Analysis Lesson 14
MAT320/MAT640 Analysis
with Professor Sormani
Spring 2022

Review for Exam 2
on Sequences, Continuity, and Limits

Your work for today's lesson will go in a googledoc you create entitled MAT320S22-Lesson14-Lastname-Firstname with your last name and your first name. The googledoc will be shared with the professor sormanic@gmail.com as an editor. Put any questions you have inside your doc and email me to let me know it is there. Be sure to complete one page of HW on paper and take a selfie holding up a few pages.

You will schedule your exam when you feel ready for it!

- **The parts are timed because there are cheating websites that provide solutions in 30 minutes. Set a timer and upload the photos for each part on time (before 25 minutes) even if it is incomplete. You can finish it for homework.**
- **All work must be done on paper and selfies taken holding up the first sheet. It may help to have two colors but please don't use red.**
- **Each student has a different exam which you will find at the top of your googledoc when it is time to start.**
- **If a part is submitted late there will be a zero on that part.**
- **If you do a different problem (not the one assigned to you) it is a zero.**
- **You may consult notes but there is very little time to search through them so try to have key definitions handy to consult quickly.**
- **All proofs must be done in the format taught in this course.**
- **You may not seek help from people or programs during the exam.**

Scoring: There is very little time for each part, so the exam is designed for students to score well even when submitting incomplete work. Students may then complete their work for homework to earn more points.

- **Part I is 18 points plus up to 10 points of extra credit.**
- **Part 2 is 18 points plus up to 10 points of extra credit.**
- **Part 3 is 32 points plus no extra credit**
- **Part 4 is 32 points plus no extra credit**

The exam has 100 points plus up to 20 points of extra credit.

- **A+ for over 100,**
- **A and A- for 90-100**
- **B+ and B for 80-90**
- **C+ and B- for 70-80**
- **C- and C for 60-70**

- **F for below 60 (recommended to withdraw)**

Basically scoring is -2 points for each error but some errors are very serious so they will have -5 points. Examples of very serious errors:

- **if you write something like 2=4 or 2>4 this is very serious (-5pts)**
- **if you write contradiction or ⊗ when there is no contradiction (-5pts) It is better to write *"I cannot find a contradiction".***
- **if you just "fudge" a proof to make it appear to work hoping I won't see there is an error (-5pts). It is better to write *"Something is Wrong Here"* or "I ran out of Time".**

More details on grading is below. As mentioned above, it is a full deduction and zero on a part if the part is submitted late or you do someone else's exam part instead of your own. Practice taking photos and uploading them quickly into a google doc with the google docs app on your phone or tablet.

The homework for today's lesson is to create a study sheet and complete the sampl e exams (which have solutions provided for you to check your own work). Please practice in a timed setting.

On Exam 2 we have more than proofs. There will be lots of definitions and ideas to understand. It is important to make a study sheet that you can consult during the exam with the definitions. Below we provide a thorough review of the topics in two playlists.

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Parts I-II short questions about given sequences: bounded, increasing, decreasing, liminf, limsup, etc and extra credit proofs

****

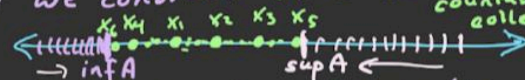
Review for Parts I-2 Playlist which is a 2 hour selection of key videos from prior lessons. You may wish to watch them at a faster playback speed and then slow down for the parts you don't remember as well.

Sup, Inf, and the Archimedean Property

✱ Defn: $\sup A = \min\{B \mid B \text{ is an upper bound for } A\}$

✱ Defn: $\inf A = \max\{B \mid B \text{ is an lower bound for } A\}$

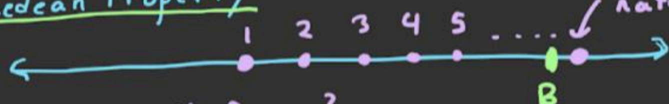
Today we consider $A = \{x_n \mid n \in \mathbb{N}\}$ countable collection of points



Using the Archimedean Property which says that $\mathbb{N}$ has no upper bound

$\mathbb{N} = \{1, 2, 3, 4, 5 \ldots\}$

Archimedean Property:



$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$

There is not an upper bound for $\mathbb{N}$.

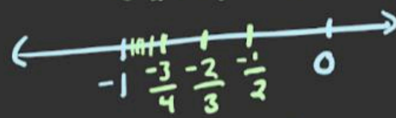
If you try to find a upper bound B such that $\forall n \in \mathbb{N},\ n \leq B$ ← Is False ⊗ so use this to create a contradiction ⊗.

we will fail because there will always be another natural number bigger than B.

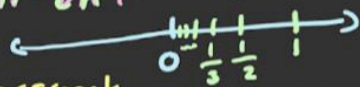
Examples of Countable Sets

$A=\{\frac{-1}{n} \mid n\in\mathbb{N}\}$
$=\{\frac{-1}{1},\frac{-1}{2},\frac{-1}{3},\frac{-1}{4}\dots\}$

$A=\{\frac{-n}{n+1} \mid n\in\mathbb{N}\}$
$=\{\frac{-1}{1+1},\frac{-2}{2+1},\frac{-3}{3+1},\dots\}$
$=\{\frac{-1}{2},\frac{-2}{3},\frac{-3}{4},\dots$

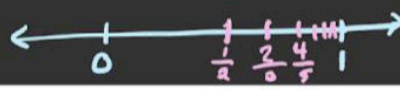


$A=\{\frac{1}{n} \mid n\in\mathbb{N}\}$



$A=\{\frac{n}{n+1} \mid n\in\mathbb{N}\}$
$=\{\frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{4}{5}\dots\}$

Classwork
Find sup, inf, max + min if they exist for each!



Examples of Countable Sets

$A = \{\frac{-1}{n} \mid n \in \mathbb{N}\}$
$= \{\frac{-1}{1}, \frac{-1}{2}, \frac{-1}{3}, \frac{-1}{4} \dots\}$

$-1 \quad -\frac{1}{2} \quad 0$

$A = \{\frac{-n}{n+1} \mid n \in \mathbb{N}\}$
$= \{\frac{-1}{1+1}, \frac{-2}{2+1}, \frac{-3}{3+1}, \dots\}$
$= \{\frac{-1}{2}, \frac{-2}{3}, \frac{-3}{4}, \dots$

$-1 \quad \frac{-3}{4} \quad \frac{-2}{3} \quad \frac{-1}{2} \quad 0$

$A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$

$0 \quad \frac{1}{3} \quad \frac{1}{2} \quad 1$

$A = \{\frac{n}{n+1} \mid n \in \mathbb{N}\}$
$= \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5} \dots\}$

$0 \quad \frac{1}{2} \quad \frac{2}{3} \quad \frac{4}{5} \quad 1$

classwork
Find sup, inf, max + min if they exist for each!

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Examples of Countable Sets

$\sup(A) = \min\{B \mid B \text{ is an upper bound for } A\}$

$A = \{\frac{-1}{n} \mid n \in \mathbb{N}\}$
$= \{\frac{-1}{1}, \frac{-1}{2}, \frac{-1}{3}, \frac{-1}{4} \dots\}$

$\sup A = 0$ no max

$-1 \quad -\frac{1}{2} \quad 0$ upperbounds

$A = \{\frac{-n}{n+1} \mid n \in \mathbb{N}\}$
$= \{\frac{-1}{1+1}, \frac{-2}{2+1}, \frac{-3}{3+1}, \dots\}$
$= \{\frac{-1}{2}, \frac{-2}{3}, \frac{-3}{4}, \dots$

$-1 \quad \frac{-3}{4} \quad \frac{-2}{3} \quad \frac{-1}{2} \quad 0$ max $=$ sup $= \frac{-1}{2}$

$A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ sup $= 1 =$ max

$0 \quad \frac{1}{3} \quad \frac{1}{2} \quad 1$

$A = \{\frac{n}{n+1} \mid n \in \mathbb{N}\}$
$= \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5} \dots\}$

no max sup $= 1$

$0 \quad \frac{1}{2} \quad \frac{2}{3} \quad \frac{4}{5} \quad 1$

classwork
Find sup, inf, max + min if they exist for each!

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Examples of Countable Sets

$\inf A = \max\{b \mid b \text{ is an lower bound for } A\}$

$A = \{\frac{-1}{n} \mid n \in \mathbb{N}\}$
$= \{\frac{-1}{1}, \frac{-1}{2}, \frac{-1}{3}, \frac{-1}{4} \dots\}$

lower $-1 = \inf = \min$ $\sup A = 0$ no max

lower bounds $-1 \quad -\frac{1}{2} \quad 0$ upperbounds

$A = \{\frac{-n}{n+1} \mid n \in \mathbb{N}\}$
$= \{\frac{-1}{1+1}, \frac{-2}{2+1}, \frac{-3}{3+1}, \dots\}$
$= \{\frac{-1}{2}, \frac{-2}{3}, \frac{-3}{4}, \dots$

$\inf = -1$ no min

$-1 \quad \frac{-3}{4} \quad \frac{-2}{3} \quad \frac{-1}{2} \quad 0$ max $=$ sup $= \frac{-1}{2}$

Defn B is an upper Bound for A
$\forall x \in A \quad x \le B$

Defn of supremum:
$\sup A = \min\{B \mid B \text{ is an upper bound for } A\}$
the smallest upper bound

Useful for Justifications

Rules of Inequalities

I $a \geq b$ AND $b \geq c \Rightarrow a \geq c$
II $a \geq b$ AND $c \in \mathbb{R} \Rightarrow a+c \geq b+c$
III $a \geq b$ AND $c > 0 \Rightarrow ac \geq bc$
IV $a \geq b$ AND $c < 0 \Rightarrow ac \leq bc$

Archimedean Principle
There is no upper bound for $\mathbb{N}$.
$\forall n \in \mathbb{N} \; n \le k$ is false.

Prove $\sup(A) = t = 0$
where $A = \{\frac{-1}{n} \mid n \in \mathbb{N}\}$

Part I: Show t is an upper bound
① Given $x \in A$

Classwork
Try This

final $x \le t$

Part II: Show t is the smallest
① Assume on the contrary that $\exists$ a smaller upper bound $B < t$.

