |
Row Operation:
R4 + (0.35294117647058826)R3
|    4.0000     4.0000     1.0000     1.0000    11.0000 |
|    0.0000    -2.0000    -1.2500     1.7500    -2.7500 |
|    0.0000     0.0000     2.1250    -3.3750    -2.1250 |
|    0.0000     0.0000     0.0000    -0.0588     0.0000 |
```

```
Now going reduced echelon
R2 + (57.374999999999986)R3
|     4.0000      4.0000      1.0000      1.0000     11.0000 |
|     0.0000     -2.0000     -1.2500      1.7500     -2.7500 |
|     0.0000      0.0000      2.1250      0.0000     -2.1250 |
|     0.0000      0.0000      0.0000     -0.0588      0.0000 |
R1 + (-29.749999999999993)R3
|     4.0000      4.0000      1.0000      1.0000     11.0000 |
|     0.0000     -2.0000     -1.2500      0.0000     -2.7500 |
|     0.0000      0.0000      2.1250      0.0000     -2.1250 |
|     0.0000      0.0000      0.0000     -0.0588      0.0000 |
R0 + (-16.999999999999996)R3
|     4.0000      4.0000      1.0000      0.0000     11.0000 |
|     0.0000     -2.0000     -1.2500      0.0000     -2.7500 |
|     0.0000      0.0000      2.1250      0.0000     -2.1250 |
|     0.0000      0.0000      0.0000     -0.0588      0.0000 |
R1 + (-0.5882352941176471)R2
|     4.0000      4.0000      1.0000      0.0000     11.0000 |
|     0.0000     -2.0000      0.0000      0.0000     -4.0000 |
|     0.0000      0.0000      2.1250      0.0000     -2.1250 |
|     0.0000      0.0000      0.0000     -0.0588      0.0000 |
R0 + (0.47058823529411764)R2
|     4.0000      4.0000      0.0000      0.0000     12.0000 |
|     0.0000     -2.0000      0.0000      0.0000     -4.0000 |
|     0.0000      0.0000      2.1250      0.0000     -2.1250 |
|     0.0000      0.0000      0.0000     -0.0588      0.0000 |
R0 + (-2.0)R1
|     4.0000      0.0000      0.0000      0.0000      4.0000 |
|     0.0000     -2.0000      0.0000      0.0000     -4.0000 |
|     0.0000      0.0000      2.1250      0.0000     -2.1250 |
|     0.0000      0.0000      0.0000     -0.0588      0.0000 |
Final Reduced Echelon Matrix
|     1.0000      0.0000      0.0000      0.0000      1.0000 |
|     0.0000      1.0000      0.0000      0.0000      2.0000 |
|     0.0000      0.0000      1.0000      0.0000     -1.0000 |
|     0.0000      0.0000      0.0000      1.0000     -0.0000 |
Out[19]:
[1.0, 2.0, -1.0, -0.0]
```

b) $2[(n-1)+(n-2)+\cdots(1)]$ are the number of multiplications that take generally.

c) Almost Twice the multiplications are required in hybrid method.

(Q#7)

Custom Method:

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ 26 \\ -19 \\ -34 \end{bmatrix}$$

A X B

$AX = B$

$LUX = B$

$UX = Y$ —— (I)

$LY = B$ —— (II)

Now, IA

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix}$$

$R_2 - (2)R_1$; $R_3 - (1/2)R_1$; $R_4 + 1R_1$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1/2 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & -12 & 8 & 1 \\ 0 & 2 & 3 & -14 \end{bmatrix}$$

$R_3 - 3R_2$
$R_4 + \frac{1}{2}R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1/2 & 3 & 1 & 0 \\ -1 & -1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 4 & -13 \end{bmatrix}$$

$R_4 - 2R_3$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1/2 & 3 & 1 & 0 \\ -1 & -1/2 & 2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix}}_{U}$$

$LY = B$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1/2 & 3 & 1 & 0 \\ -1 & -1/2 & 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 16 \\ 26 \\ -19 \\ -34 \end{bmatrix}$$

$$= \begin{cases} y_1 = 16 \\ y_2 = 26 - 2y_1 = -6 \\ y_3 = -19 - \frac{1}{2}y_1 - 3y_2 = -9 \end{cases}$$

$$y_4 = -34 + y_1 + \frac{y_2}{2} + 2y_3 = -3$$

$$UX = Y$$

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -9 \\ -3 \end{bmatrix}$$

From

R_4 $\quad -3x_4 = -3$

$$x_4 = 1$$

R_3 $\quad 2x_3 - 5x_4 = -9$

$$2x_3 = -4$$

$$x_3 = -2$$

R_2 $\quad -4x_2 + 2x_3 + 2x_4 = -6$

$$x_2 = 1$$

R_1 $\quad 6x_1 - 2x_2 + 2x_3 + 4x_4 = 16$

$$x_1 = 3$$

$$\{3, 1, -2, 1\}$$

Verifying.

