Differential Geometry
MAT432 and MAT733
Professor Sormani
Week 12 Differentiation of Vector Fields
Part I is a priority.

The first submission (with notes, questions, and attempts of homework) is due
Sun Nov 26 at 10pm
The resubmission (with corrections and completed homework for Part I) is due
Sat Dec 2 at 12 noon (extra week for Thanksgiving)
Part I should be completed on time.
Extra Credit and Part II may be submitted later

Googledocs: All work will be submitted by sharing your googledoc for this week with the professor using the correct title on that doc stating the course number, the week number and your name:

MAT432F23-Week12-YourNameHere

MAT733F22-Week12-YourNameHere

Please include a selfie.

Week 12
This lesson has two parts.
Part I should be done on time

Part I: Differentiation, Lagrange Multipliers, and Vector Fields
Part II: Covariant Derivatives and Christoffel Symbols

Note there is a small error in this lesson. If you find it, email me and also tell me about it at the top of your googledoc for extra credit.

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Part I: Differentiation, Lagrange Multipliers, and Vector Fields
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Week 11 Part I must be completed before starting this part.

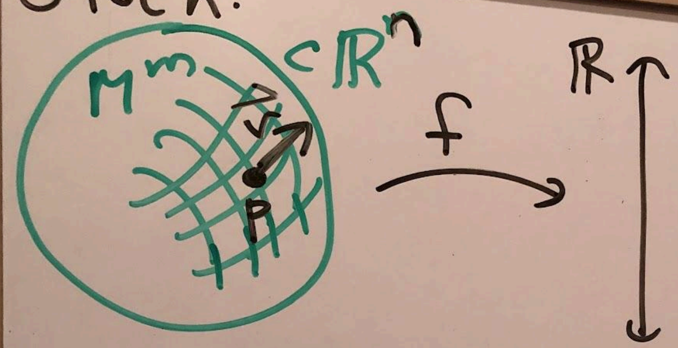
In Video DiffonMPart1a we review:

Defn of directional derivative V(f) of a function f on M by a vector V in T_pM is d/dt f(C(t)) at t=0 where C(t) is any curve such that C(0)=p and C'(0)=v.

Thm: This does not depend on the choice of curve. In fact V(f)=<V, grad f>.

Thm: If a function h on M has a maximum at p then V(h)=0 for every V in T_p(M). Thus grad h is perpendicular to the tangent space T_pM.

Given:



Defn If $\vec{v} \in T_pM$ the derivative of f in the direction $\vec{v}$, $\vec{v}(f)$, is defined to be

$$= \frac{d}{dt} f(c(t))\Big|_{t=0}$$

where

$c(0) = p$

$c'(0) = \vec{v}$

Thm: $\vec{v}(f) = \vec{v} \cdot \nabla f$

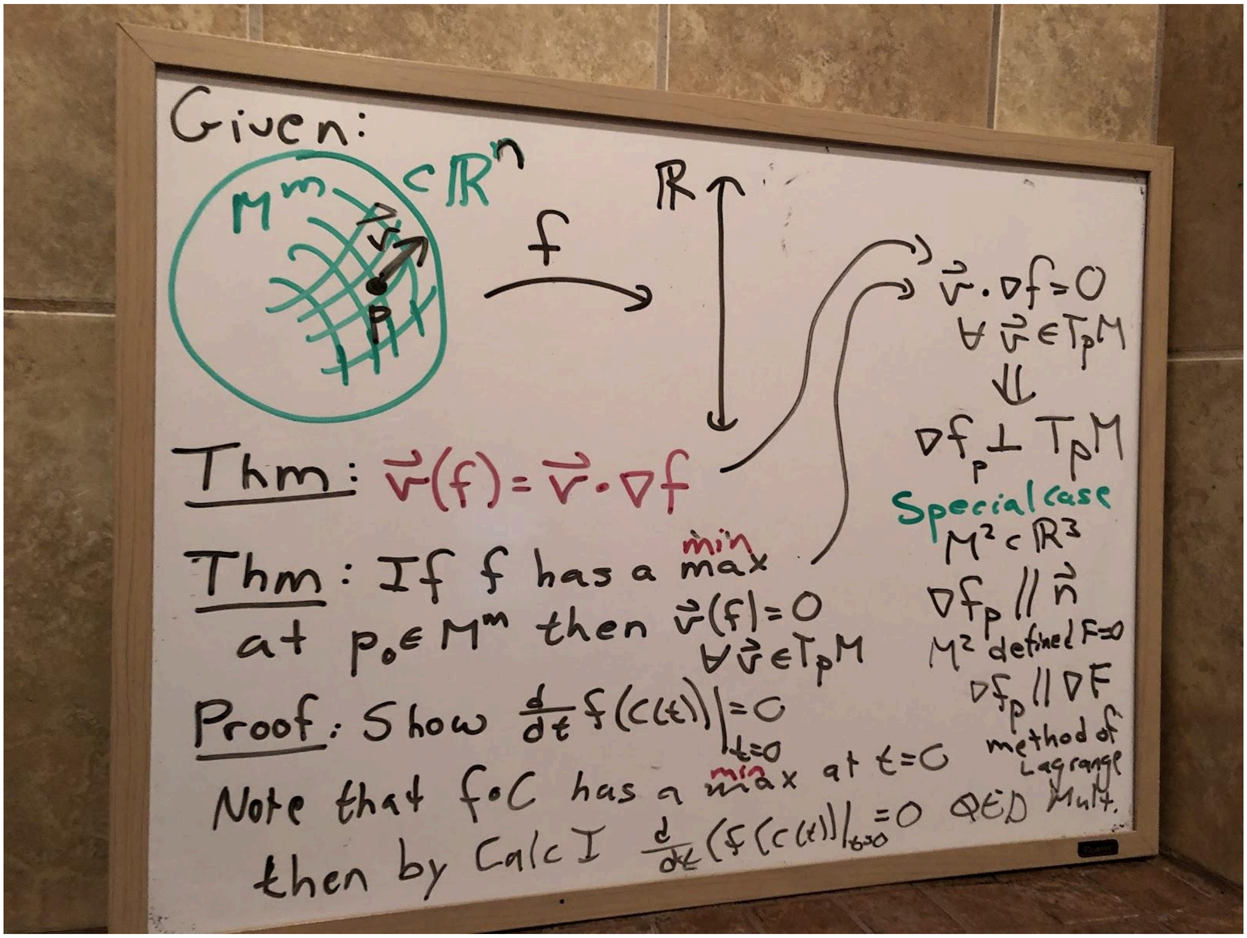
Pf:

$$\vec{v}(f) = \frac{d}{dt} f(c(t))\Big|_{t=0}$$

$$= \frac{d}{dt} f\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}\Bigg|_{t=0} = \begin{pmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{pmatrix} \begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix}$$

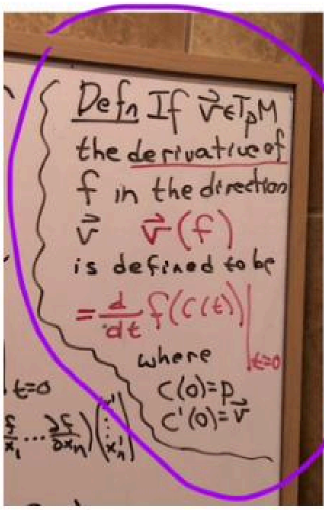
$$= Df_{c(0)}\, c'(0) = \nabla f \cdot \vec{v}$$

QED



HW1: Suppose M is just the sphere of radius 5 and consider the equator C(s)=(5cos(s),5sin(s),0)
and let V=C'(0)
and let f(x,y,z)=2x+3y+4z
HW1a: Write the implicit formula for M
HW1b: Check C lies on M using this formula.
HW1c: Find v(f) using the definition of v(f) and plot at p=C(0).
HW1d: Find grad f using partial derivatives.
HW1e: Find <grad f, v> by taking the dot product.
Ask a question if HW1e is not equal to HW1c.

Hint for HW1: the defn of V(f) is on the first photo above:



Defn of $\vec{v}(f)$

$$v(f) = \frac{d}{dt} f(c(t))\Big|_{t=0}$$

HW1: Suppose M is just the sphere of radius 5 and consider
the equator C(s)=(5cos(s),5sin(s),0)
and let V=C'(0)
and let f(x,y,z)=2x+3y+4z
HW1a: Write the implicit formula for M
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HW1d: Find grad f using partial derivatives.
HW1e: Find <grad f, v> by taking the dot product.
Ask a question if HW1e is not equal to HW1c.

$\boxed{\text{HW 1a}}$ $M = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \,\middle|\, x^2 + y^2 + z^2 = 5^2 \right\}$

$\boxed{\text{HW 1b}}$ To check C lies on M

use $C(s) = (\underset{x}{5\cos(s)}, \underset{y}{5\sin(s)}, \underset{z}{0})$

Fill in work

$(5\cos(s))^2 + (5\sin(s))^2 + (0)^2 = \ldots = 5^2$ ← if you get 5^2 then it is in M.

$\boxed{\text{HW 1c}}$ Find v(f) using the definition (circled above)

$v(f) = \frac{d}{ds} f(c(s))\Big|_{s=0} = \frac{d}{ds} 2(5\cos(s)) + 3(5\sin(s)) + 4(0)\Big|_{s=0} =$

$= 10(-\sin(s)) + 15(\cos(s)) + 0 \Big|_0 = \ldots$ keep going

Now do HW1d yourself
When you do HW1e take the dot product carefully and make sure you get the same answer as HW1c because we proved in the theorem above that these are the same. If it does not work, ask me a question by sending me an email telling me you have a question.

Solution to HW1:

HW1: Suppose M is just the sphere of radius 5 and consider the equator C(s)=(5cos(s),5sin(s),0)
and let V=C'(0)
and let f(x,y,z)=2x+3y+4z
HW1a: Write the implicit formula for M
HW1b: Check C lies on M using this formula.
HW1c: Find v(f) using the definition of v(f) and plot at p=C(0).
HW1d: Find grad f using partial derivatives.
HW1e: Find <grad f, v> by taking the dot product.
Ask a question if HW1e is not equal to HW1c.

$\boxed{HW1a}$ $M = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x^2+y^2+z^2=5^2 \right\}$

$\boxed{HW1b}$ To check C lies on M

use $C(s) = (5\cos(s), 5\sin(s), 0)$ with $x = 5\cos(s)$, $y = 5\sin(s)$, $z = 0$

Fill in work

$(5\cos(s))^2 + (5\sin(s))^2 + (0)^2 = \ldots = 5^2$ ← if you get 5^2 then it is in M.

Solution:

$5^2\cos^2 s + 5^2\sin^2 s + 0$

$= 5^2(\cos^2 s + \sin^2 s)$

$= 5^2(1) = 5^2$

$\boxed{HW1c}$ Find v(f) using the definition (circled above)

$v(f) = \frac{d}{ds} f(C(s))\Big|_{s=0} = \frac{d}{ds}\, 2(5\cos(s)) + 3(5\sin(s)) + 4(0)\Big|_{s=0} =$

$= 10(-\sin(s)) + 15(\cos(s)) + 0\Big|_0 = \ldots$ keep going

Solution:

$= 10\cdot 0 + 15\cdot 1 + 0 = \boxed{15}$

Now do HW1d yourself
When you do HW1e take the dot product carefully and make sure you get the same answer as HW1c because we proved in the theorem above that these are the same. If it does not work, ask me a question by sending me an email telling me you have a question.

Solution

$\boxed{HW1d}$ $\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x}(2x+3y+4z) \\ \frac{\partial}{\partial y}(2x+3y+4z) \\ \frac{\partial}{\partial z}(2x+3y+4z) \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

$\boxed{HW1e}$ $\langle \nabla f, v \rangle = \nabla f \cdot v$

Recall $\vec{v} = C'(0)$

$\vec{v} = \frac{d}{ds}\begin{pmatrix} 5\cos s \\ 5\sin s \\ 0 \end{pmatrix}\Bigg|_{s=0} = \begin{pmatrix} -5\sin s \\ 5\cos s \\ 0 \end{pmatrix}\Bigg|_{s=0}$

$\vec{v} = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} = 2\cdot 0 + 3\cdot 5 + 4\cdot 0 = 15$

HW2: Suppose M is just the sphere of radius 5 about the origin and consider the C(s)=(3cos(s), 4, 3sin(s))
and let V=C'(0)
and let f(x,y,z)=8x+9y+10z
HW2a: Write the implicit formula for M
HW2b: Check C lies on M using this formula.
HW2c: Find v(f) using the definition of v(f) and plot at p=C(0).
HW2d: Find grad f using partial derivatives.