Solve for n

Classwork
Try This

$\otimes$ Archimedean Principle

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Defn B is an upper Bound for A
$\forall x \in A \quad x \le B$

Defn of supremum:
$\sup A = \min\{B \mid B \text{ is an upper bound for } A\}$
the smallest upper bound

Useful for Justifications

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I $a \geq b$ AND $b \geq c \Rightarrow a \geq c$
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IV $a \geq b$ AND $c < 0 \Rightarrow ac \leq bc$

Archimedean Principle
There is no upper bound for $\mathbb{N}$.
$\forall n \in \mathbb{N} \; n \le k$ is false.

Prove $\sup(A) = t = 0$
where $A = \{\frac{-1}{n} \mid n \in \mathbb{N}\}$

Part I: Show 0 is an upper bound

① Given $x \in A$, $x = -\frac{1}{n}$	① by defn of A
② $-1 \le 0$	② ordering of $\mathbb{R}$
③ $-\frac{1}{n} \le 0 \cdot \frac{1}{n}$	③ $a \le b$ and $c = \frac{1}{n} > 0 \Rightarrow ac \le bc$
final $\frac{-1}{n} \le 0$	final $0 \cdot \frac{1}{n} = 0$

Part II: Show 0 is the smallest
① Assume on the contrary that $\exists$ a smaller upper bound $B < 0$

Solve for n

$\otimes$ Archimedean Principle

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Defn B is an upper Bound for A
$\forall x \in A \quad x \leq B$

Defn of supremum:
$\sup A = \min\{B \mid B \text{ is an upper bound for } A\}$
the smallest upper bound

Useful for Justifications

Rules of Inequalities

I $a \geq b$ AND $b \geq c \Rightarrow a \geq c$
II $a \geq b$ AND $c \in \mathbb{R} \Rightarrow a+c \geq b+c$
III $a \geq b$ AND $c > 0 \Rightarrow ac \geq bc$
IV $a \geq b$ AND $c < 0 \Rightarrow ac \leq bc$

Archimedean Principle
There is no upper bound for $\mathbb{N}$.
$\forall n \in \mathbb{N} \; n \leq k$ is false.

Thm
$B < 0 \Rightarrow \frac{1}{B} < 0$
$\boxed{\text{HW 1}}$ Prove

Classwork
prove
$\sup\left\{\frac{n}{n+1} \mid n \in \mathbb{N}\right\} = 1$

Prove $\sup(A) = t = 0$
where $A = \left\{\frac{-1}{n} \mid n \in \mathbb{N}\right\}$

Part I: Show 0 is an upper bound

(1) Given $x \in A$ $x = \frac{-1}{n}$	(1) by defn of A
(2) $-1 \leq 0$	(2) ordering of $\mathbb{R}$
(3) $-\frac{1}{n} \leq 0 \cdot \frac{1}{n}$	(3) $a \leq b$ and $c = \frac{1}{n} > 0 \Rightarrow ac \leq bc$
(final) $\frac{-1}{n} \leq 0$	(final) $0 \cdot \frac{1}{n} = 0$

Part II: Show 0 is the smallest

(1) Assume on the contrary that $\exists$ a smaller upper bound $\boxed{B < 0}$	(1) Ind Hyp
(2) $\forall x \in A \quad x \leq B$	(2) Defn of Upper Bound
(3) $\forall n \in \mathbb{N} \; \frac{-1}{n} \in A$	(3) by defn of set A
(3.5) $\forall n \in \mathbb{N} \; \frac{-1}{n} \leq B$	(3.5) steps 2+3
(4) $\forall n \in \mathbb{N} \; -1 \leq Bn$	(4) $a \leq b$ and $c = \frac{1}{n} > 0$, $ac \leq bc$
(5) $\forall n \in \mathbb{N}, \; -1\left(\frac{1}{B}\right) \geq Bn\left(\frac{1}{B}\right)$	(5) $a \leq b$ and $c = \frac{1}{B} < 0 \Rightarrow ac \geq bc$
(6) $\forall n \in \mathbb{N} \; \frac{-1}{B} \geq n$	(6) $B\left(\frac{1}{B}\right) = 1$
(7) So $\frac{-1}{B}$ is an upper bound for $\mathbb{N}$.	(7) Defn upper bound.

$\otimes$ Archimedean Principle
because $\mathbb{N}$ has no upper bound!

Thus our indirect hyp. is false.

Thus 0 is the smallest upper bound

Thus $0 = \sup A$

Prove $\sup\left\{\frac{n}{n+1} \mid n \in \mathbb{N}\right\} = 1$

Part I 1 is an upper bound $\forall x \in A, x \le 1$

① Given $x \in \left\{\frac{n}{n+1} \mid n \in \mathbb{N}\right\}$ — ① Given

② $\exists n \in \mathbb{N}$ s.t. $x = \frac{n}{n+1}$ — ② by defn of A

?

$\frac{n}{n+1} \le 1$

final $x \le 1$

solve up for n to a true fact

Prove $\sup\left\{\frac{n}{n+1} \mid n \in \mathbb{N}\right\} = 1$

Part I 1 is an upper bound $\forall x \in A, x \le 1$

① Given $x \in \left\{\frac{n}{n+1} \mid n \in \mathbb{N}\right\}$ — ① Given

② $\exists n \in \mathbb{N}$ s.t. $x = \frac{n}{n+1}$ — ② by defn of A

③ $0 \le 1$ — ③ by ordering of $\mathbb{R}$

④ $n \le (n+1)$ — ④ $a \le b$ and $c = n \in \mathbb{R}$ $\Rightarrow a + c \le b + c$

⑤ $\frac{n}{n+1} \le 1$ — ⑤ $a \le b$ and $c = \frac{1}{n+1} > 0$ $\Rightarrow a \cdot c \le b \cdot c$ no switch

final $x \le 1$ — final Sub step 2 $x = \frac{n}{n+1}$

Done with Part I

Part II 1 is the smallest upper bound for A

(1) Assume on the contrary $\exists$ an upper bound $B<1$ — (1) Indirect Hypothesis

(2) $\forall x \in A$, $x \leq B$ — (2) Defn Upper Bound

(3) by defn A

try writing what we know about A

solve for n

(8) Archimedes if possible.

Part II 1 is the smallest upper bound for A

(1) Assume on the contrary $\exists$ an upper bound $B<1$ — (1) Indirect Hypothesis

(2) $\forall x \in A$, $x \leq B$ — (2) Defn Upper Bound

(3) $\forall n \in \mathbb{N}$, $\frac{n}{n+1} \leq B$ — (3) by defn $A = \left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$

(4) $\forall n \in \mathbb{N}$, $n \leq B(n+1)$ — (4) $a \leq b$ AND $c = n+1 > 0$ $\Rightarrow ac \leq bc$

(5) $\forall n \in \mathbb{N}$, $n \leq Bn + B$ — (5) distribution (algebra)

(6) $\forall n \in \mathbb{N}$, $n - Bn \leq B$ — (6) $a \leq b$ AND $c = -Bn \in \mathbb{R}$ $\Rightarrow a+c \leq b+c$

(7) $\forall n \in \mathbb{N}$, $n(1-B) \leq B$ — (7) factoring.

(8) $\forall n \in \mathbb{N}$, $n \leq B\frac{1}{(1-B)}$ — (8) $a \leq b$ AND $c = \frac{1}{1-B} > 0$

No switch

because $B<1 \Rightarrow 0<1-B$ $\Rightarrow \frac{1}{1-B} > 0$

(8) Archimedes!

Thus our indirect Hyp is false so 1 is the smallest upper bound $\sup A = 1$ QED

Prove $-1 = \inf\left\{\frac{-n}{n+1} \mid n \in \mathbb{N}\right\} = \inf A$

Part I: -1 is a lower bound

Given $x \in A$ Show $-1 \leq x$

① Given $x \in A$	① Given
② $\exists n \in \mathbb{N}$ s.t $x = \frac{-n}{n+1}$	② Defn of A.
③ $0 \geq -1$	③ ordering of reals
④ $-n \geq -n-1$	④ $a \geq b$ and $c = -n \in \mathbb{R}$ $\Rightarrow a + c \geq b + c$
⑤ $-n \geq -1(n+1)$	⑤ factoring
⑥ $\frac{-n}{n+1} \geq -1$	⑥ $a \geq b$ and $c = \frac{1}{n+1} > 0$ $\Rightarrow ac \geq bc$ because $n > 0$ so $n+1 > 0$ so $\frac{1}{n+1} > 0$
final $x \geq -1$	⑦ by step 2

Done w/ Part I.

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Prove $-1 = \inf\left\{\frac{-n}{n+1} \mid n \in \mathbb{N}\right\} = \inf A$

Part II Show -1 is the biggest lower bound

① Assume on the contrary that $\exists$ a lower bound $b > -1$. (-1 is not the biggest)	① Indirect Hypothesis
② $\forall x \in A \quad x \geq b$	② defn of lower bound
③ $\forall n \in \mathbb{N} \quad \frac{-n}{n+1} \geq b$	③ by defn of A

$\downarrow$ solve for n + find a contradiction

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Prove $-1 = \inf\left\{ \frac{-n}{n+1} \mid n \in \mathbb{N} \right\} = \inf A$

Part II Show -1 is the biggest lower bound

① Assume on the contrary that $\exists$ a lower bound $b > -1$. (−1 is not the biggest) — ① Indirect Hypothesis

② $\forall x \in A \quad x \geq b$ — ② defn of lower bound

③ $\forall n \in \mathbb{N} \quad \frac{-n}{n+1} \geq b$ — ③ by defn of A

④ $\forall n \in \mathbb{N} \quad -n \geq (n+1)b$ — ④ $a \geq b$ and $c = n+1 > 0 \Rightarrow ac \geq bc$

⑤ $\forall n \in \mathbb{N} \quad -n \geq n \cdot b + b$ — ⑤ distribution

⑥ $\forall n \in \mathbb{N} \quad -n \cdot b - n \geq b$ — ⑥ $a \geq b$ and $c = nb \in \mathbb{R} \Rightarrow a + c \geq b + c$

⑦ $\forall n \in \mathbb{N} \quad n(-b-1) \geq b$ — ⑦ Factoring

⑧ $\forall n \in \mathbb{N} \quad n \leq \frac{b}{-b-1}$ — ⑧ $a \geq b$ and $c = \frac{1}{-b-1} < 0$ by $\begin{cases} b > -1 \\ -b < 1 \\ -b-1 < 0 \\ \frac{1}{-b-1} < 0 \end{cases}$ $\Rightarrow ac \leq bc$

⊗ Archimedes: the naturals have no upper bound.