$$6(3) + (-2)(1) + (2)(-2) + (4)(1) = 16$$

$$12(3) + (-8)(1) + (6)(-2) + (10)(1) = 26$$

$$3(3) + (-13)(1) + (9)(-2) + (3)(1) = -19$$

$$-6(3) + (4)(1) + (1)(-2) + (-18)(1) = -34$$

a) Crout LU

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$$

$$A=\begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} & l_{21}u_{14}+u_{24} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} & l_{31}u_{14}+l_{32}u_{24}+u_{34} \\ l_{41}u_{11} & l_{41}u_{12}+l_{42}u_{22} & l_{41}u_{13}+l_{42}u_{23}+l_{43}u_{33} & l_{41}u_{14}+l_{42}u_{24}+l_{43}u_{34}+u_{44} \end{bmatrix}$$

$u_{11}=6$; $u_{12}=-2$, $u_{13}=2$, $u_{14}=4$

$l_{21}=2$; $l_{31}=1/2$, $l_{41}=-1$, $u_{22}=-4$

$l_{32}=3$; $l_{42}=-1/2$, $l_{43}=2$, $u_{23}=2$

$u_{24}=2$; $u_{33}=2$, $u_{34}=-5$, $u_{44}=-3$

$$L=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1/2 & 3 & 1 & 0 \\ -1 & -1/2 & 2 & 1 \end{bmatrix}, U=\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

Same steps as previos part

$LY = B$

$UX = Y$

$X = \{3, 1, -2, 1\}$

(Q#8)

a) Jacobi:

$x^0 = [0, 0, 0, 0]$

eq. ① $x_1^1 = -1.3 + 4x_2 - x_4 = (-1.3) + (4)(0) - (0) = -1.3$

eq. ⑪ $x_2^1 = -1.8 - x_3 + 4x_4 = -1.8$

eq. (iv) $x_3^1 = -0.5 - x_2 + 4x_1 = -0.5$

eq. (iii) $x_4^1 = -1 - x_1 + 4x_3 = -1.0$

$x^1 = [-1.3, -1.8, -0.5, -1.0]$

$x_1^2 = -7.5$

$x_2^2 = -5.3$

$x_3^2 = -3.9$

$x_4^2 = -3.7$

$x^2 = [-7.5, -5.3, -3.1, -3.7]$

$x^3 = [-18.5, -12.7, -25.2, -8.3$

$x^4 = [-43.8, -9.8, -63., -15.4]$

$x^5 = [-25.1, -0.4, -165.9, -18.8$

b) Error

$= \max\{|x_1^5 - x_1^*|, |x_2^5 - x_2^*|, |x_3^5 - x_3^*|, |x_4^5 - x_4^*|\}$

$= \max\{|0.4125 - (-25.1)|, |0.6125 - (-0.4)|, |0.5357 + 165.9|, |+18.8 + 0.7375|\}$

$= 166.435690$

b) Gauss- Seidal

$x^0 = [0, 0, 0, 0]$

$x^1 = [-1.3, -1.8, -3.9, 0.3]$

$x^2 = [-8.8, 3.3, -39., 9]$

$x^3 = [2.9, 73.2, -62.1, 32.1]$

$x^4 = [259.4, 188.7, 848.4, -132.]$

$x^5 = [885.5, -1378.2, 4999, -1414.5]$

d) Error

Error = 4919.164290

e) Jacobi Method is slower. Gauss Seidal is faster because on every iteration value is updated for solution vector. So, Bigger values are substituted resulting in immediate increase.