HW2e: Find <grad f, v> by taking the dot product.
Ask a question if HW2e is not equal to HW2c

HW3a: Suppose M is just the xy plane lying in 3 dimensional space and V is a vector tangent to that plane. Show: V(f)=Df V where Df is the derivative matrix for f(x,y).
Hint: use the chain rule and f(x,y,z)=f(x,y) means f does not depend on z.
HW3b: Use HW3a to write a very short proof that V(f)=<V, grad f>.
Hint: Write Df using partial derivatives and find V(f)=DfV. Write grad f using partial derivatives and take the dot product of v with grad f. Check these are the same.

In Video DiffonMPart1b we review:

Thm: If a function h on M has a maximum at p and M is defined implicitly by a single equation then grad h is parallel to grad F
And so one may use the method of Lagrange multipliers to optimize over a restraining surface.

Thm: If a function h on M has a maximum at p and if M is defined implicitly with k equations then grad h is in the span of the rows of DF (which we have seen in the past span all k dimensions perpendicular to T_pM). So one may use the method of multiple Lagrange multipliers to optimize over a restraining surface.

Method of Lagrange Multipliers

Thm: If $\boxed{M^2 = \{\vec{x} : F(\vec{x}) = 0\} \subset \mathbb{R}^3}$ where $F : \mathbb{R}^3 \to \mathbb{R}$

and if $h : M^2 \to \mathbb{R}$ has a max or min at $p \in M$

then $\vec{v}(h) = 0 \ \forall \vec{v} \in T_pM$ so $\nabla h \cdot \vec{v} = 0 \ \forall v \in T_pM$

so ∇h is $\perp T_pM$

∇h is $\parallel$ to ∇F

CONSTRAINT

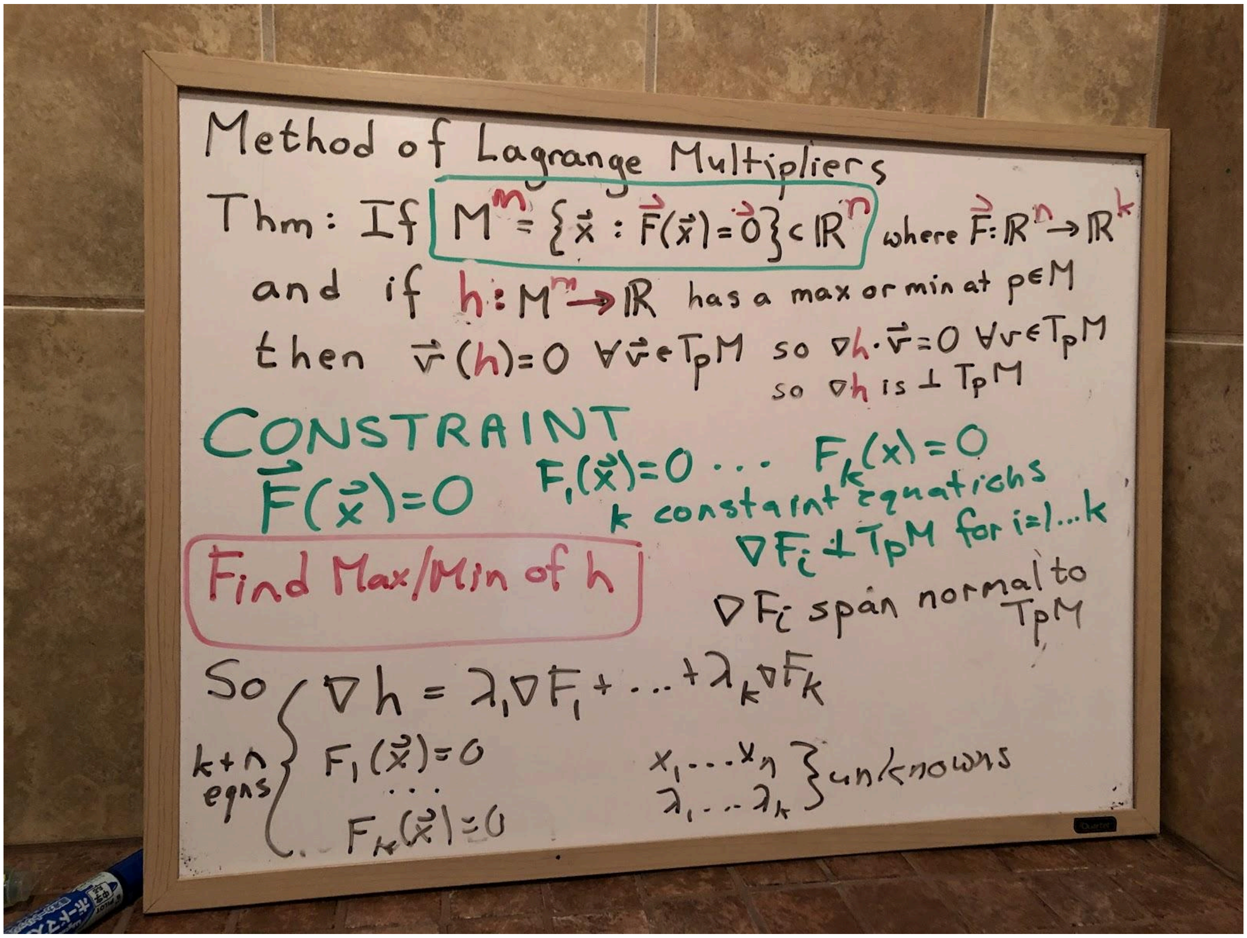
$F(\vec{x}) = 0$ Constraint Equation

MAXIMIZE h "subject to the constraint"

$$\left(\begin{array}{l} \nabla h = \lambda \nabla F \quad \lambda \text{ is a Lagrange Multiplier} \\ F(\vec{x}) = 0 \end{array} \right)$$

Solve the equations

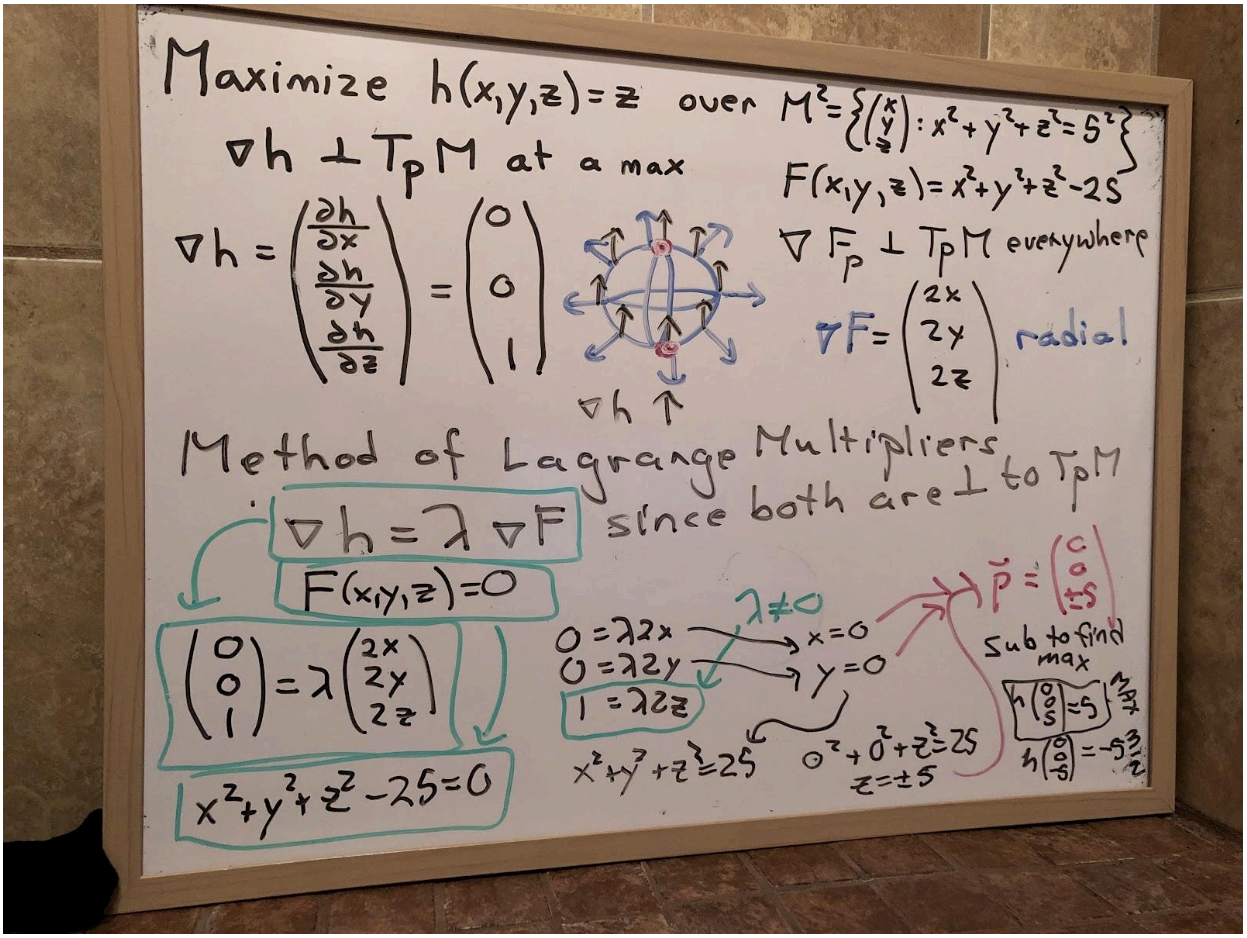
for $\vec{x}$ to find critical pts



Classwork:

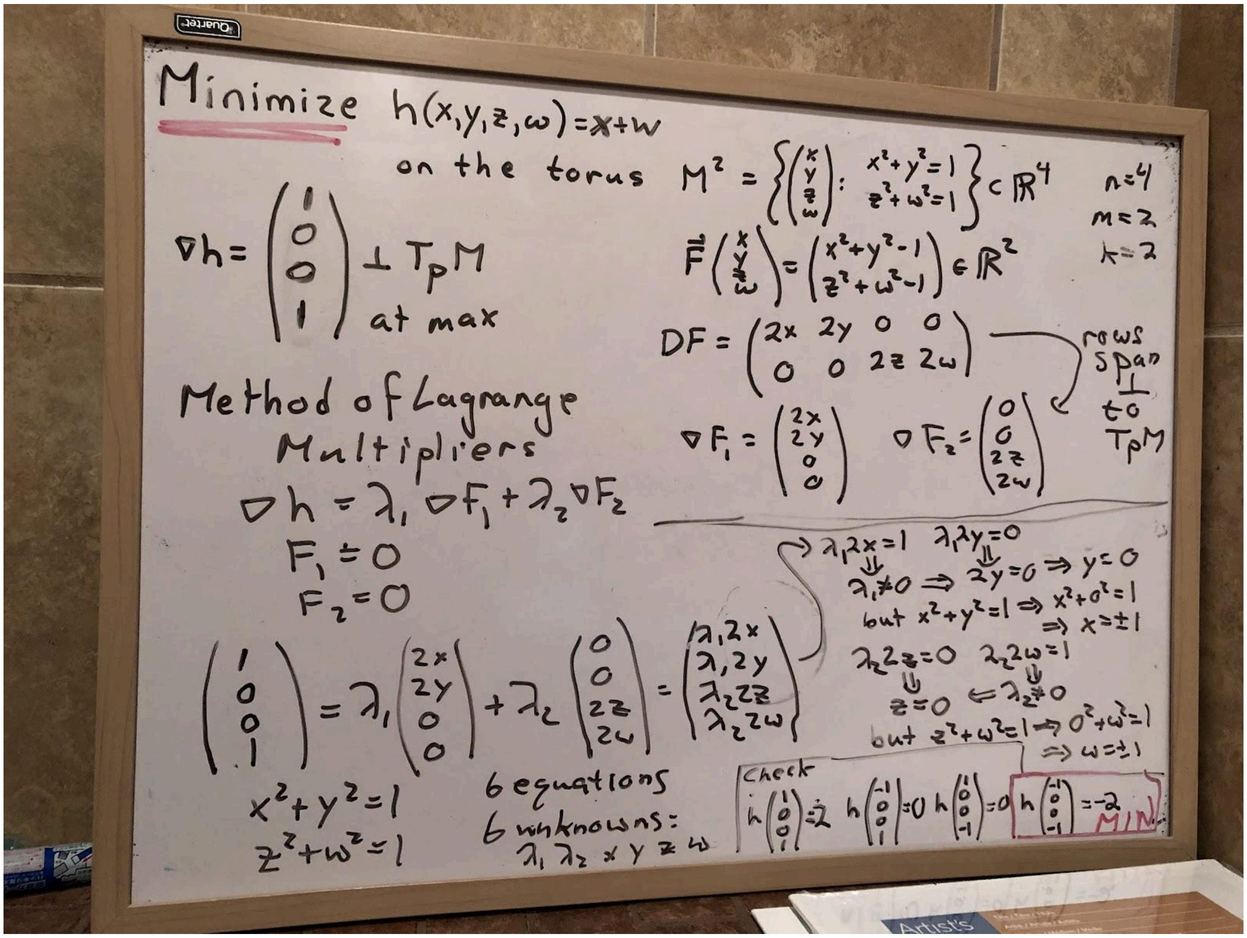
HW4: Maximize h(x,y,z)=z over the sphere of radius 5.

