Proof of Isochron Formula

The equation for an isochron is:

$$\frac{D_0+\Delta P}{D_{ref}} = \frac{\Delta P}{P_i-\Delta P}\frac{P_i-\Delta P}{D_{ref}} + \frac{D_0}{D_{ref}}$$

The following is a proof that this equation is valid.

The number of atoms at any time t for a radioactive material is given by the formula

$$1:\ N = N_0 e^{-\lambda t} \rightarrow 1A:\ N_0 = Ne^{\lambda t}$$

Where:

N=Number of atoms at time t

N_0 = Original # of atoms

Lambda = decay constant (lambda = ln(2)/half-life)

t = time elapsed

For a daughter element, the amount of daughter produced over any period of time will equal the amount of parent that decayed over that same period of time (assuming that there is only one decay pathway for the parent; if there are multiple pathways, you would simply weight delta P by the likelihood of the pathway to produce the daughter. The resulting math is the same once the weight is applied)

$$2:\ D^* = \Delta P = N_0 - N$$

Where:

D*=The amount of daughter produced

ΔP = Amount of parent lost, equivalent to D*

N_0 = Original amount of parent

N = Amount of parent at time t

Substituting 1A into equation 2 yields:

$$3:\ D^* = Ne^{\lambda t} - N = N(e^{\lambda t} - 1)$$

The amount of daughter isotope today equals the amount that was there originally plus the amount produced. Therefore we can say:

$$4: D = D_0 + D^* = D_0 + N(e^{\lambda t} - 1)$$

In a sample where isochron dating will be used, there will be a stable isotope of the daughter element (D_ref). The amount of this isotope is constant over time. Dividing equation 4 by this amount of stable isotope yields:

$$5:\ \frac{D}{D_{ref}} = \frac{N}{D_{ref}}\left(e^{\lambda t} - 1\right) + \frac{D_0}{D_{ref}}$$

Making a substitution for D from equation 4 (remembering that D* = ΔP, from equation 2), and recognizing that the change in parent isotope is given in equation 3, we can make these substitutions into equation 5 to give us equation 6:

$$6:\ \frac{D_0+\Delta P}{D_{ref}} = \frac{\Delta P}{D_{ref}} + \frac{D_0}{D_{ref}}$$

The amount of parent isotope today is given by:

$$7.\ P = P_i - \Delta P$$

Since it is useful to have the present amount of parent in our formula, we can multiply and divide the second term by equation 7 (effectively multiplying by 1) to yield:

$$8: \frac{D_0+\Delta P}{D_{ref}} = \frac{\Delta P}{D_{ref}}\frac{P_i-\Delta P}{P_i-\Delta P} + \frac{D_0}{D_{ref}}$$

Rearranging slightly:

$$8a: \frac{D_0+\Delta P}{D_{ref}} = \frac{\Delta P}{P_i-\Delta P}\frac{P_i-\Delta P}{D_{ref}} + \frac{D_0}{D_{ref}}$$

QED

Recognize that:

D_0+ΔP = Daughter isotope **today**

P_i-ΔP = Parent isotope **today**

D_ref = Stable daughter isotope **today**