Thus our indirect hypothesis was false and -1 is the biggest lower bound QED

Monotone Sequences

$\{x_j \mid j \in \mathbb{N}\} = \{x_1, x_2, x_3, x_4 \dots\}$

Defn: $\{x_j \mid j \in \mathbb{N}\}$ is strictly increasing

if $x_1 < x_2 < x_3 \dots$

$x_j < x_{j+1}$ for all $j \in \mathbb{N}$.

Defn: It is strictly decreasing

if $x_1 > x_2 > x_3 \dots$

$x_j > x_{j+1}$ for all $j \in \mathbb{N}$.

Monotone Sequences

$\{x_j \mid j \in \mathbb{N}\} = \{x_1, x_2, x_3, x_4 \dots\}$

Defn: $\{x_j \mid j \in \mathbb{N}\}$ is "not-strictly increasing" "nondecreasing"

if $x_1 \leq x_2 \leq x_3 \dots$

$x_j \leq x_{j+1}$ for all $j \in \mathbb{N}$.

Defn: It is "not-strictly decreasing" "nonincreasing"

if $x_1 \geq x_2 \geq x_3 \dots$

$x_j \geq x_{j+1}$ for all $j \in \mathbb{N}$.

Monotone Sequences

$\{x_j \mid j \in \mathbb{N}\} = \{x_1, x_2, x_3, x_4 \ldots\}$

Defn: $\{x_j \mid j \in \mathbb{N}\}$ is strictly increasing if $x_j < x_{j+1}$ for all $j \in \mathbb{N}$

Defn: It is strictly decreasing if $x_j < x_{j+1}$ for all $j \in \mathbb{N}$

Defn: It is strictly monotone if it is decreasing or increasing.

Thm: If $\{x_j \mid j \in \mathbb{N}\}$ is increasing and bounded above then $\exists L \in \mathbb{R}$ s.t. $x_j \to L$ and $L = \sup\{x_j \mid j \in \mathbb{N}\}$

Example: $x_j = \dfrac{5j-1}{j+2}$

Monotone Convergence Theorem

A bounded monotone sequence converges.

Proof has two cases: I increasing, II decreasing

Thm I: If $\{x_j \mid j \in \mathbb{N}\}$ is increasing and bounded above then $\exists L \in \mathbb{R}$ s.t. $x_j \to L$ and $L = \sup\{x_j \mid j \in \mathbb{N}\}$

Monotone Convergence Theorem

A bounded monotone sequence converges.

Proof has two cases: I increasing, II decreasing

Thm II: If $\{x_j \mid j \in \mathbb{N}\}$ is decreasing and bounded below then $\exists L \in \mathbb{R}$ s.t. $x_j \to L$ and $L = \inf\{x_j \mid j \in \mathbb{N}\}$

Bolzano-Weierstrass Theorem

on the Real Line

Subsequences

liminf and limsup

A sequence is a set in which every point has been assigned a unique natural number, and every natural number has a point.

$$\{x_j \mid j \in \mathbb{N}\} = \{x_1, x_2, x_3, x_4, x_5, \dots\}$$

we keep track of the order.

Set $\{1,2,3\} = \{3,2,1\}$ ← Contrast with Set!

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A subsequence is a list of points from the original sequence in the same order.

Example: Even Subseq $= \{x_2, x_4, x_6 \dots\}$

Odd Subseq $= \{x_1, x_3, x_5, x_7 \dots\}$

Any selection $\{x_{j_1}, x_{j_2}, x_{j_3}, \dots\}$

j_k

Does a subsequence converge?

$$\{(-1)^j \mid j \in \mathbb{N}\} = \{(-1)^1, (-1)^2, (-1)^3, (-1)^4, \dots\}$$
$$= \{-1, +1, -1, +1, -1, +1 \dots\}$$

we know this sequence does not converge

but does it has a subseq. that converges?

Yes, even subseq $= \{(-1)^{2k} \mid k \in \mathbb{N}\} = \{1, 1, 1 \dots\}$

$j_k = 2k$ $\qquad \lim_{k\to\infty} x_{j_k} = \lim_{k\to\infty} 1 = 1$

also odd subseq $= \{(-1)^{2k-1} \mid k \in \mathbb{N}\} = \{-1, -1, -1 \dots\}$

$j_k = 2k-1$ $\qquad \lim_{k\to\infty} x_{j_k} = \lim_{k\to\infty} -1 = -1$

Some sequences have a converging subsequence and different subsequences can have different limits.

Classwork: Prove the even subsequence of $\{x_j \mid j \in \mathbb{N}\} = \{\frac{(-1)^j j}{j+1} \mid j \in \mathbb{N}\}$ converges to 1.

Even subseq is $j_k = 2k$

$$\{x_{2k} \mid k \in \mathbb{N}\} = \left\{\frac{(-1)^{2k}(2k)}{(2k)+1} \;\Big|\; k \in \mathbb{N}\right\}$$

Show:

$$\forall \varepsilon > 0 \ \exists N_\varepsilon \text{ s.t. } \forall k \geq N_\varepsilon \ |x_{2k} - 1| < \varepsilon$$

$$\forall \varepsilon > 0 \ \exists N_\varepsilon \text{ s.t. } \forall k \geq N_\varepsilon \ \left|\frac{(-1)^{2k}(2k)}{(2k)+1} - 1\right| < \varepsilon$$

must use parenthesis around $(2k) = j$ or $(2k+1)$ for odd

Some sequences have no converging subsequences!

Example $x_j = j^2$

$\{j^2 \mid j \in \mathbb{N}\} = \{1^2, 2^2, 3^2, 4^2 \dots\}$

$= \{1, 4, 9, 16, \dots\}$

This seq $\lim_{j\to\infty} j^2 = \infty$

We will prove later that this sequence has no conv. subseq.

Part 1 HW4 imitate the proof below to prove any subsequence of the sequence above also diverges to infinity (not required)

Thm: If $\lim_{j\to\infty} x_j = L$ then for any subsequence $\lim_{k\to\infty} x_{j_k} = L$ too.

Given: $x_j \to L$ $\quad \forall \varepsilon > 0\ \exists N_\varepsilon^x$ s.t. $\forall j \geq N_\varepsilon^x\ |x_j - L| < \varepsilon$

Show: $x_{j_k} \to L$ $\quad \forall \varepsilon > 0\ \exists N_\varepsilon'$ s.t. $\forall k \geq N_\varepsilon'\ |x_{j_k} - L| < \varepsilon$

for any subseq

Proof:

Set up yourself

pause!

limsup and liminf

Given a sequence $\{x_j \mid j \in \mathbb{N}\}$
that has an upper bound, B,
$\forall j \in \mathbb{N} \quad x_j \le B$
$\sup\{x_j \mid j \in \mathbb{N}\}$ exists and $\le B$
by Continuoum hypothesis.

$s_1 = \sup\{x_1, x_2, x_3, x_4 \dots\} \le B$
$s_2 = \sup\{x_2, x_3, x_4 \dots\} \le B$
which is bigger? $s_1 \ge s_2$
$s_3 = \sup\{x_3, x_4, x_5 \dots\} \le B$
$s_1 \ge s_2 \ge s_3$

$s_k = \sup\{x_k, x_{k+1}, x_{k+2}, \dots\} \le B$

sequence of sups
$s_1 \ge s_2 \ge s_3 \ge \dots \ge s_k \ge \dots$

if we assumes $\{x_j \mid j \in \mathbb{N}\}$
has a lower bound b
$s_1 \ge s_2 \ge \dots \ge s_k \ge \dots \ge b$
decreasing and bounded below

so by the monotone conv thm
final $\lim_{k \to \infty} s_k$ exists. final just

Defn $\limsup_{j \to \infty} x_j = \lim_{k \to \infty} s_k$

Thm: $\limsup_{j \to \infty} x_j$ exists
if x_j is bounded above and below.

limsup

Write the first six terms of $\{(-1)^j \mid j \in \mathbb{N}\}$ ✓

find the limit if it exists ✓

find a converging subsequence if it has one ✓

find the liminf and lim sup ✓ Classwork

$x_1 = (-1)^1 = -1 \quad x_2 = (-1)^2 = 1 \quad x_3 = (-1)^3 = -1 \quad x_4 = 1 \quad x_5 = -1 \quad x_6 = 1$

no limit (alternating between -1 and 1)

because has two subseq that conv to diff limits

even sub $x_{2j} \to 1$ odd subseq $x_{2j-1} \to -1$

$$\liminf_{j\to\infty} x_j = \lim_{k\to\infty} \underbrace{\inf\{x_k, x_{k+1} \dots\}}_{y_k} = \lim_{k\to\infty} -1 = -1$$

$$y_1 = \inf\{x_1, x_2 \dots\} = \inf\{-1, 1, -1, 1 \dots\} = -1$$

$$y_2 = \inf\{x_2, x_3 \dots\} = \inf\{1, -1, 1, -1 \dots\} = -1$$

$$\dots$$

$$y_k = \inf\{x_k \dots\} = -1$$

$$\limsup_{j\to\infty} x_j = \lim_{k\to\infty} \sup\{- - -\} = \lim_{k\to\infty} 1 = 1$$

$$s_1 = \quad - - - \quad = 1$$

$$s_2 = \quad - - - \quad = 1$$

Write the first six terms of $\left\{ \frac{(-1)^j j}{j+1} \mid j \in \mathbb{N} \right\}$ ✓

find the limit if it exists ✓

find a converging subsequence if it has one

find the liminf and lim sup. Classwork

$x_1 = \frac{-1}{2}$ $x_2 = \frac{2}{3}$ $x_3 = \frac{-3}{4}$ $x_4 = \frac{4}{5}$ $x_5 = \frac{-5}{6}$ $x_6 = \frac{6}{7}$