Solution explained in Video DiffonMPart2a:



HW5: Minimize h(x,y,z,w)=x+w on the torus defined by x^2+y^2=1 and z^2+w^2=1.

Solution explained in Video DiffonMPart2b:

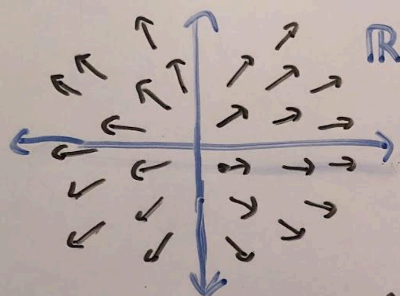


Vector Fields and Differentiation:

In Video DiffonMPart3a we review the next two photos:

Vector Fields

$\mathbb{R}^2$



$\hat{x}_1 = \begin{pmatrix}1\\0\end{pmatrix}$

$\hat{x}_2 = \begin{pmatrix}0\\1\end{pmatrix}$

$$\vec{v} = \begin{pmatrix} v_1(x,y) \\ v_2(x,y)\end{pmatrix} = v_1(x,y)\hat{x}_1 + v_2(x,y)\hat{x}_2$$

$\mathbb{R}^m$

$$\vec{v} = \begin{pmatrix} v_1(\vec{x}) \\ v_2(\vec{x}) \\ \vdots \\ v_m(\vec{x})\end{pmatrix} = v_1\hat{x}_1 + \dots + v_m\hat{x}_m$$

$v_i(x_1, x_2 \dots x_m)$

$v_i : \mathbb{R}^m \to \mathbb{R}$

$\vec{v} = \mathbb{R}^m \to \mathbb{R}^m$

$$D_{\vec{v}}\vec{w} = \begin{pmatrix} v(w_1)\\ \cdots \\ v(w_m)\end{pmatrix} \in \mathbb{R}^m$$

at a point p

$$D_{\vec v}\vec w\Big|_p = \begin{pmatrix} \frac{d}{dt} w_1(c(t))\big|_{t=0} \\ \vdots \\ \frac{d}{dt} w_m(c(t))\big|_{t=0}\end{pmatrix}$$

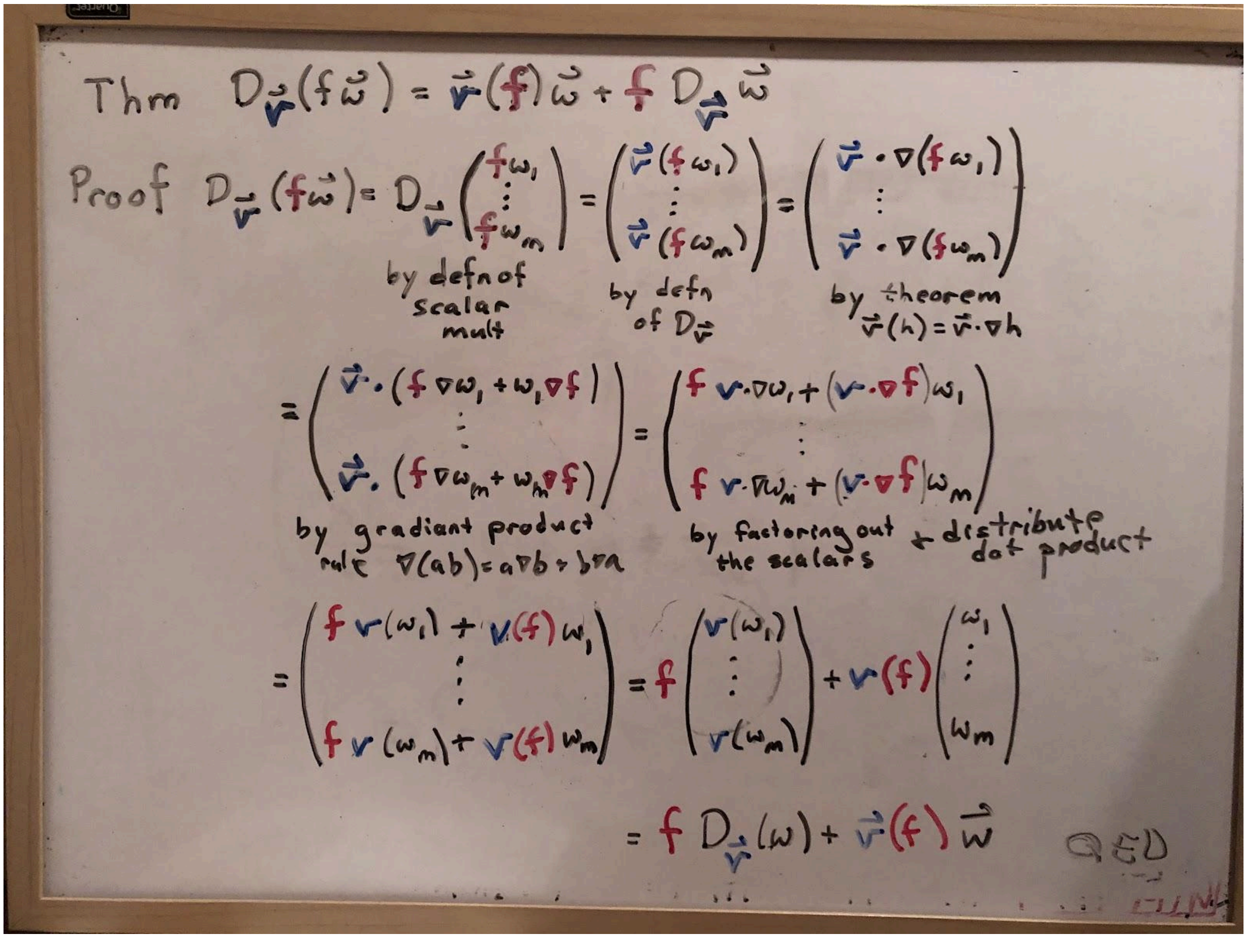
where $c(0)=p$

$c'(0)=\vec v$

Thm

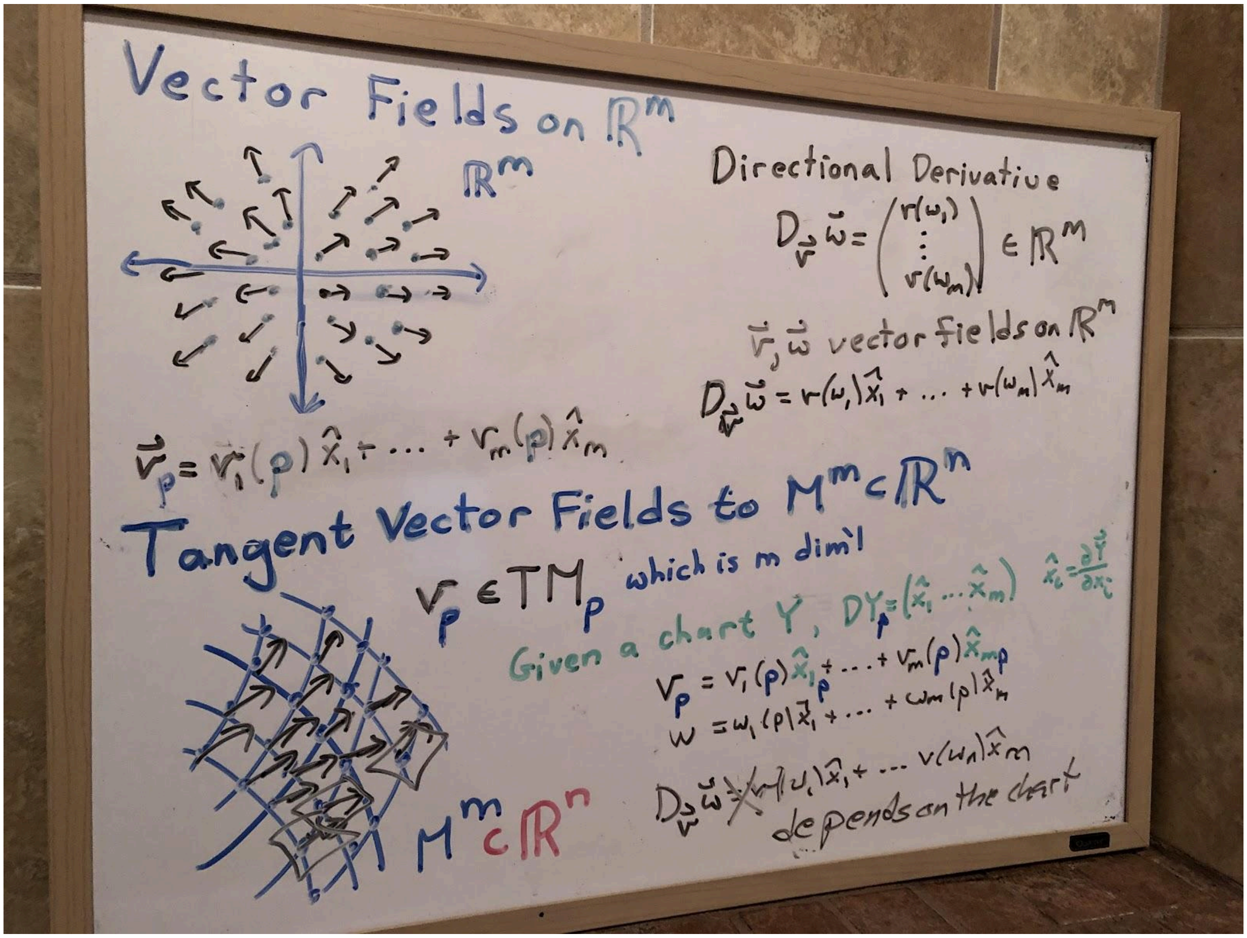
$$D_{f\vec v}\vec w = \begin{pmatrix} f\vec v(w_1)\\ \vdots \\ f\vec v(w_m)\end{pmatrix}$$

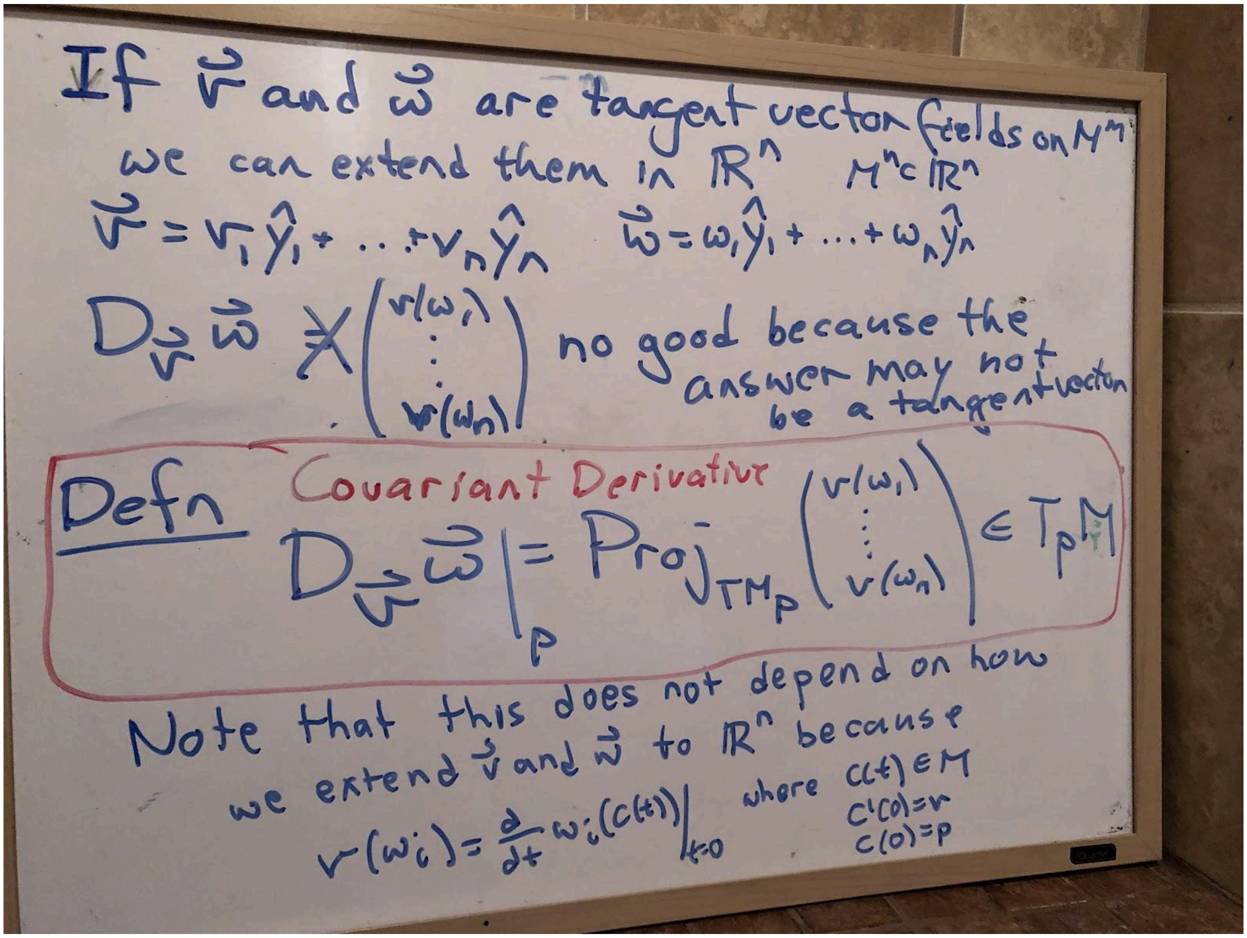
$$= f D_{\vec v}\vec w$$



Recall a vector field in Euclidean space is an assignment of a vector at every point in the space. So if w is a vector field, w_p, is a vector. In vector calc you learned to take the divergence and curl of vector fields. And also given a vector w at a point p you found the directional derivative of the vector field in the direction w at p
v_p(w)= d/dt w_{C(t)} at t=0 where C(0)=p and C'(p)=v
which was found just by differentiating each term
and so the answer was another vector of the same dimension as v and w.

