Exam 2 Review

Sec 2.7-2.11, 3.1-3.4, Chapters 4 and 5

NOTE: This review sheet was updated 10/16/14 to exclude the material from Chapter 6 (Proof by Contradiction). Two problems were removed (#8 and #12 on the previous version). No new problems were added.

1. Translate each statement to symbols. Is the statement true or false?
   You may use the following in your answers:
   E(n): the number n is even
   O(n): the number n is odd
   $\mathbb{R}^+$: the set of positive real numbers
   $\mathbb{R}^-$: the set of negative real numbers
   F: The set of functions
   G: The set of polynomial functions
   a. For every integer x, the number $4x + 7$ is odd.
   b. There exists a real number x such that all real numbers greater than x are positive.
   c. There is no real number y such that all real numbers less than y are negative.
   d. There exists a real number z such that all positive numbers are greater than z.
   e. For every function f, if f is a polynomial then f' is a polynomial.
   f. If f and g are polynomials then so is the sum $f + g$.
2. Translate to English. Is the statement true or false?
   a. $\forall x \in \mathbb{R}, (x \in \mathbb{N} \Rightarrow x \in \mathbb{Z})$
   b. $\forall m, n \in \mathbb{Z}\, (E(m) \wedge O(n)) \Rightarrow O(m \cdot n)$
   c. $\forall f \in F, \sim (f \in G) \Rightarrow \sim (f = 2x^2 + 6x + 1)$
3. Find the negation of each sentence. For a) and b), give your answer in words. For c) and d), give your answer first in symbols, then in words.
   a. a and b are both positive
   b. every integer is either prime or composite
   c. $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\ y^2 = x$
   d. $(p \text{ is even}) \Rightarrow (q \text{ is odd})$
4. How many lists of length 5 are there, taken from the set $\{A, B, C, D, E, F, G, H\}$, if
   a. repetition is not allowed
   b. repetition is not allowed and the last two entries must be vowels
   c. repetition is allowed
   d. repetition is allowed, and the list must contain at least one repeated letter
5.
   a. You have three vampire books, four steampunk novels, and five post-apocalyptic teen romances. How many ways are there of arranging them on a shelf if books of the same type must be kept together?
   b. How many 5 element subsets are there from a set with 8 elements?
   c. Find $|\{X \in P(A) : A = \{0, 1, 2, 3, 4, 5, 6\} \text{ and } |X| = 3\}|$

d. After expanding $(a + b)^5$, what is the coefficient of the term a^3b^2?
e. In a class consisting of 12 men and 15 women, how many different ways are there of choosing a group consisting of 3 men and 6 women?

Prove each proposition. Clearly state whether you are using a direct or contrapositive proof.

6. Proposition. For $a \in \mathbb{Z}$, if $a^2 + 2a + 6$ is odd, then a is odd.
7. Proposition. If $x + y$ is even, then x and y have the same parity.
8. Proposition. Suppose $a, b \in \mathbb{Z}$. If $a|b$ then $a^2|b^2$.
9. Proposition. If $n \in \mathbb{Z}$ then $5n^2 + 3n$ is even.
10. Proposition. Let $a, b \in \mathbb{Z}, n \in \mathbb{N}$. If $a \equiv b (mod\, n)$ then $a^3 \equiv b^3 (mod\, n)$.
11. Proposition. For any integer n, if n^2 is odd then n is odd.

Exam 2 Review ANSWER KEY

If you discover an error please let me know, either in class, on the OpenLab, or by email to jreitz@citytech.cuny.edu. Corrections will be posted on the "Exam Reviews" page.

1. a. $\forall x \in \mathbb{Z},\ O(4x + y)$ b. $\exists x \in \mathbb{R}\, \forall y \in \mathbb{R}\, (y > x \Rightarrow y \in \mathbb{R}^{+})$
   c. $\sim \exists y \in \mathbb{R}\, \forall x \in \mathbb{R}\, (x < y \Rightarrow x \in \mathbb{R}^{-})$ d. $\exists x \in \mathbb{R}\, \forall y \in \mathbb{R}^{+}\ y > x$
   e. $\forall f \in F,\ (f \in G \Rightarrow f' \in G)$ f. $f, g \in G \Rightarrow (f + g) \in G$
2. a. For every real number *x*, if *x* is a natural number then x is an integer. TRUE.
   b. For all integers *m* and *n*, if *m* is even and *n* is odd, then *mn* is odd. FALSE.
   c. For every function *f*, if *f* is not a polynomial then *f* is not equal to $2x^2 + 6x + 1$. TRUE.
3. a. *a* is negative or *b* is negative.
   b. There is an integer that is neither prime nor composite.
   c. $\exists x \in \mathbb{R}\, \forall y \in \mathbb{R},\ y^2 \neq x$. There is a real number *x* such that for all real numbers *y*, we have $y^2 \neq x$.
   d. $(p\ is\ even) \wedge (q\ is\ even)$. Both p and q are even.
4. a. $\frac{8!}{3!} = 6720$ b. $2 \cdot 1 \cdot 6 \cdot 5 \cdot 4 = 240$ c. $8^5 = 32768$ d) $8^5 - \frac{8!}{3!} = 26048$
5. a. $3! \cdot (3! \cdot 4! \cdot 5!) = 103680$ b. $\binom{8}{5} = 56$ c. $\binom{7}{3} = 35$ d. $\binom{5}{3} = 10$
   e. $\binom{12}{3} \cdot \binom{15}{6} = 1101100$