There is no limit because it has two subsequences with different limits

odds $\lim\limits_{k\to\infty} x_{2k-1} = -1$ evens $\lim\limits_{k\to\infty} x_{2k} = 1$

$$\liminf_{j\to\infty} x_j = \lim_{k\to\infty} \inf\{x_k, x_{k+1}, \ldots\} = \lim_{k\to\infty} -1 = -1$$

$$\limsup_{j\to\infty} x_j = \lim_{k\to\infty} \sup\{x_k, x_{k+1}, \ldots\} = \lim_{k\to\infty} +1 = +1$$

Observe $\liminf\limits_{j\to\infty} x_j = -1 = \lim\limits_{k\to\infty} x_{2k-1}$

$\limsup\limits_{j\to\infty} x_j = +1 = \lim\limits_{k\to\infty} x_{2k}$

Bolzano-Weierstrass Theorem

Thm If $\{x_j \mid j \in \mathbb{N}\} \subset [b, B]$

then $\exists$ subseq x_{j_k} s.t. $\lim_{k\to\infty} x_{j_k} = L \in [b, B]$

The proof follows from the following three theorems.

- <u>Thm I</u>: If $\{x_j \mid j \in \mathbb{N}\} \subset [b, B]$ then $\limsup_{j\to\infty} x_j$ exists $\in [b, B]$
- <u>Thm II</u> If $\{y_k \mid k \in \mathbb{N}\} \subset [b, B]$ has a limit $y_k \to y$ then $\lim_{k\to\infty} y_k = y \in [b, B]$ (proof is HW)
- <u>Thm III</u> If $\limsup_{j\to\infty} x_j = L$ then $\exists$ subseq $x_{j_k} \to L$.

<u>Thm III</u> If $\limsup_{j\to\infty} x_j = L$ then $\exists$ subseq $x_{j_k} \to L$.

① $\lim_{k\to\infty} s_k = L$ where $s_k = \sup\{x_k, x_{k+1}, \dots\}$ — ① by given and defn limsup

② s_k is an upper bound for $\{x_k, x_{k+1} \dots\}$ and $s_k - \frac{1}{k}$ is not an upper bound — ② by defn of sup

③ $\exists\ x_{j_k} \in \{x_k, x_{k+1} \dots\}$ s.t. $s_k - \frac{1}{k} < x_{j_k}$ — ③ by defn of upper bound

~~Later~~ we can erase any x_{j_k} that are out of order.

Thm III If $\limsup_{j\to\infty} x_j = L$ then $\exists$ subseq $x_{j_k} \to L$.

(1) $\boxed{\lim_{k\to\infty} s_k = L}$ where $s_k = \sup\{x_k, x_{k+1}, \dots\}$ — (1) by given and defn limsup

(2) s_k is an upper bound for $\{x_k, x_{k+1} \cdots\}$ and $s_k - \frac{1}{k}$ is not an upper bound — (2) by defn of sup

(3) $\exists$ $x_{j_k} \in \{x_k, x_{k+1}, \dots\}$ s.t. $\boxed{s_k - \frac{1}{k} < x_{j_k} \le s_k}$ — (3) by defn of upper bound

(4) $\boxed{\lim_{k\to\infty} s_k - \frac{1}{k}} = \lim_{k\to\infty} s_k - \lim_{k\to\infty} \frac{1}{k} = L - 0 = \boxed{L}$ — (4) $\lim_{k\to\infty} a_k - b_k = a_\infty - b_\infty$ and $\lim_{k\to\infty} \frac{1}{k} = 0$

(5) $\lim_{k\to\infty} x_{j_k} = L$ — (5) by sandwich theorem

QED

HW Complete the Sample Exam II Parts I-II Check solutions after doing each sample problem.

Sample Exam II

Part I Let $x_n = \frac{-1}{n}$

1. Write first five terms + plot.
2. Is x_n increasing?
3. Is x_n decreasing?
4. Is x_n monotone?
5. Is x_n bounded above?
6. Is x_n bounded below?
7. Is x_n bounded?
8. Does x_n converge?
9. What is the limit?

Extra Credit: Prove it is monotone.

Sample Exam II

Part I Let $x_n = \frac{-1}{n}$

1. Write first five terms + plot?
2. Is x_n increasing?
3. Is x_n decreasing?
4. Is x_n monotone?
5. Is x_n bounded above?
6. Is x_n bounded below?
7. Is x_n bounded?
8. Does x_n converge?
9. What is the limit?

Extra Credit: Prove it is monotone. Prove inc

Prove $x_{n+1} \geq x_n \ \forall n \in \mathbb{N}$

(1)

$\frac{-1}{1} = -1 \quad \frac{-1}{2} \quad \frac{-1}{3} \quad \frac{-1}{4} \quad \frac{-1}{5}$

(2) Yes increasing $\rightarrow$

(3) No

(4) Yes increasing $\Rightarrow$ monotone

(5) $x_n \leq 0$ So Yes bounded above by 0

(6) $-1 \leq x_n$ So Yes bounded below

(7) Yes bounded because bounded above + bounded below.

(8) Inc + Bounded above so $\lim_{n\to\infty} x_n = \sup x_n$ by Monotone Conv Thm Yes

(9) $\lim_{n\to\infty} \frac{-1}{n} = 0$

Given $x_n = \frac{-1}{n}$

Show $x_{n+1} \geq x_n \ \forall n \in \mathbb{N}$

Proof:

(1)	$0 \geq -1$	(1) by ordering of reals
(2)	$-n \geq -n-1$	(2) $a \geq b$ and $c = -n \in \mathbb{R} \Rightarrow a+c \geq b+c$
(3)	$-n \geq -1(n+1)$	(3) factor $-n-1 = -1(n+1)$
(4)	$\frac{-n}{n+1} \geq -1$	(4) $a \geq b$ and $c = \frac{1}{n+1} > 0 \Rightarrow ac \geq bc$
(final)	$\frac{-1}{n+1} \geq \frac{-1}{n}$	(final) $a \geq b$ and $c = \frac{1}{n} > 0 \Rightarrow ac \geq bc$

QED

See video explaining the above solution.

This part is 18% with 2pts per question plus up to 10% for extra credit.

Sample Exam II

Part II: Let $x_j = \frac{(-1)^j}{j}$

1. Find first six terms and plot.
2. Find an upper bound.
3. Find a lower bound.
4. Does $\lim_{k \to \infty} x_{2k}$ exist?
   If so, what is it?
5. Does $\lim_{k \to \infty} x_{2k+1}$ exist?
   If so, what is it?
6. What is $s_n = \sup\{x_n, x_{n+1}, \ldots\}$?
7. What is $\limsup_{j \to \infty} x_j = \lim_{n \to \infty} s_n$?
8. What is $\liminf_{j \to \infty} x_j$?
9. Does $\lim_{j \to \infty} x_j$ exist?
   If so, what is it?

Extra Credit: prove it has no limit or prove it converges.

Remember a sequence has a limit if it converges to the limit and a sequence has no limit if it does not converge.

Sample Exam II

Part II: Let $x_j = \frac{(-1)^j}{j}$

1. Find first six terms and plot
2. Find an upper bound 1
3. Find a lower bound -1
4. Does $\lim_{k\to\infty} x_{2k}$ exist? Yes
   If so, what is it? 0
5. Does $\lim_{k\to\infty} x_{2k+1}$ exist? Yes
   If so, what is it? 0
6. What is $s_n = \sup\{x_n, x_{n+1}, \dots\}$?
7. What is $\limsup_{j\to\infty} x_j = \lim_{n\to\infty} s_n$? 0
8. What is $\liminf_{j\to\infty} x_j$? 0
9. Does $\lim_{j\to\infty} x_j$ exist? Yes
   If so, what is it? 0

Show work

Show work

(1)

$x_1 = \frac{(-1)^1}{1} = -1$ $\quad x_2 = \frac{(-1)^2}{2} = \frac{1}{2}$ $\quad x_3 = \frac{(-1)^3}{3} = \frac{-1}{3}$

$x_4 = \frac{(-1)^4}{4} = \frac{1}{4}$ $\quad x_5 = \frac{(-1)^5}{5} = \frac{-1}{5}$ $\quad x_6 = \frac{(-1)^6}{6} = \frac{1}{6}$

(2) 1 is an upper bound

$x_j = \frac{(-1)^j}{j} \leq \frac{1}{j} \leq 1$

(3) -1 is a lower bound

$x_j = \frac{(-1)^j}{j} \geq \frac{-1}{j} \geq -1$ see graph above

(4) $\lim_{k\to\infty} \frac{(-1)^{2k}}{2k} = \lim_{k\to\infty} \frac{1}{2k} = 0$ Yes

(5) $\lim_{k\to\infty} \frac{(-1)^{2k+1}}{2k+1} = \lim_{k\to\infty} \frac{-1}{2k+1} = 0$ Yes

(6) $s_n = \sup\{x_n, x_{n+1}, \dots\}$
$= \text{an even term} \begin{cases} x_n \\ x_{n+1} \end{cases}$

(7) $\limsup_{j\to\infty} x_j = \lim_{k\to\infty} x_{2k} = 0$

(8) $\liminf_{j\to\infty} x_j = \underset{\text{terms}}{\text{odd}} = \lim_{k\to\infty} x_{2k+1} = 0$

(9) $\lim_{j\to\infty} x_j = 0$ because both limsup + liminf are 0

Extra Credit: Prove the sequence has no limit or prove it converges

$\lim_{j\to\infty} \frac{(-1)^j}{j} = 0$ so we prove it converges

Show: $\forall \varepsilon > 0 \ \exists N_\varepsilon$ s.t. $\forall j \geq N_\varepsilon$ $\ |(-1)^j \frac{1}{j} - 0| < \varepsilon$

(1) Given any $\varepsilon > 0$ Choose $N_\varepsilon = \boxed{1/\varepsilon + 1}$ — (1) well defined because $\varepsilon > 0$

(2) Whenever $j \geq N_\varepsilon$ we have — (2) by choice of N_ε in step 1

$\frac{1}{\varepsilon} < j$ — $j \geq \frac{1}{\varepsilon} + 1 > \frac{1}{\varepsilon}$

(3) $1 < \varepsilon j$ — (3) $a < b$ and $c = \varepsilon > 0 \Rightarrow ac < bc$

(4) $\frac{1}{j} < \varepsilon$ — (4) $a < b$ and $c = \frac{1}{j} > 0 \Rightarrow ac < bc$

solve up

(5) $\left|\frac{(-1)^j}{j}\right| < \varepsilon$ — (5) defn of abs value and $(-1)^j = \pm 1$

(final) $\left|\frac{(-1)^j}{j} - 0\right| < \varepsilon$ — (final) $a - 0 = a$ QED

This part is 18% with 2 pts per question and 10pts of extra credit.