In Video DiffonMPart3b we introduce tangent vector fields and their derivative as in the following photos:


Vector Fields on $\mathbb{R}^m$
$\mathbb{R}^m$
Directional Derivative
$D_{\vec{v}}\vec{w} = \begin{pmatrix} v(w_1) \\ \vdots \\ v(w_m) \end{pmatrix} \in \mathbb{R}^m$
$\vec{v}, \vec{w}$ vector fields on $\mathbb{R}^m$
$D_{\vec{v}}\vec{w} = v(w_1)\hat{x}_1 + \dots + v(w_m)\hat{x}_m$
$\vec{v}_p = \vec{v}_1(p)\hat{x}_1 + \dots + v_m(p)\hat{x}_m$
Tangent Vector Fields to $M^m \subset \mathbb{R}^n$
$v_p \in TM_p$ which is m dim'l
Given a chart Y, $DY_p = (\hat{x}_1 \dots \hat{x}_m)$
$\hat{x}_i = \frac{\partial \vec{Y}}{\partial x_i}$
$v_p = v_1(p)\hat{x}_{1p} + \dots + v_m(p)\hat{x}_{mp}$
$w = w_1(p)\hat{x}_1 + \dots + w_m(p)\hat{x}_m$
$D_{\vec{v}}\vec{w} \cancel{= v}(w_1)\hat{x}_1 + \dots v(w_m)\hat{x}_m$
depends on the chart
$M^m \subset \mathbb{R}^n$



HW6: Classwork: Find the vector fields corresponding to the two columns of DY of the patch in Week11 HW6 and sketch them. Call the one vector field v and the other w. Check if the directional derivative of w in the direction v is tangent or not. Take the projection the Projection map you found in Week11 HW6' to find D_v w at p

In Video DiffonMPart4 we complete the classwork.

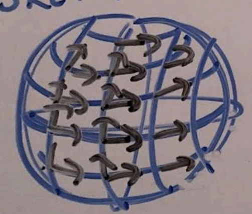
Classwork

$$M = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x^2 + y^2 + z^2 = 25 \right\} \quad \Psi\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} \sqrt{25-y^2-z^2} \\ y \\ z \end{pmatrix} \quad p = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$$

Find the vector fields = columns of $D\Psi$

$$D\Psi = \begin{pmatrix} \frac{-y}{\sqrt{25-y^2-z^2}} & \frac{-z}{\sqrt{25-y^2-z^2}} \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \vec{v} = \begin{pmatrix} \frac{-y}{\sqrt{\sim}} \\ 1 \\ 0 \end{pmatrix} \quad \vec{\omega} = \begin{pmatrix} \frac{-z}{\sqrt{\sim}} \\ 0 \\ 1 \end{pmatrix}$$

Sketch

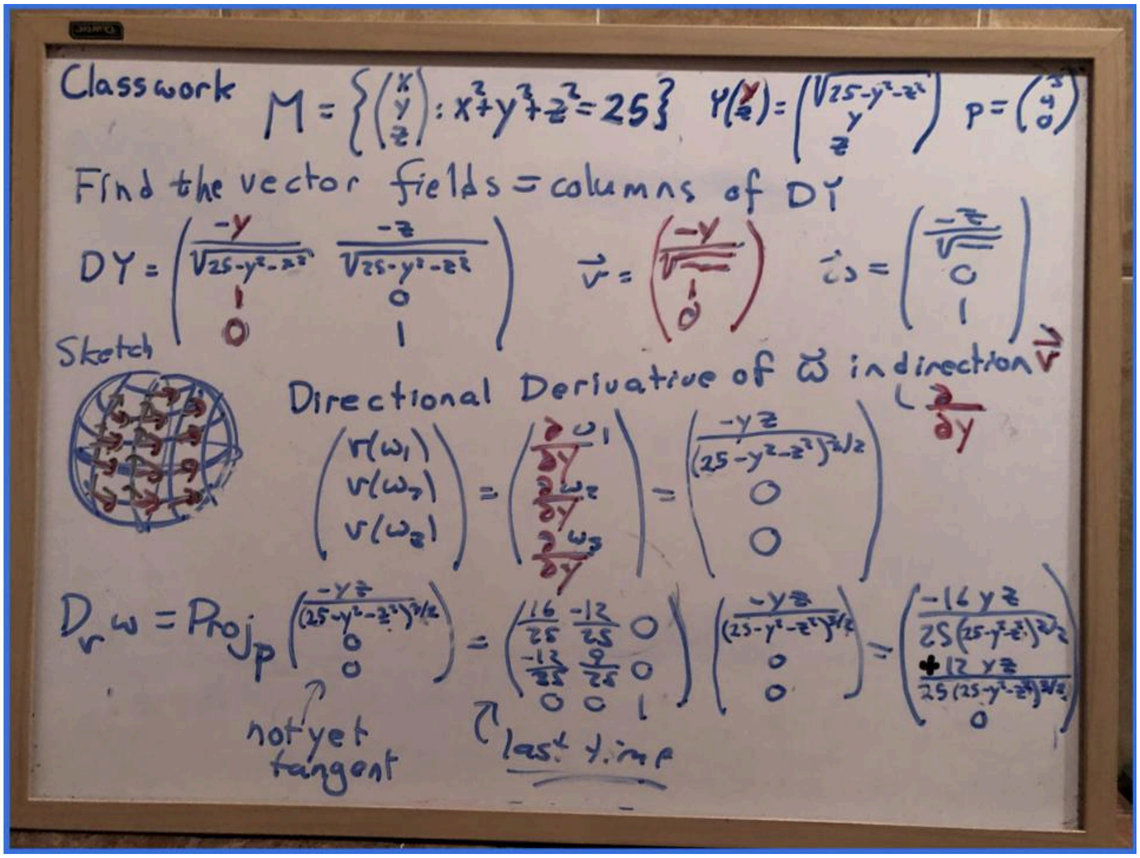


Directional Derivative of $\vec{\omega}$ in direction $\vec{v}$ $\left(\frac{\partial}{\partial y} \right.$

$$\begin{pmatrix} v(\omega_1) \\ v(\omega_2) \\ v(\omega_3) \end{pmatrix} = \begin{pmatrix} \frac{\partial \omega_1}{\partial y} \\ \frac{\partial \omega_2}{\partial y} \\ \frac{\partial \omega_3}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{-yz}{(25-y^2-z^2)^{3/2}} \\ 0 \\ 0 \end{pmatrix}$$

$$\frac{\partial}{\partial y} \frac{-z}{\sqrt{25-y^2-z^2}} = \frac{\partial}{\partial y} -z\left(25-y^2-z^2\right)^{-1/2}$$

$$= -z\left(\frac{-1}{\cancel{2}}\right)\left(25-y^2-z^2\right)^{-3/2}(-\cancel{2}y)$$



Not done in the video, we should plug in the values of y and z at p=(3,4,0)
And see that

D_v w= (0, 0, 0)

It is rather special that its value is the 0 vector and you may or may not get the 0 vector when studying other manifolds in your final project. If you do get 0 it is because you have an especially nice patch which behaves very well at the given point p.

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Part II: Covariant Derivatives and Christoffel Symbols

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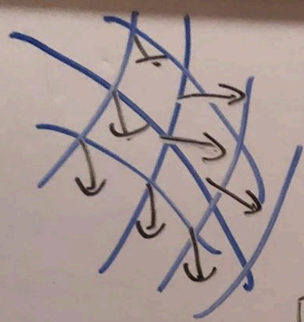
Skip this part if you are behind schedule and proceed to the next lesson.

Week 11 Parts I-II must be completed and fixed before starting this part.

We introduce the covariant derivative and prove the product rule for the covariant derivative of a function times a vector field in Video DiffonMPart5a :



$M^m \subset \mathbb{R}^n$

$\vec{v}, \vec{w}$ are tangent vector fields on M

extend to $\mathbb{R}^n$

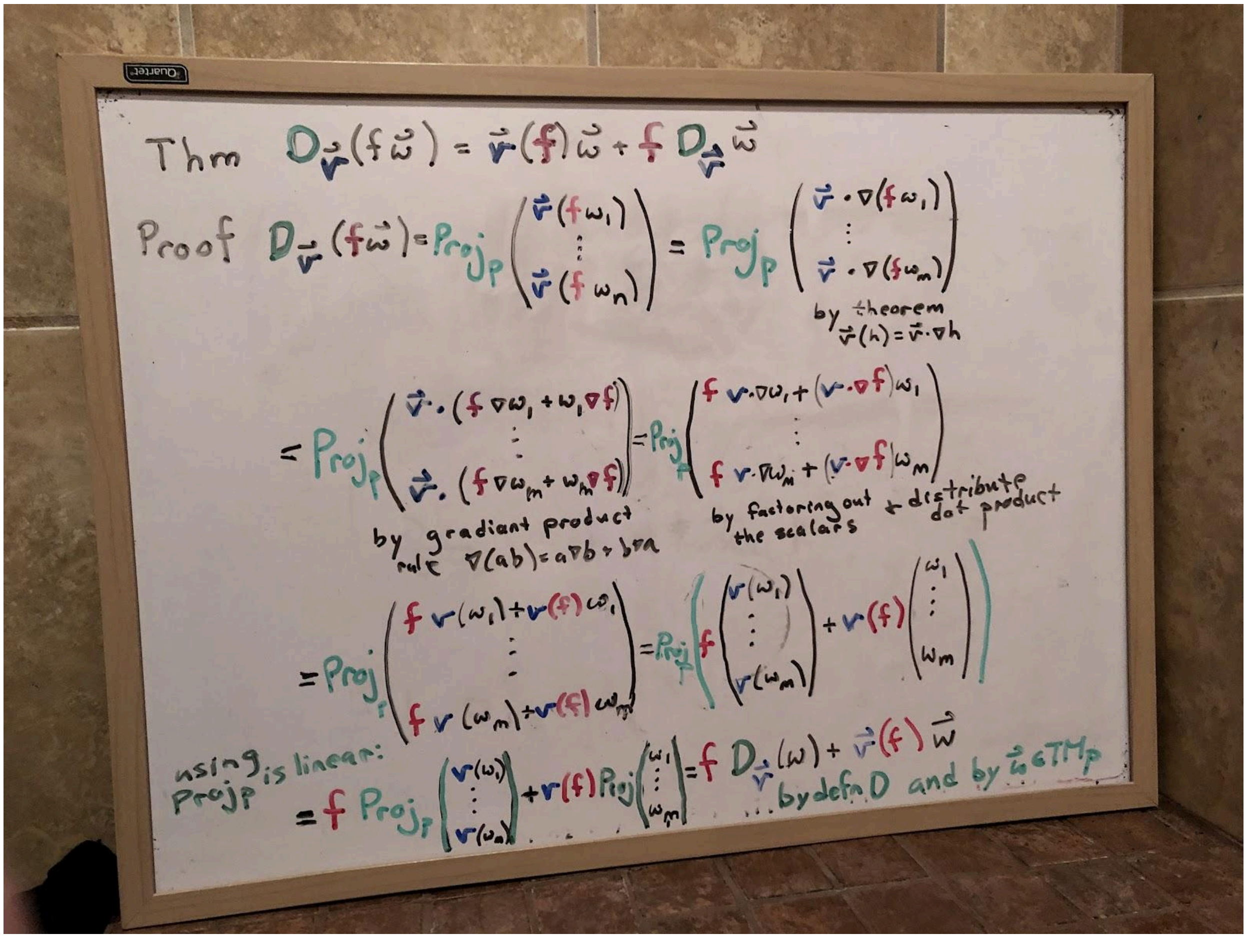
Directional Derivative $\begin{pmatrix} v(w_1) \\ \vdots \\ v(w_m) \end{pmatrix}$ might not be in TM_p

Defn Covariant Derivative

$$D_{\vec{v}}\vec{w}\Big|_p = \mathrm{Proj}_{TM_p}\begin{pmatrix} v(w_1) \\ \vdots \\ v(w_n) \end{pmatrix} \in T_pM$$

Note that this does not depend on how we extend $\vec{v}$ and $\vec{w}$ to $\mathbb{R}^n$ because

$v(w_i) = \frac{d}{dt} w_i(c(t))\Big|_{t=0}$ where $c(t) \in M$, $c'(0) = v$, $c(0) = p$

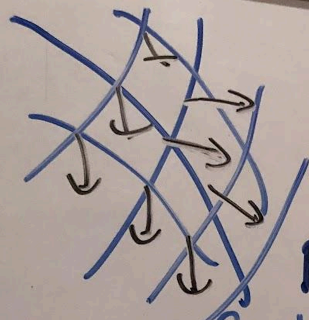


Defn: Given w in T_pM, the covariant derivative in the direction w of the vector field v at p is:
D_w v = D/dt V_{C(t)} at t=0 which is defined to be
= Proj_p (d/dt V_{C(t)} at t=0) where the projection is to T_pM
That is you first differentiate the components to find the vector d/dt V_{C(t)} at t=0
And then project that vector into T_pM.