NOTE: For problems requiring you to prove something, there is usually more than one correct answer, and it is often possible to use more than one type of proof correctly. The following are examples of correct solutions, yours may be different.

6. Proposition. For $a \in \mathbb{Z}$, if $a^2 + 2a + 6$ is odd, then a is odd.
   I will use contrapositive proof.
   Proof. Suppose *a* is even. Then $a = 2b$ for some integer *b*, by the definition of even number. Thus $a^2 + 2a + 6 = (2b)^2 + 2(2b) + 6 = 4b^2 + 4b + 6 = 2(2b^2 + 2b + 3)$. So $a^2 + 2a + 6 = 2c$, where *c* is the integer $c = 2b^2 + 2b + 3$. Therefore $a^2 + 2a + 6$ is an even number, by the definition of even number. ☐
7. Proposition. If $x + y$ is even, then *x* and *y* have the same parity.
   I will use contrapositive proof.
   Proof. Suppose *x* and *y* have opposite parity. There are two cases, first where x is odd and y is even, and second where y is odd and x is even.
   (NOTE: The proofs of these two cases will look ***exactly*** *the same, except that 'x' will be switched with 'y'. Because of this, I do NOT need to write out both cases -- instead, I simply use the*

phrase "without loss of generality" or "WLOG" and choose one of them. This blue paragraph is not part of the proof).

Without loss of generality, assume x is odd and y is even. Then $x = 2a + 1$ and $y = 2b$, by the definitions of 'odd' and 'even' respectively. Therefore $x + y = 2a + 1 + 2b = 2(a + b) + 1$, and so $x + y = 2c + 1$ for the integer $c = a + b$. Thus $x + y$ is odd, by the definition of odd number. ☐

8. Proposition. Suppose $a, b \in \mathbb{Z}$. If $a|b$ then $a^2|b^2$.
   I will use direct proof.
   Proof. Suppose $a|b$. Then $b = ac$ for some integer *c*, by the definition of divides, and so $b^2 = (ac)^2 = a^2c^2$. Thus $b^2 = a^2k$ for the integer $k = a^2$. Therefore $a^2|b^2$ by the definition of divides. ☐
9. Proposition. If $n \in \mathbb{Z}$ then $5n^2 + 3n$ is even.
   I will use direct proof.
   Proof. Suppose $n \in \mathbb{Z}$. There are two cases, *n* is odd or *n* is even.
   Case 1. Suppose *n* is odd. Then $n = 2a + 1$ for some integer *a* (by the definition of odd number), and $5n^2 + 3n = 5(2a + 1)^2 + 3(2a + 1) = 2(10a^2 + 13a + 4)$. Thus $5n^2 + 3n = 2k$ for the integer $k = 10a^2 + 13a + 4$, and so $5n^2 + 3n$ is even by the definition of even number.
   Case 2. Suppose *n* is even. Then $n = 2a$ for some integer *a* (by the definition of even number), and $5n^2 + 3n = 5(2a)^2 + 3(2a) = 2(10a^2 + 3a)$, and so $5n^2 + 3n$ is even by the definition of even number. ☐
10. Proposition. Let $a, b \in \mathbb{Z}, n \in \mathbb{N}$. If $a \equiv b (mod\, n)$ then $a^3 \equiv b^3 (mod\, n)$.
   I will use direct proof.
   Proof. Suppose $a \equiv b(mod\, n)$. Then $n|(a - b)$, by the definition of congruence, and so $a - b = nk$ for some integer *k*, by the definition of divides. This gives $a = nk + b$, and so $a^3 = (nk + b)^3 = n^3k^3 + 3n^2k^2b + 3nkb^2 + b^3$. Subtracting b^3 we have $a^3 - b^3 = n(n^2k^3 + 3n\ k^2b + 3kb^2)$, and so $n|(a^3 - b^3)$ by the definition of divides. Therefore $a^3 \equiv b^3(mod\, n)$ by the definition of congruence. ☐
11. Proposition. For any integer *n*, if n^2 is odd then *n* is odd.
   I will use contrapositive proof.
   Proof. Suppose *n* is even. Then $n = 2b$ for some integer *b*, by the definition of even number. Therefore $n^2 = (2b)^2 = 2(2b^2)$, and so $n^2 = 2k$ for the integer $k = 2b^2$. Thus n^2 is even, by the definition of even number. ☐