Sample Exam II

Part II: Let $x_j = \frac{(-1)^j j}{j+1}$

1. Find first six terms and plot.
2. Find an upper bound.
3. Find a lower bound.
4. Does $\lim_{k \to \infty} x_{2k}$ exist?
   If so, what is it?
5. Does $\lim_{k \to \infty} x_{2k+1}$ exist?
   If so, what is it?
6. What is $s_n = \sup\{x_n, x_{n+1}, \dots\}$?
7. What is $\limsup_{j \to \infty} x_j = \lim_{n \to \infty} s_n$?
8. What is $\liminf_{j \to \infty} x_j$?
9. Does $\lim_{j \to \infty} x_j$ exist?
   If so, what is it?

Extra Credit: Prove it has no limit or prove it converges

Sample Exam II

Part II: Let $x_j = \frac{(-1)^j j}{j+1}$

1. Find first six terms and plot.
2. Find an upper bound. 1
3. Find a lower bound. −1
4. Does $\lim_{k\to\infty} x_{2k}$ exist? Yes
   If so, what is it? 1
5. Does $\lim_{k\to\infty} x_{2k+1}$ exist? Yes
   If so, what is it? −1
6. What is $s_n = \sup\{x_n, x_{n+1}, \ldots\}$?
7. What is $\limsup_{j\to\infty} x_j = \lim_{n\to\infty} s_n$?
8. What is $\liminf_{j\to\infty} x_j$?
9. Does $\lim_{j\to\infty} x_j$ exist? No
   If so, what is it?

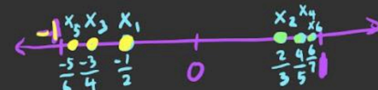
Must show work

We have two subseq with different limits!

Extra Credit:
Prove it has no limit
or prove it converges

① $x_1 = \frac{(-1)^1 1}{1+1} = \frac{-1}{2}$ $x_2 = \frac{(-1)^2 2}{2+1} = \frac{2}{3}$ $x_3 = \frac{(-1)^3 3}{3+1} = \frac{-3}{4}$

$x_4 = \frac{(-1)^4 4}{4+1} = \frac{4}{5}$ $x_5 = \frac{(-1)^5 5}{5+1} = \frac{-5}{6}$ $x_6 = \frac{(-1)^6 6}{6+1} = \frac{6}{7}$



② 1 is an upper bound

$x_j = \frac{(-1)^j j}{j+1} \le \frac{j}{j+1} \le \frac{j+1}{j+1} = 1$

③ −1 is a lower bound

④ $\lim_{k\to\infty} \frac{(-1)^{2k} 2k}{2k+1} = \lim_{k\to\infty} \frac{+2k}{2k+1} = 1$ Yes

⑤ $\lim_{k\to\infty} \frac{(-1)^{2k+1}(2k+1)}{(2k+1)+1} = \lim_{k\to\infty} \frac{-(2k+1)}{2k+2} = -1$ Yes

⑥ $s_n = \sup\{x_n, \ldots\} = 1$ always have x_{2j} near 1 for $j \to \infty$

⑦ $\limsup_{j\to\infty} x_j = \lim_{n\to\infty} 1 = 1$

⑧ $\liminf_{j\to\infty} x_j = \lim_{n\to\infty} -1 = -1$

① Assump on the contrary $\exists L$ s.t. $x_j \to L$.
② $\forall \varepsilon > 0\ \exists N_\varepsilon$ s.t. $\forall j \ge N_\varepsilon$ $|x_j - L| < \varepsilon$ so, $x_j - \varepsilon < L < x_j + \varepsilon$
③ $x_{2j} \ge \frac{2}{3}$
④ $x_{2j+1} \le \frac{-1}{2}$
⑤ $L > x_j - \varepsilon \ge \frac{2}{3} - \varepsilon > 0$ for all $j \ge N_{1/2}$
⑥ $L < x_j + \varepsilon \le \frac{-1}{2} + \varepsilon \le 0$ for all $j \ge N_{1/2}$

① Indirect Hypothesis.
② Defn of limit
③ no time to explain but see
④ number line above
⑤ by step 2 using $\varepsilon = \frac{1}{2} > 0$ and step 3
⑥ by step 2 using $\varepsilon = \frac{1}{2} > 0$ and step 4
⑧ steps 5 + 6 Thus ind Hyp false. x_j has no lim QED

More Sample Part I if you want more practice:

5:13 AM Mon Mar 15 50%

MAT320S21

MAT320S21 | Axiomatic Geometry | Scratchwork

Sample Exam II

Part I Let $x_n = \frac{n+1}{n}$

1. Write first five terms + plot
2. Is x_n increasing?
3. Is x_n decreasing?
4. Is x_n monotone?
5. Is x_n bounded above?
6. Is x_n bounded below?
7. Is x_n bounded?
8. Does x_n converge?
9. What is the limit?

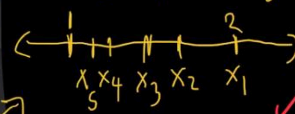
Extra Credit: Prove it is monotone.

Sample Exam II

Part I Let $x_n = \frac{n+1}{n}$

① Write first five terms + plot

① $x_1 = \frac{1+1}{1} = 2$ ✓ $x_2 = \frac{3}{2}$ ✓ $x_3 = \frac{4}{3}$ ✓ $x_4 = \frac{5}{4}$ ✓ $x_5 = \frac{6}{5}$ ✓



✓ (+2)

② Is x_n increasing? No ✓ (+2)

③ Is x_n decreasing? Yes ✓ (+2)

④ Is x_n monotone? Yes because dec ✓ (+2)

⑤ Is x_n bounded above? Yes $\frac{n+1}{n} \le \frac{n}{n} + \frac{1}{n} \le 1+1 \le 2$ ✓ (+2)

⑥ Is x_n bounded below? Yes $\frac{n+1}{n} \ge \frac{n}{n} \ge 1$ ✓ (+2)

⑦ Is x_n bounded? Yes because bounded above and below ✓ (+2)

⑧ Does x_n converge? Yes ✓ (+2)

⑨ What is the limit? 0 $\lim_{n \to \infty} \frac{n+1}{n} = \cancel{0}\ 1$ (-2)