Thm scalar product rule for covariant differentiation
D_w(fv)=w(f)v+ fD_wv

In Video DiffonMPart5b we prove

Thm: inner product rule for covariant differentiation
v(g(w,z))=g(D_v w,z)+g(v,D_v z)



$M^m \subset \mathbb{R}^n$

$\vec{z}, \vec{v}, \vec{w}$ are tangent vector fields

Recall $g(\vec{v},\vec{w}) = \vec{v}\cdot\vec{w} = \sum_{i=1}^{n} v_i w_i$

first fund form

inner product

Riemannian metric

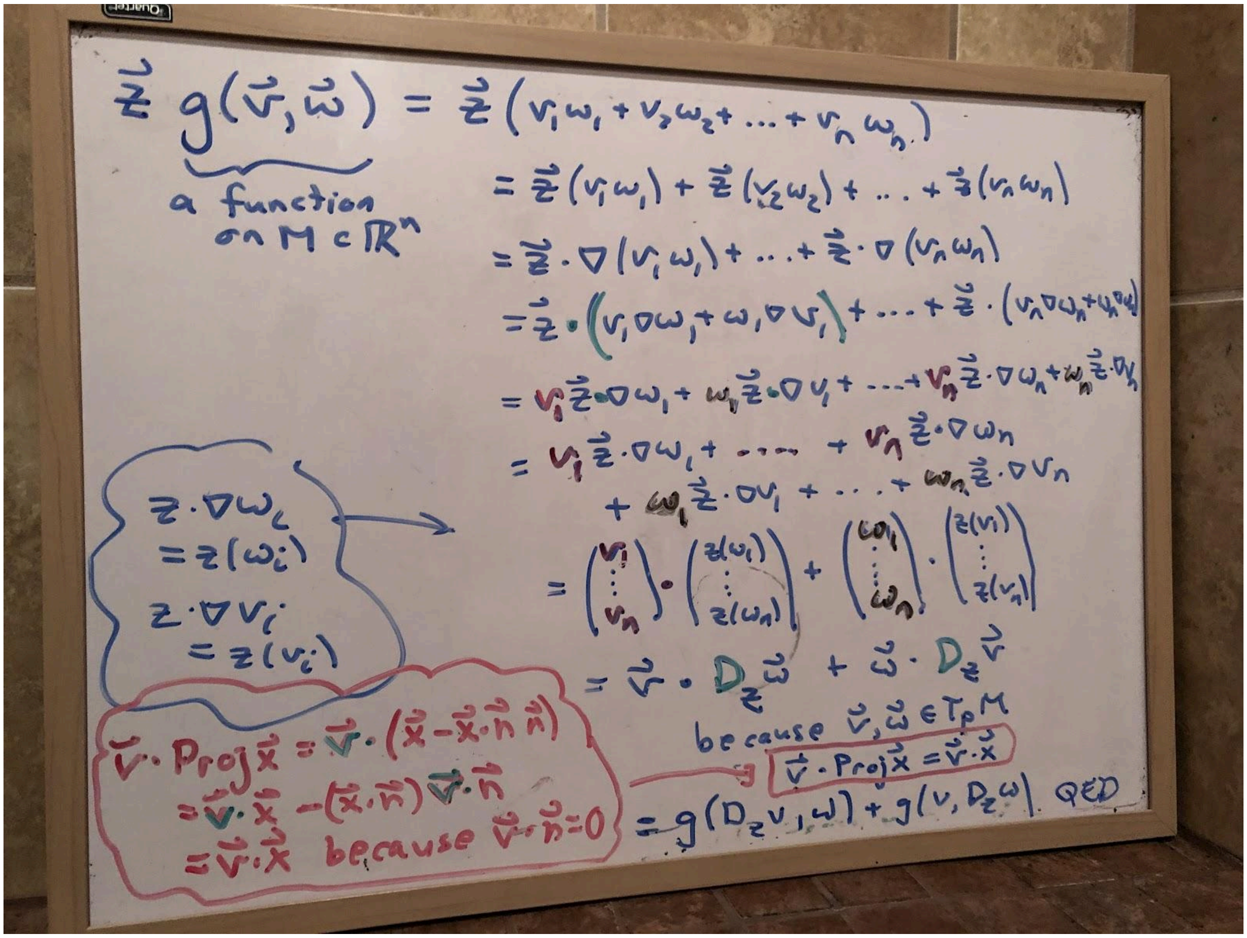
is a function on M

Defn Covariant Derivative

$$D_{\vec{v}}\vec{w}\Big|_p = \mathrm{Proj}_{TM_p}\begin{pmatrix} v(w_1) \\ \vdots \\ v(w_n) \end{pmatrix} \in T_pM$$

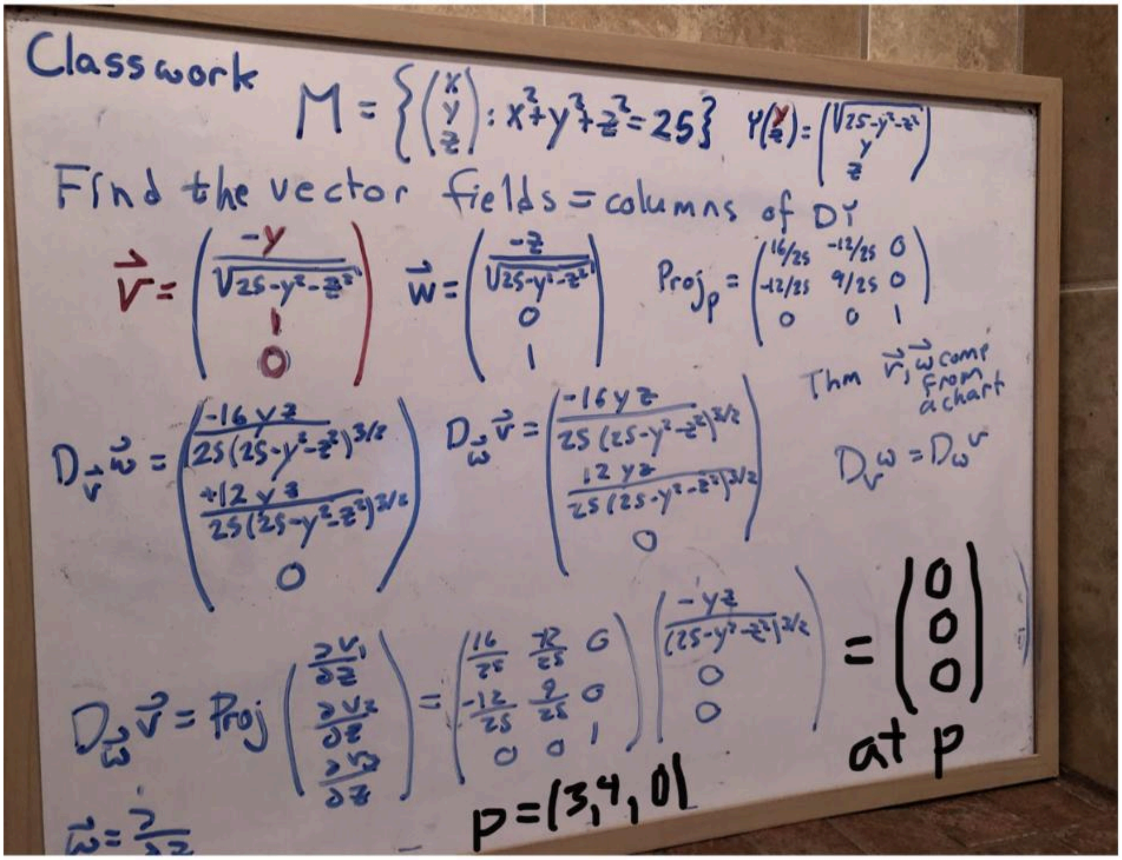
What is $\vec{z}\, g(\vec{v},\vec{w})$?

By defn $\frac{d}{dt} g(\vec{v},\vec{w})\Big|_{c(t)}\Big|_{t=0}$ where $c(0)=p$, $c'(0)=z$



HW7: Classwork1: Recall HW6 above in Part I where we found the vector fields corresponding to the two columns of DY of the patch and called the one vector field v and the other w. Using the the Projection map, you found D_v w at p. Now continue:
1a) find D_w v at p
and observe it agrees with D_v w
1b) find D_v v at p

In Video DiffonMPart6a we complete the classwork 1a:



Note D_v w = D_w v when v and w are columns of a chart. This is essentially the same as saying that we can differentiate with respect to different variables in any order. They are not always equal to 0 at p.

In Video DiffonMPart6b we complete the classwork 1b

Classwork

$$M = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x^2+y^2+z^2 = 25 \right\} \qquad \Psi\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} \sqrt{25-y^2-z^2} \\ y \\ z \end{pmatrix}$$

Find the vector fields = columns of $D\Psi$

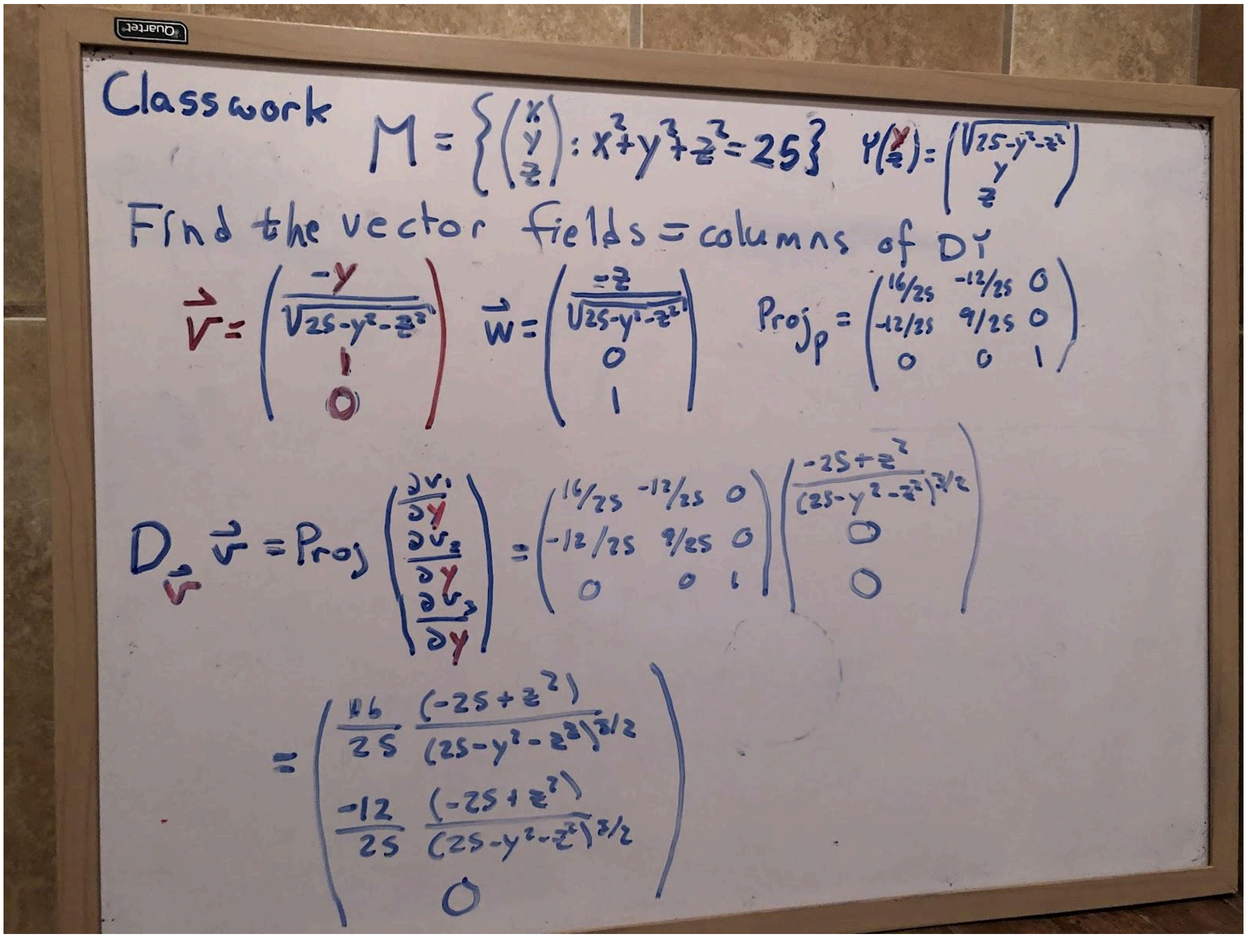
$$\vec{v} = \begin{pmatrix} \frac{-y}{\sqrt{25-y^2-z^2}} \\ 1 \\ 0 \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} \frac{-z}{\sqrt{25-y^2-z^2}} \\ 0 \\ 1 \end{pmatrix} \qquad \mathrm{Proj}_p = \begin{pmatrix} 16/25 & -12/25 & 0 \\ -12/25 & 9/25 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D_{\vec{v}}\vec{v} = \mathrm{Proj}\begin{pmatrix} \frac{\partial v_1}{\partial y} \\ \frac{\partial v_2}{\partial y} \\ \frac{\partial v_3}{\partial y} \end{pmatrix} = \begin{pmatrix} 16/25 & -12/25 & 0 \\ -12/25 & 9/25 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{-25+z^2}{(25-y^2-z^2)^{3/2}} \\ 0 \\ 0 \end{pmatrix}$$

Scratch:

$$\frac{\partial v_1}{\partial y} = \frac{\partial}{\partial y} -y(25-y^2-z^2)^{-1/2} \underset{\text{product rule}}{=} -1(25-y^2-z^2)^{-1/2} - y\left(\frac{-1}{2}\right)(25-y^2-z^2)^{-3/2}(-2y) \text{ by chain rule}$$

$$= \frac{-(25-y^2-z^2)^1 - y^2}{(25-y^2-z^2)^{3/2}} = \frac{-25+z^2}{(25-y^2-z^2)^{3/2}}$$



HW8: **Classwork 2:**
Continuing with v and w as in classwork 1 check
vg(v,w)= g(D_v v,w) + g(v,D_vw) at p
by finding the value on both sides.

In Video DiffonMPart6c we complete classwork 2 as in the following photos:

Classwork

$p=(3,4,0)$ $\quad M = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x^2+y^2+z^2=25 \right\}$ $\quad u=(25-y^2-z^2)^{1/2}$

$$\vec{v} = \frac{\partial}{\partial y} = \begin{pmatrix} \frac{-y}{u} \\ 1 \\ 0 \end{pmatrix} \quad \vec{w} = \frac{\partial}{\partial z} = \begin{pmatrix} \frac{-z}{u} \\ 0 \\ 1 \end{pmatrix} \quad \underset{\text{at } p}{D_{\vec{v}}\vec{w} = D_{\vec{w}}\vec{v}} = \begin{pmatrix} \frac{-16yz}{25u^3} \\ \frac{12yz}{25u^3} \\ 0 \end{pmatrix} \text{ and } \underset{\text{at } p}{D_{\vec{v}}\vec{v}} = \begin{pmatrix} \frac{16(-25+z^2)}{25u^3} \\ \frac{-12(-25+z^2)}{25u^3} \\ 0 \end{pmatrix}$$

Verify that $v(g(v,w)) = g(D_v v, w) + g(v, D_v w)$

Used the Proj Matrix at p

$x=3 \quad y=4 \quad z=0$

$$u=(25-4^2-0^2)^{1/2} = (25-16)^{1/2} = 9^{1/2} = 3$$

Classwork

$p=(3,4,0)$ $M = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x^2+y^2+z^2 = 25 \right\}$ $u = (25-y^2-z^2)^{1/2}$

$\vec{v} = \begin{pmatrix} -\frac{4}{3} \\ 1 \\ 0 \end{pmatrix}$ $\frac{\partial}{\partial y}$ $\vec{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\frac{\partial}{\partial z}$ $\underset{\text{at } p}{D_{\vec{v}}\vec{w} = D_{\vec{w}}\vec{v}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $\underset{\text{at } p}{D_{\vec{v}}\vec{v}} = \begin{pmatrix} -16/27 \\ 12/27 \\ 0 \end{pmatrix}$

Verify that $v(g(v,w)) = g(D_v v, w) + g(v, D_v w)$

Used the Proj Matrix at p

$x=3$ $y=4$ $z=0$ $u = (25-4^2-0^2)^{1/2}$
$= (25-16)^{1/2} = 9^{1/2} = 3$

Check they are tangent

$\vec{n} = \begin{pmatrix} 6 \\ 8 \\ 0 \end{pmatrix}$ $\frac{-16}{27} \cdot 6 + \frac{12}{27} 8$
$=$

Classwork

$p=(3,4,0)$ $\quad M = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x^2+y^2+z^2=25 \right\}$ $\quad u=(25-y^2-z^2)^{1/2}$

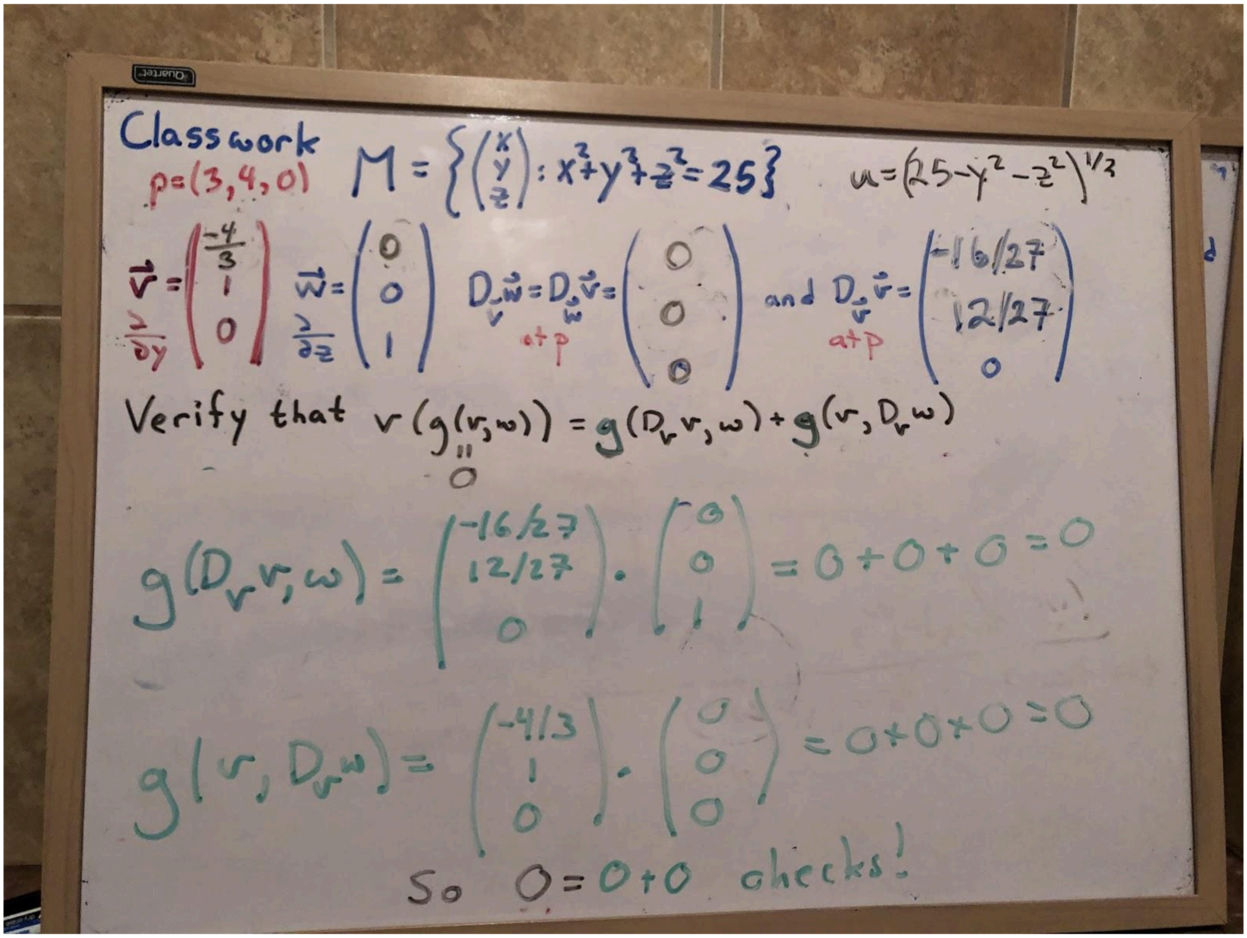
$$\vec{v} = \frac{\partial}{\partial y} = \begin{pmatrix} -\frac{4}{3} \\ 1 \\ 0 \end{pmatrix} \quad \vec{w} = \frac{\partial}{\partial z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \underset{\text{at } p}{D_{\vec{v}}\vec{w} = D_{\vec{w}}\vec{v}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } \underset{\text{at } p}{D_{\vec{v}}\vec{v}} = \begin{pmatrix} -16/27 \\ 12/27 \\ 0 \end{pmatrix}$$

Verify that $v(g(v,w)) = g(D_v v, w) + g(v, D_v w)$

$$v(\underbrace{g(v,w)}_{\text{function}}) = \frac{\partial}{\partial y}\left(\begin{pmatrix} -y/u \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -z/u \\ 0 \\ 1 \end{pmatrix} \right)$$

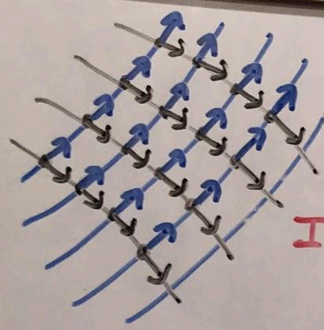
$$= \frac{\partial}{\partial y}\left(\frac{yz}{25-y^2-z^2} \right) = \frac{(z)(25-y^2-z^2)-(yz)(-2y)}{(25-y^2-z^2)^2} \quad \text{Quotient Rule}$$

$$= \frac{25z+y^2z-z^3}{(25-y^2-z^2)^2} = 0$$



<<take a break>>

In Video [DiffonMPart7a]() we introduce the Christoffel Symbols:



Given a chart $Y: U \to W \subset M^m$

$DY_p = \left(\hat{x}_1 \cdots \hat{x}_m\right)$ Columns span TM_p

each column is a vector field on $W \subset M$

If $\vec{v}, \vec{w}, \vec{z}$ are tangent vector fields to M

$$\left.\begin{aligned} \vec{v} &= v_1 \hat{x}_1 + \dots + v_m \hat{x}_m \\ \vec{w} &= w_1 \hat{x}_1 + \dots + w_m \hat{x}_m \\ \vec{z} &= z_1 \hat{x}_1 + \dots + z_m \hat{x}_m \end{aligned}\right\} \text{Depends on the chart}$$

Defn Covariant Derivative

$$D_{\vec{v}} \vec{w} \Big|_p = \mathrm{Proj}_{TM_p} \begin{pmatrix} v(w_1) \\ \vdots \\ v(w_n) \end{pmatrix} \in T_p M$$