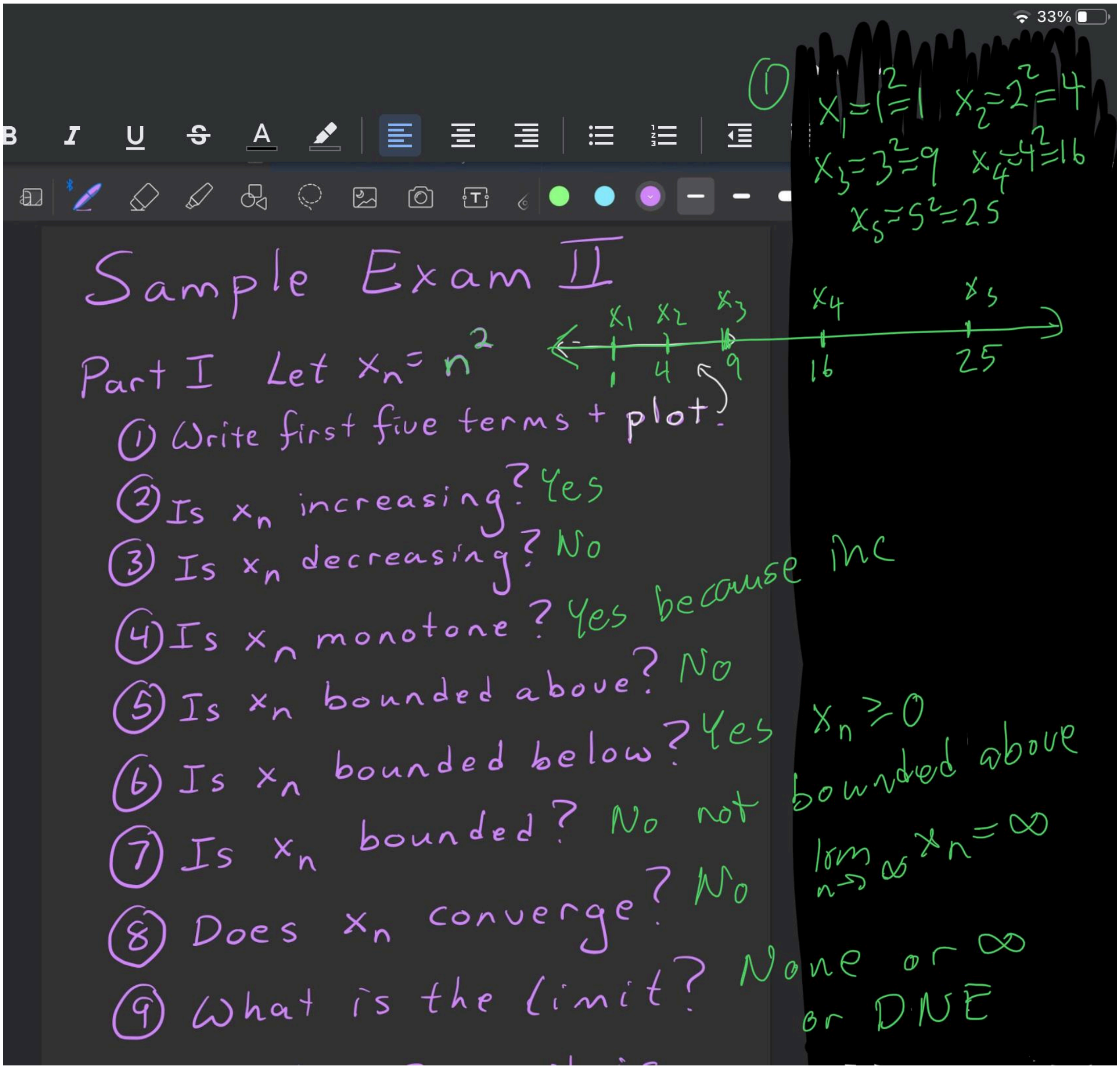
16 out of 18

Sample Exam II

Part I Let $x_n = n^2$

1. Write first five terms + plot.
2. Is x_n increasing?
3. Is x_n decreasing?
4. Is x_n monotone?
5. Is x_n bounded above?
6. Is x_n bounded below?
7. Is x_n bounded?
8. Does x_n converge?
9. What is the limit?

Extra Credit: Prove it is monotone.



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Part III: an epsilon-delta continuity or limit proof for a specific function,
Part IV: an epsilon delta continuity or limit proof for a combination of functions

****

Watch both playlists: [Limit Review Playlist] and [Continuity Review Playlist]

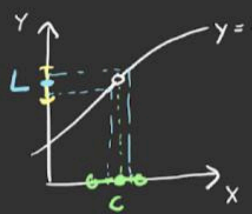
Limits $\lim_{x \to c} f(x) = L$

$\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0$ s.t. $0 < d(x,c) < \delta_\varepsilon \Rightarrow d(f(x), L) < \varepsilon$

error around the target limit

we can find an estimate about c

$x \neq c$ $\quad \underbrace{x \in B(c, \delta_\varepsilon)}_{\text{how carefully must we aim or estimate } x \text{ close to } c,\ x \neq c} \Rightarrow \underbrace{f(x) \in B(L, \varepsilon)}_{\text{to guarantee } f(x) \text{ lands in the target with error } < \varepsilon.}$



If we choose a smaller $\varepsilon > 0$

then we take δ_ε smaller too.

Classwork: Prove that $\lim_{x \to 2} 5x = 10$

Show: $\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0$ s.t. $0 < |x-2| < \delta_\varepsilon \Rightarrow |5x-10| < \varepsilon$

$10-\varepsilon < 5x < 10+\varepsilon$

Structure of the Proof:

① Given any $\varepsilon > 0$
Choose $\delta_\varepsilon = \boxed{} > 0$

① Must justify why $\delta_\varepsilon > 0$

② Whenever $0 < |x-2| < \delta_\varepsilon$
we have ...

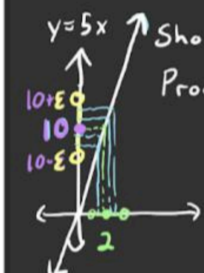
② Use choice of δ_ε

Solve upwards to find δ_ε

Justify down

final $|5x-10| < \varepsilon$

Classwork: Prove that $\lim_{x \to 2} 5x = 10$



Show: $\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0$ s.t. $0 < |x-2| < \delta_\varepsilon \Rightarrow |5x-10| < \varepsilon$

Proof:
① Given any $\varepsilon > 0$
Choose $\delta_\varepsilon = \boxed{} > 0$

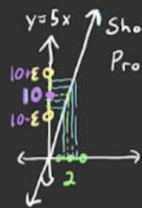
② Whenever $0 < |x-2| < \delta_\varepsilon$

Solve up for $|x-2|$ to find δ_ε

Justify down

final $|5x-10| < \varepsilon$

Classwork: Prove that $\lim_{x\to 2} 5x = 10$



Show: $\forall \varepsilon>0\ \exists \delta_\varepsilon>0$ s.t. $0<|x-2|<\delta_\varepsilon \Rightarrow |5x-10|<\varepsilon$

Proof:

① Given any $\varepsilon>0$. Choose $\delta_\varepsilon = \boxed{\varepsilon/5} > 0$ — ① $\varepsilon/5>0$ because $\varepsilon>0$

② Whenever $0<|x-2|<\delta_\varepsilon$ we have $|x-2|<\varepsilon/5$ — ② Choice of δ_ε in Step 1

③ $5|x-2|<\varepsilon$ — ③ $a<b$ and $c=5>0 \Rightarrow ac<bc$

④ $|5(x-2)|<\varepsilon$ — ④ $a|b|=|ab|$ when $a\geq 0$

final $|5x-10|<\varepsilon$ — final Distribution and $5\cdot 2=10$

QED

$\boxed{\text{HW1}}$ Prove that $\lim_{x\to 8} \frac{x}{4} = 2$

12:13 AM Tue Mar 9

MAT320S21

MAT320S21 | Axiomatic Geometry

Classwork: If $\lim_{t\to c} f(t) = F$ and $\lim_{t\to c} g(t) = G$

epsilon ε δ delta

then $\lim_{t\to c} (2f(t)+4g(t)) = 2F+4G$

Given: $\forall \varepsilon>0\ \exists \delta_\varepsilon^f>0$ s.t. $0<|t-c|<\delta_\varepsilon^f \Rightarrow |f(t)-F|<\varepsilon$

$\forall \varepsilon>0\ \exists \delta_\varepsilon^g>0$ s.t. $0<|t-c|<\delta_\varepsilon^g \Rightarrow |g(t)-G|<\varepsilon$

Show: $\forall \varepsilon>0\ \exists \delta_\varepsilon>0$ s.t. $0<|t-c|<\delta_\varepsilon \Rightarrow |(2f(t)+4g(t))-(2F+4G)|<\varepsilon$

need parentheses

Set up the proof using the show line

Given: $\forall \varepsilon > 0 \quad \exists \delta_\varepsilon^f > 0$ s.t $0 < |t-c| < \delta_\varepsilon^f \Rightarrow |f(t) - F| < \varepsilon$

$\forall \varepsilon > 0 \quad \exists \delta_\varepsilon^g > 0$ s.t $0 < |t-c| < \delta_\varepsilon^g \Rightarrow |g(t) - G| < \varepsilon$

Show: $\forall \varepsilon > 0 \quad \exists \delta_\varepsilon > 0$ s.t $0 < |t-c| < \delta_\varepsilon \Rightarrow |(2f(t)+4g(t)) - (2F+4G)| < \varepsilon$

need parentheses

Proof: ① Given any $\varepsilon > 0$

Choose $\delta_\varepsilon = \boxed{} > 0$

② Whenever $0 < |t-c| < \delta_\varepsilon$

we have ____

Solve up

Justify Down

final $|(2f(t)+4g(t)) - (2F+4G)| < \varepsilon$

12:21 AM Tue Mar 9

MAT320S21

MAT320S21 Axiomatic Geometry

need parentheses

Proof: ① Given any $\varepsilon > 0$

Choose $\delta_\varepsilon = \boxed{} > 0$

② Whenever $0 < |t-c| < \delta_\varepsilon$

we have $|f(t) - F| < \frac{\varepsilon}{4}$ AND $|g(t) - G| < \varepsilon/8$

$2|f(t) - F| < \varepsilon/2$ AND $4|g(t) - G| < \varepsilon/2$

$|2f(t) - 2F| < \frac{\varepsilon}{2}$ AND $|4g(t) - 4G| < \frac{\varepsilon}{2}$

$|2f(t) - 2F| + |4g(t) - 4G| < \varepsilon$

$\geq$ triangle ineq

$|2f(t) - 2F + 4g(t) - 4G| < \varepsilon$

$|2f(t) + 4g(t) - 2F - 4G| < \varepsilon$

Solve up

final $|(2f(t)+4g(t)) - (2F+4G)| < \varepsilon$

12:26 AM Tue Mar 9

Classwork: If $\lim_{t\to c} f(t)=F$ and $\lim_{t\to c} g(t)=G$

ε epsilon δ delta

then $\lim_{t\to c}(2f(t)+4g(t))=2F+4G$

Given: $\forall\varepsilon>0\ \exists\delta^f_\varepsilon>0$ s.t $0<|t-c|<\delta^f_\varepsilon \Rightarrow |f(t)-F|<\varepsilon$

$\forall\varepsilon>0\ \exists\delta^g_\varepsilon>0$ s.t $0<|t-c|<\delta^g_\varepsilon \Rightarrow |g(t)-G|<\varepsilon$

Show: $\forall\varepsilon>0\ \exists\delta_\varepsilon>0$ s.t $0<|t-c|<\delta_\varepsilon \Rightarrow |(2f(t)+4g(t))-(2F+4G)|<\varepsilon$

need parentheses

Proof: ① Given any $\varepsilon>0$ Choose $\delta_\varepsilon=\boxed{\min\{\delta^f_{\varepsilon/4},\ \delta^g_{\varepsilon/8}\}}>0$

② Whenever $0<|t-c|<\delta_\varepsilon$ we have $0<|t-c|<\delta^f_{\varepsilon/4}$ AND $0<|t-c|<\delta^g_{\varepsilon/8}$