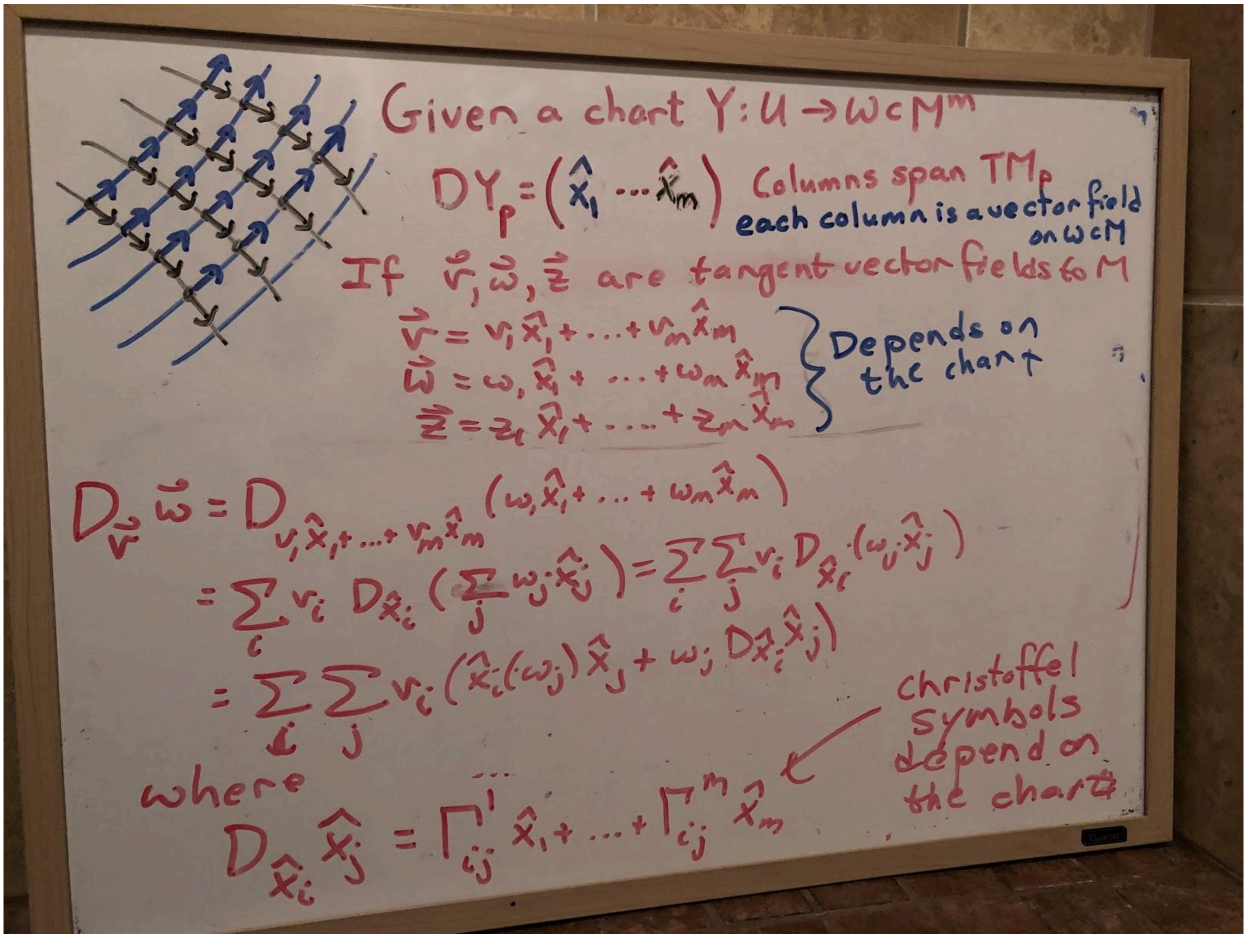
$$D_{\vec{v}} \vec{w} = a_1 \hat{x}_1 + \dots + a_m \hat{x}_m$$

We want a nice formula for $a_1, \dots a_m$ dep on $\vec{v} + \vec{w}$

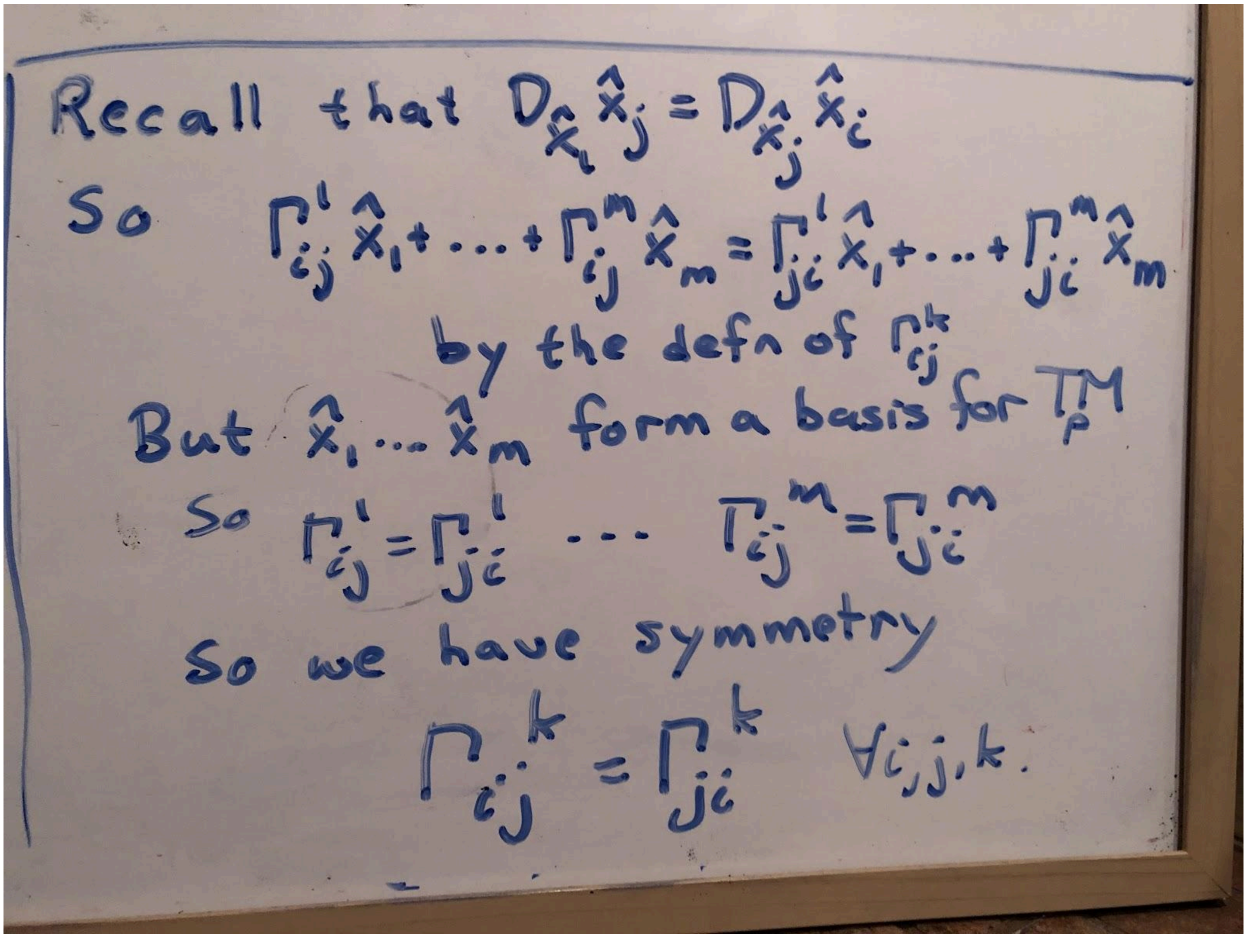
Christoffel symbols

depend on the chart

$$D_{\hat{x}_i} \hat{x}_j = \Gamma^1_{ij} \hat{x}_1 + \dots + \Gamma^m_{ij} \hat{x}_m$$

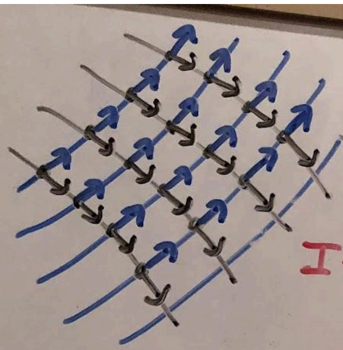


Not mentioned in the video is the symmetry of the Christoffel Symbols:



Also not mentioned is that these Christoffel symbols are the entries of a tensor. A tensor is a more general notion than a matrix. It includes matrices which have only two indices (like for example the metric tensor g_ij). Tensors can also have more indices ijk or ijkl or even more. When multiplying tensors you have to indicate which indices are involved. You will see this later when you learn about the curvature tensor.

In Video DiffonMPart7b we present a formula for the Christoffel symbols depending on a metric tensor:



Given a chart $Y: U \to W \subset M^m$

$DY_p = \left(\hat{x}_1 \cdots \hat{x}_m\right)$ Columns span TM_p

each column is a vector field on $W \subset M$

If $\vec{v}, \vec{w}, \vec{z}$ are tangent vector fields to M

$$\left.\begin{aligned} \vec{v} &= v_1 \hat{x}_1 + \dots + v_m \hat{x}_m \\ \vec{w} &= w_1 \hat{x}_1 + \dots + w_m \hat{x}_m \\ \vec{z} &= z_1 \hat{x}_1 + \dots + z_m \hat{x}_m \end{aligned}\right\} \text{Depends on the chart}$$

$$D_{\vec{v}} \vec{w} = D_{v_1 \hat{x}_1 + \dots + v_m \hat{x}_m} \left(w_1 \hat{x}_1 + \dots + w_m \hat{x}_m \right)$$

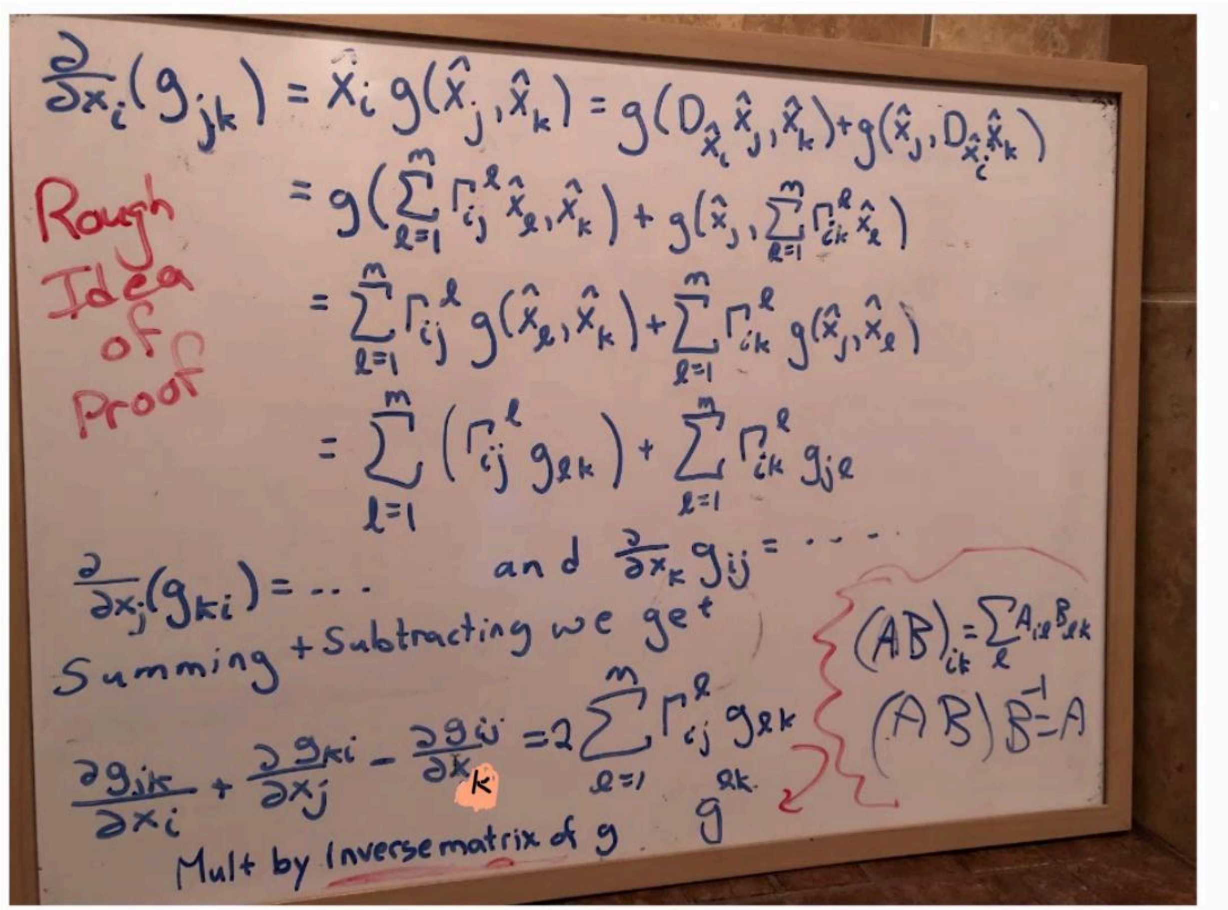
$$= \sum_i v_i \, D_{\hat{x}_i} \left(\sum_j w_j \hat{x}_j \right) = \sum_i \sum_j v_i \, D_{\hat{x}_i} \left(w_j \hat{x}_j \right)$$

$$= \sum_i \sum_j v_i \left(\frac{\partial}{\partial x_i}(w_j) \hat{x}_j + w_j \sum_k \Gamma_{ij}^k \hat{x}_k \right)$$

$$\Gamma_{ij}^k = \sum_l \frac{1}{2} g^{kl} \left(\frac{\partial g_{li}}{\partial x_j} + \frac{\partial}{\partial x_i} g_{lj} - \frac{\partial}{\partial x_l} g_{ij} \right)$$

Christoffel symbols

g^{kl} is the inverse of g_{ij}



The proof of the formula for Christoffel Symbols also uses the symmetry when cancelling terms after subtracting.