③ $|f(t)-F|<\frac{\varepsilon}{4}$ AND $|g(t)-G|<\varepsilon/8$

④ $2|f(t)-F|<\varepsilon/2$ AND $4|g(t)-G|<\varepsilon/2$

⑤ $|2f(t)-2F|<\frac{\varepsilon}{2}$ AND $|4g(t)-4G|<\frac{\varepsilon}{2}$

⑥ $|2f(t)-2F|+|4g(t)-4G|<\varepsilon$

VI triangle ineq

⑦ $|2f(t)-2F+4g(t)-4G|<\varepsilon$

⑧ $|2f(t)+4g(t)-2F-4G|<\varepsilon$

final $|(2f(t)+4g(t))-(2F+4G)|<\varepsilon$

solve up

Justify down

① Given $\delta^f_{\varepsilon/4}>0$ and $\delta^g_{\varepsilon/8}>0$

② by choice of δ_ε in step 1 and defn of min

③ Given

④ $a<b$ and $c=2>0$ $\Rightarrow ac<bc$ also $c=4>0$

⑤ ______

⑥ $\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$

⑦ triangle ineq

⑧+⑨ algebra QED

Defn: $f: X \to Y$ is continuous at c

if $\lim\limits_{x \to c} f(x) = f(c)$

$\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0$ s.t. $d(x,c) < \delta_\varepsilon \Rightarrow d(f(x), f(c)) < \varepsilon$

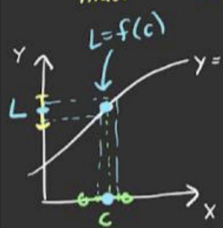
$x \in B(c, \delta_\varepsilon) \Rightarrow f(x) \in B(f(c), \varepsilon)$

error around the target limit

we can find an estimate about c

how carefully must we aim or estimate x close to c $x \neq c$

to guarantee f(x) lands in the target with error $< \varepsilon$.



no holes

Thm: $\lim\limits_{x \to c^-} f(x) = L = \lim\limits_{x \to c^+} f(x)$

iff $\lim\limits_{x \to c} f(x) = L$

Consequence: $\lim\limits_{x \to c^-} f(x) \neq \lim\limits_{x \to c^-} f(x)$

then $\lim\limits_{x \to c} f(x)$ DNE

Does Not Exist

Thm: If $f(x)$ and $g(x)$ are continuous at x_0 then $f(x)+g(x)$ is too.

Classwork

Given: $\forall \varepsilon>0\ \exists \delta_\varepsilon^f>0$ s.t $|x-x_0|<\delta_\varepsilon^f \Rightarrow |f(x)-f(x_0)|<\varepsilon$

$\forall \varepsilon>0\ \exists \delta_\varepsilon^g>0$ s.t $|x-x_0|<\delta_\varepsilon^g \Rightarrow |g(x)-g(x_0)|<\varepsilon$

Show: $\forall \varepsilon>0\ \exists \delta_\varepsilon>0$ s.t $|x-x_0|<\delta_\varepsilon$

$\Longrightarrow \left|\left(f(x)+g(x)\right)-\left(f(x_0)+g(x_0)\right)\right|<\varepsilon$

Thm: If $f(x)$ and $g(x)$ are continuous at x_0 then $f(x)+g(x)$ is too.

Classwork

Given: $\forall \varepsilon>0\ \exists \delta_\varepsilon^f>0$ s.t $|x-x_0|<\delta_\varepsilon^f \Rightarrow |f(x)-f(x_0)|<\varepsilon$

$\forall \varepsilon>0\ \exists \delta_\varepsilon^g>0$ s.t $|x-x_0|<\delta_\varepsilon^g \Rightarrow |g(x)-g(x_0)|<\varepsilon$

Show: $\forall \varepsilon>0\ \exists \delta_\varepsilon>0$ s.t $|x-x_0|<\delta_\varepsilon$

$\Longrightarrow \left|\left(f(x)+g(x)\right)-\left(f(x_0)+g(x_0)\right)\right|<\varepsilon$

Always Try Before copying!

Proof: ① Given any $\varepsilon>0$

Choose $\delta_\varepsilon = \boxed{\min\{\delta^f_{\varepsilon/2}, \delta^g_{\varepsilon/2}\}} > 0$

② Whenever $|x-x_0|<\delta_\varepsilon$

we have

$|x-x_0|<\delta^f_{\varepsilon/2}$ AND $|x-x_0|<\delta^g_{\varepsilon/2}$

$|f(x)-f(x_0)|<\frac{\varepsilon}{2}$ AND $|g(x)-g(x_0)|<\frac{\varepsilon}{2}$

$|f(x)-f(x_0)|+|g(x)-g(x_0)|<\varepsilon$

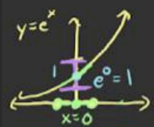
$|f(x)-f(x_0)+g(x)-g(x_0)|<\varepsilon$

final $\left|\left(f(x)+g(x)\right)-\left(f(x_0)+g(x_0)\right)\right|<\varepsilon$

Classwork fill in justifications

Fill in Justifications

Classwork Prove $f(x) = e^x$ is continuous at $x = 0$.



Prove $\lim_{x \to 0} e^x = 1$

Show: $\forall \varepsilon > 0 \; \exists \delta_\varepsilon > 0$ s.t $|x-0| < \delta_\varepsilon \Rightarrow |e^x - 1| < \varepsilon$

Set up Proof:

① Given any $\varepsilon > 0$
Choose $\delta_\varepsilon = \boxed{} > 0$

② Whenever $|x-0| < \delta_\varepsilon$ we have

solve up

final $|e^x - 1| < \varepsilon$

① Given any $\varepsilon > 0$
Choose $\delta_\varepsilon = \boxed{} > 0$

② Whenever $|x-0| < \delta_\varepsilon$ we have

$0 - \delta_\varepsilon < x < 0 + \delta_\varepsilon$

?

$\ln(1-\varepsilon) < x < \ln(1+\varepsilon)$

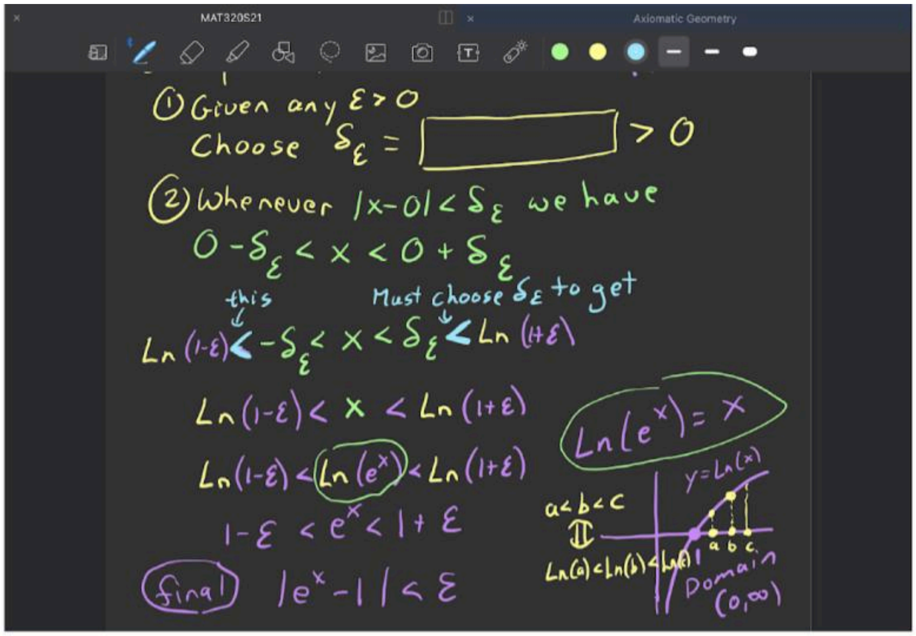
$\ln(e^x) = x$

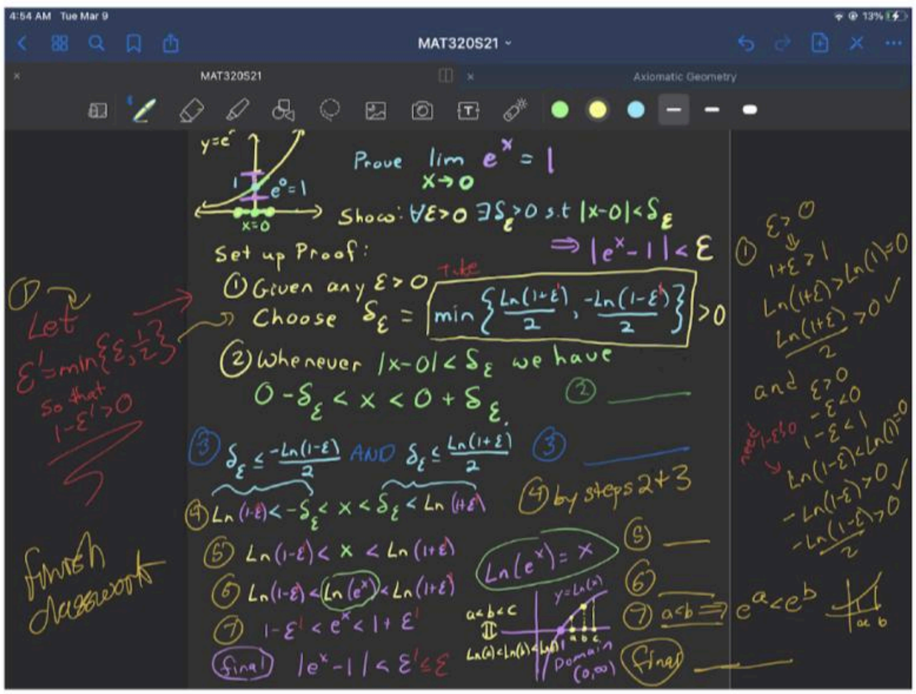
$\ln(1-\varepsilon) < \ln(e^x) < \ln(1+\varepsilon)$

$1 - \varepsilon < e^x < 1 + \varepsilon$

final $|e^x - 1| < \varepsilon$

$a < b < c$
$\Updownarrow$
$\ln(a) < \ln(b) < \ln(c)$





HW Complete Sample Exam II Parts III-IV

MAT320 # Exam II Sormani

SAMPLE

Part III

Prove $f(x) = 5x$

is continuous at $x = 2$

Be sure to write the

show line: $\forall \varepsilon > 0 \ldots$

Partial credit for the correct

proof structure:

first step

second step

final step

Be sure to number your

statements and justifications

and write the formula for δ_ε.

MAT320 Exam II Sormani

Part III 2nd SAMPLE

Prove

$$\lim_{x \to 2} f(x) = 1 \text{ when } f(x) = e^{x-2}.$$

Be sure to write the show line: $\forall \varepsilon > 0 \ldots$

Partial credit for the correct proof structure:

first step
second step
final step

Be sure to number your statements and justifications and write the formula for δ_ε.

MAT 320 Sormani

Exam II

SAMPLE

Part IV Prove:

If $\lim_{x \to 5} f(x) = 2$ and $\lim_{x \to 5} g(x) = 1$

then $\lim_{x \to 5} 2f(x) + 4g(x) = 8$

Be sure to write out your givens with δ_ε^f and δ_ε^g and write your show.

Partial credit for the correct proof structure: first step
second step
final step

Be sure to number your statements and justifications and write the formula for δ_ε

MAT 320 Exam II Sormani

2nd SAMPLE

Part IV Prove:

If $f(x)$ and $g(x)$ are continuous at x_0

then $2f(x)+4g(x)$ is continuous at x_0.

Be sure to write out your givens with δ^f_ε and δ^g_ε and write your show.

Partial credit for the correct proof structure: first step
second step
final step

Be sure to number your statements and justifications and write the formula for δ_ε

Solutions to check yourself are in this playlist with photos below. Email me if you did it a different way which might also be correct.

Sample Exam II

Part III Samples

Prove: $\lim_{x \to 2} e^{x-2} = 1$

Solution was in the review

Prove: $f(x)=5x$ is continuous at $x=2$

Prove: $\lim_{x \to 2} e^{x-2} = 1$

Show: $\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0$ s.t. $0 < |x-2| < \delta_\varepsilon \Rightarrow |e^{x-2} - 1| < \varepsilon$

Proof Structure:

(1) Given any $\varepsilon > 0$ Choose $\delta_\varepsilon : \boxed{} > 0$

(2) Whenever $0 < |\underline{x-2}| < \delta_\varepsilon$

Solve up for $\underline{x-2}$

Later Justify Down

Final $|e^{x-2} - 1| < \varepsilon$

$x \to 2$

Show: $\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0$ s.t. $0 < |x-2| < \delta_\varepsilon \Rightarrow |e^{x-2} - 1| < \varepsilon$

Proof Structure:

① Given any $\varepsilon > 0$ Choose $\delta_\varepsilon = \boxed{\min\{Ln(1+\varepsilon), -Ln(1-\varepsilon)\}} > 0$

② Whenever $0 < |x-2| < \delta_\varepsilon$

$$-\delta_\varepsilon < x-2 < \delta_\varepsilon$$

$$Ln(1-\varepsilon) < x-2 < Ln(1+\varepsilon)$$

$$Ln(1-\varepsilon) < Ln(e^{x-2}) < Ln(1+\varepsilon)$$

$$1-\varepsilon < e^{x-2} < 1+\varepsilon$$

Final $|e^{x-2} - 1| < \varepsilon$

$\delta_\varepsilon < Ln(1+\varepsilon)$

$-\delta_\varepsilon > Ln(1-\varepsilon)$

$\delta_\varepsilon < -Ln(1-\varepsilon)$

Solve up for $x-2$

Still need to justify!

Show: $\forall \varepsilon > 0\ \exists \delta_\varepsilon > 0$ s.t. $0 < |x-2| < \delta_\varepsilon \Rightarrow |e^{x-2} - 1| < \varepsilon$

Proof Structure:

Let $\varepsilon' = \min\{\varepsilon, 1/2\}$

(1) Given any $\varepsilon > 0$ Choose $\delta_\varepsilon = \boxed{\min\{Ln(1+\varepsilon'), -Ln(1-\varepsilon')\}} > 0$

(2) Whenever $0 < |x-2| < \delta_\varepsilon$

$-\delta_\varepsilon < x-2 < \delta_\varepsilon$

(3) $Ln(1-\varepsilon') < x-2 < Ln(1+\varepsilon')$

(4) $Ln(1-\varepsilon') < Ln(e^{x-2}) < Ln(1+\varepsilon')$

(5) $1-\varepsilon' < e^{x-2} < 1+\varepsilon'$

Final $|e^{x-2} - 1| < \varepsilon' \leq \varepsilon$

(1) $\varepsilon > 0 \Rightarrow 1+\varepsilon' > 1$

$\Rightarrow Ln(1+\varepsilon') > 0$

$\varepsilon' > 0 \Rightarrow 1-\varepsilon' < 1$

$\Rightarrow Ln(1-\varepsilon') < 0$

$\Downarrow$ because $\varepsilon' \leq 1/2$

$-Ln(1-\varepsilon') > 0$

(2) $|z| < R \Leftrightarrow -R < z < R$

(3) Defn of min and Choice of δ_ε in step 1

(4) $Ln(e^a) = a$

(5) $a < b < c \Rightarrow e^a < e^b < e^c$ because e^x is an increasing function

Final $-R < z < R \Leftrightarrow |z| < R$ QED

This part is worth 32% with -2 pts for each error and -5 for a serious error.

3:16 AM Mon Mar 22 59%

MAT320S21

MAT320S21 | Axiomatic Geometry | Scratchwork

Part IV Samples

Prove in the review

Prove:

If f and g are continuous at x_0

then $2f(x)+4g(x)$ is contin. at x_0.

we do this proof now

Prove

If $\lim_{x\to 5} f(x)=2$ and $\lim_{x\to 5} g(x)=1$

then $\lim_{x\to 2} 2f(x)+4g(x)=8$

3:23 AM Mon Mar 22 57%

MAT320S21

MAT320S21 | Axiomatic Geometry | Scratchwork

Given: $\lim_{x\to 5} f(x) = 2$ $\quad \forall \varepsilon > 0 \ \exists \delta_\varepsilon^f > 0$ s.t $0 < |x-5| < \delta_\varepsilon \Rightarrow |f(x) - 2| < \varepsilon$

$\lim_{x\to 5} g(x) = 1$ $\quad \forall \varepsilon > 0 \ \exists \delta_\varepsilon^g > 0$ s.t $0 < |x-5| < \delta_\varepsilon \Rightarrow |g(x) - 1| < \varepsilon$

Show $\lim_{x\to 5} 2f(x) + 4g(x) = 8$

$\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0$ s.t $0 < |x-5| < \delta_\varepsilon \Rightarrow |2f(x) + 4g(x) - 8| < \varepsilon$

Proof Structure:

① Given any $\varepsilon > 0$ Choose $\delta_\varepsilon = \boxed{} > 0$

② Whenever $0 < |x-5| < \delta_\varepsilon$ we have . . .

work up toward our givens $|f(x) - 2| < \varepsilon$ and $|g(x) - 1| < \varepsilon$

Final $|2f(x) + 4g(x) - 8| < \varepsilon$

Proof Structure:

① Given any $\varepsilon > 0$ Choose $\delta_\varepsilon = \boxed{} > 0$

② Whenever $0 < |x-5| < \delta_\varepsilon$ we have . . .

solve up

$|f(x)-2| < \varepsilon$ and $|g(x)-1| < \varepsilon$

work up

Just down

$$|2(f(x)-2)| + |4(g(x)-1)| < \varepsilon$$

cancels!

$$|2(f(x)-2) + 4(g(x)-1) + \cancel{4} + \cancel{4} - \cancel{8}| < \varepsilon$$

$$|2(f(x)-2) + 4(g(x)-1) + 2\cdot 2 + 4\cdot 1 - 8| < \varepsilon$$

final $|2f(x) + 4g(x) - 8| < \varepsilon$ ← check out algebra works

If $\lim\limits_{x\to5} f(x)=2$ and $\lim\limits_{x\to5} g(x)=1$

then $\lim\limits_{x\to5} 2f(x)+4g(x)=8$

Given: $\lim\limits_{x\to5} f(x)=2 \quad \forall\varepsilon>0\ \exists\delta^f_\varepsilon>0$ s.t $0<|x-5|<\delta^f_\varepsilon \Rightarrow |f(x)-2|<\varepsilon$

$\lim\limits_{x\to5} g(x)=1 \quad \forall\varepsilon>0\ \exists\delta^g_\varepsilon>0$ s.t $0<|x-5|<\delta^g_\varepsilon \Rightarrow |g(x)-1|<\varepsilon$

Show $\lim\limits_{x\to5} 2f(x)+4g(x)=8$

$\forall\varepsilon>0\ \exists\delta_\varepsilon>0$ s.t $0<|x-5|<\delta_\varepsilon \Rightarrow |2f(x)+4g(x)-8|<\varepsilon$

Proof:

(1) Given any $\varepsilon>0$ Choose $\delta_\varepsilon = \boxed{\min\{\delta^f_{\varepsilon/4}, \delta^g_{\varepsilon/8}\}} >0$

(2) Whenever $0<|x-5|<\delta_\varepsilon$ we have . . .

Justify Down

$0<|x-5|<\delta^f_{\varepsilon/4}$ AND $0<|x-5|<\delta^g_{\varepsilon/8}$

$|f(x)-2|<\frac{\varepsilon}{4}$ AND $|g(x)-1|<\varepsilon/8$

$2\,|f(x)-2|<\frac{\varepsilon}{2}$ AND $4\,|g(x)-1|<\varepsilon/2$

$|2(f(x)-2)|<\frac{\varepsilon}{2}$ AND $|4(g(x)-1)|<\varepsilon/2$

Work up

$|2(f(x)-2)|+|4(g(x)-1)|<\varepsilon$

cancels!

$|2(f(x)-2)+4(g(x)-1)\cancel{+4}\cancel{+4}\cancel{-8}|<\varepsilon$

Just down

$|2(f(x)-2)+4(g(x)-1)+2\cdot2+4\cdot1-8|<\varepsilon$

final $|2f(x)+4g(x)-8|<\varepsilon$ ← check out algebra works

This part is worth 32% with -2 pts for each error and -5 for a serious error.