HW9: Classwork 3
3) Use the formula for the Christoffel symbols as sums of derivatives of g_ij to find D_v w at p and check your answer with your earlier classwork where you found D_v w at p already.

In Video DiffonMPart8a we start the classwork:

Classwork

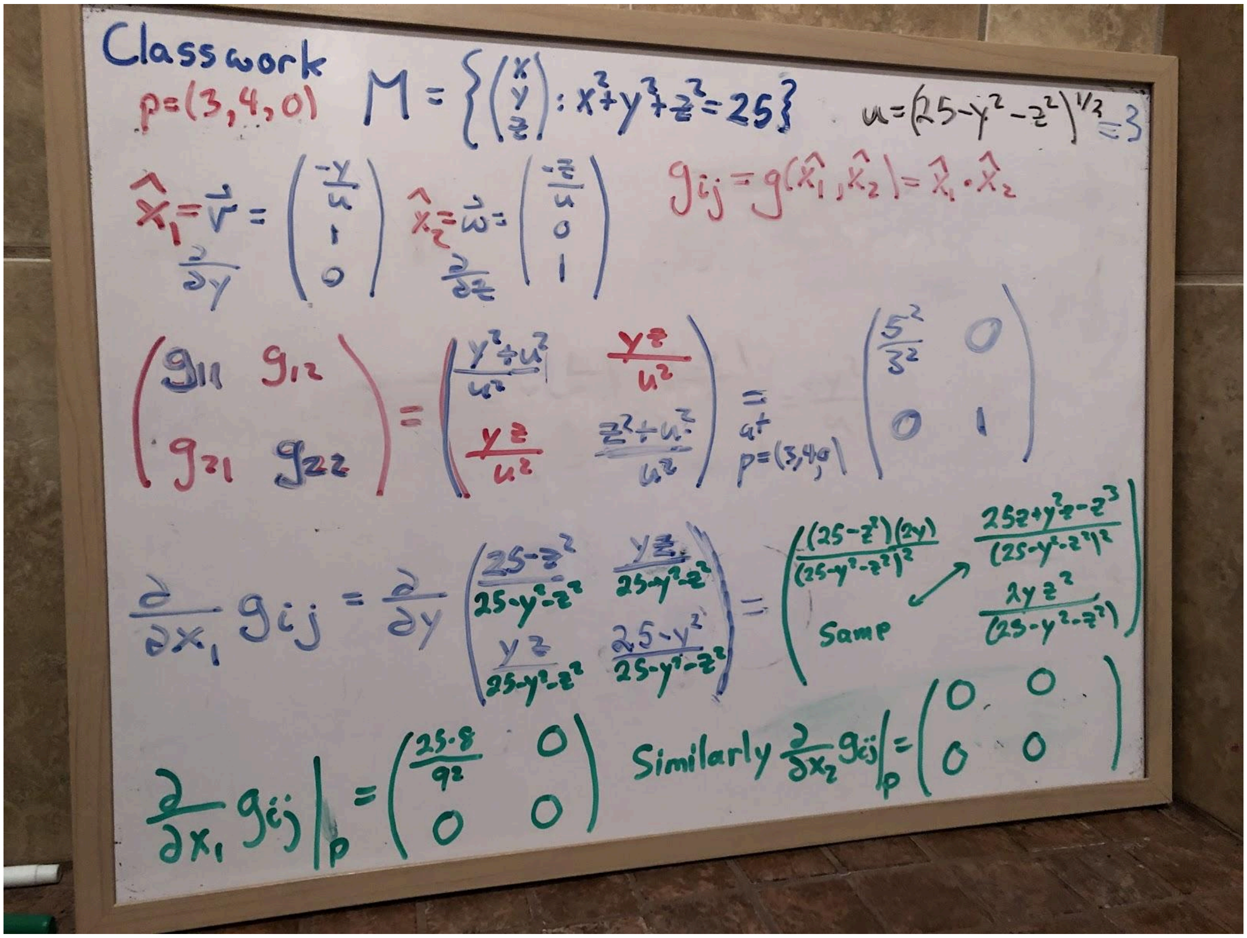
$p=(3,4,0)$ $\quad M = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x^2+y^2+z^2 = 25 \right\}$ $\quad u = (25-y^2-z^2)^{1/2} = 3$

$$\hat{x}_1 = \vec{v} = \begin{pmatrix} -\frac{y}{u} \\ 1 \\ 0 \end{pmatrix} \quad \hat{x}_2 = \vec{w} = \begin{pmatrix} -\frac{z}{u} \\ 0 \\ 1 \end{pmatrix} \quad g_{ij} = g(\hat{x}_1, \hat{x}_2) = \hat{x}_1 \cdot \hat{x}_2$$

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \frac{y^2}{u^2} + 1^2 & \frac{yz}{u^2} \\ \frac{yz}{u^2} & \frac{z^2}{u^2} + 1^2 \end{pmatrix} \underset{\text{at } p=(3,4,0)}{=} \begin{pmatrix} \frac{5^2}{3^2} & 0 \\ 0 & 1 \end{pmatrix}$$

$$\frac{y^2}{u^2} + 1 = \frac{4^2}{3^2} + 1 = \frac{4^2+3^2}{3^2} = \frac{5^2}{3^2}$$

$$\frac{yz}{u^2} = \frac{4 \cdot 0}{u^2} = 0 \qquad \frac{z^2}{u^2} + 1^2 = 0 + 1$$



The green part of the second photo is not in the video. It is just taking the partial derivatives in every entry and then evaluating at p.

In Video <u>DiffonMPart8b</u> we finish the classwork:

Classwork

$p=(3,4,0)$ $\quad M = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x^2+y^2+z^2=25 \right\}$ $\quad u=(25-y^2-z^2)^{1/2}=3$

$$\hat{x}_1 = \vec{v} = \begin{pmatrix} \frac{-y}{u} \\ 1 \\ 0 \end{pmatrix}, \quad \frac{\partial}{\partial y} \qquad \hat{x}_2 = \vec{w} = \begin{pmatrix} \frac{-z}{u} \\ 0 \\ 1 \end{pmatrix}, \quad \frac{\partial}{\partial z}$$

$$\boxed{\frac{\partial}{\partial x_1} g_{11} = \frac{25 \cdot 8}{9^2}} \qquad \frac{\partial}{\partial x_\ell} g_{ij} = 0 \text{ for all others}$$

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \frac{y^2+u^2}{u^2} & \frac{yz}{u^2} \\ \frac{yz}{u^2} & \frac{z^2+u^2}{u^2} \end{pmatrix} \underset{\text{at } p=(3,4,0)}{=} \begin{pmatrix} \frac{5^2}{3^2} & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{Inverse of } g \text{ (diagonal)} = \begin{pmatrix} 3^2/5^2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$g^{11} = \frac{3^2}{5^2} \qquad g^{12} = g^{21} = 0 \qquad g^{22} = 1$$

$$D_{\vec{v}}\vec{w} = D_{\hat{x}_1}\hat{x}_2 = \Gamma_{12}^{1}\hat{x}_1 + \Gamma_{12}^{2}\hat{x}_2 \quad \text{by defn of } \Gamma_{ij}^{k}$$

$$\Gamma_{12}^{1} = \frac{1}{2}g^{11}\left(\frac{\partial}{\partial x_2}g_{11} + \frac{\partial}{\partial x_1}g_{12} - \frac{\partial}{\partial x_1}g_{12}\right)$$

$$+ \frac{1}{2}\overset{0}{\cancel{g^{12}}}\left(\frac{\partial}{\partial x_2}g_{21} + \frac{\partial}{\partial x_1}g_{22} - \frac{\partial}{\partial x_2}g_{12}\right)$$

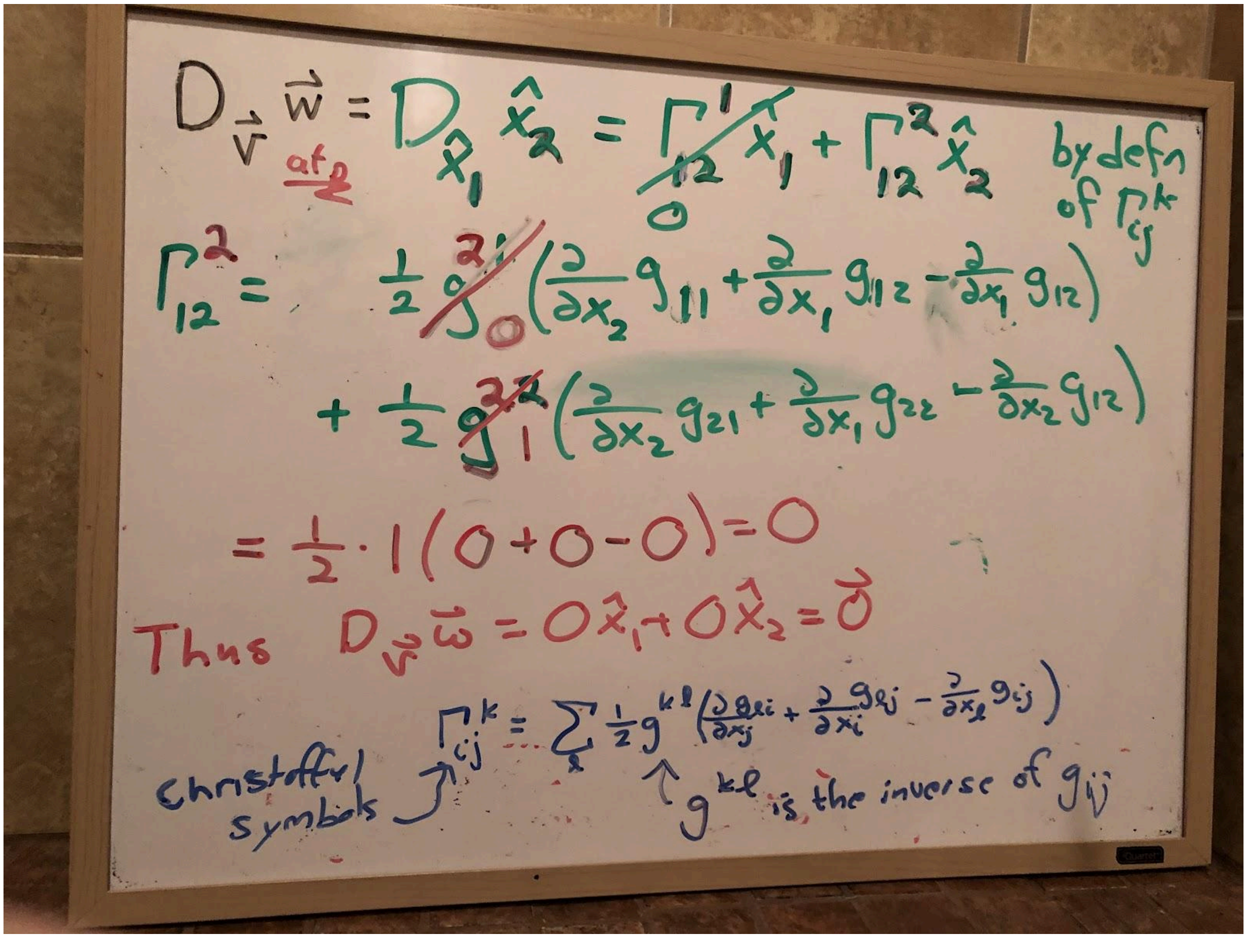
using our values

$$= \frac{1}{2}\left(\frac{3^2}{5^2}\right)(0 + 0 - 0) = 0$$

$$\Gamma_{ij}^{k} = \sum_{\ell}\frac{1}{2}g^{k\ell}\left(\frac{\partial}{\partial x_j}g_{\ell i} + \frac{\partial}{\partial x_i}g_{\ell j} - \frac{\partial}{\partial x_\ell}g_{ij}\right)$$

Christoffel symbols

$g^{k\ell}$ is the inverse of g_{ij}



Note that solutions were provided for all the homework, but be sure to ask questions if you are unsure of something or see a possible error in the videos.

A cool video I found on covariant differentiation.